HAMILTONIANS WITH PURELY DISCRETE SPECTRUM

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ABSTRACT. We discuss criteria for a self-adjoint operator on $L^2(X)$ to have empty essential spectrum. We state a general result for the case of a locally compact abelian group X and give examples for $X = \mathbb{R}^n$.

1. Let Δ be the positive Laplacian on $\mathbb{R}^n$. We set $B_a(r) = \{x \in \mathbb{R}^n \mid |x - a| \leq r\}$ and $B_a = B_a(1)$.

Proposition 1. *Let V be a real locally integrable function on $\mathbb{R}^n$ such that:*

(i) *if $\lambda > 0$ then the measure ω_λ of the set $\{x \in B_a \mid V(x) < \lambda\}$ satisfies $\lim_{a\to\infty} \omega_\lambda(a) = 0$,*
(ii) *the negative part of V satisfies $V_- \leq \mu\Delta + \nu$ for some positive real numbers μ, ν with $0 < \mu < 1$.*

Then the spectrum of the self-adjoint operator H associated to the form sum $\Delta + V$ is purely discrete.

Remark 2. Let $V_\pm = \max\{\pm V, 0\}$ and for each $\lambda > 0$ let $\Omega_\lambda = \{x \mid V_+(x) < \lambda\}$. Then $\omega_\lambda(a)$ is the measure of the set $B_a \cap \Omega_\lambda$. From Lemma 5 it follows that the condition (i) is equivalent to

$$\lim_{a\to\infty} \int_{B_a} \frac{\mathrm{d}x}{1 + V_+(x)} = 0. \tag{1}$$

Remark 3. From Lemma 7 we get $\lim_{a\to\infty} \omega_\lambda(a) = 0$ if $\int_{\Omega_\lambda} \omega_\lambda^p \mathrm{d}x < \infty$ for some $p > 0$. Thus Theorems 1 and 3 from [S] are consequences of Proposition 1. In the case $V \geq 0$ Proposition 1 is a consequence of Theorem 2.2 from [MS]. More general results will be obtained below. Note, however, that our techniques are not applicable in the framework considered in Theorem 2 from [S] and in [WW].

Proposition 1 is very easy to prove if condition (1) is replaced by $\lim_{x\to\infty} V_+(x) = \infty$. In fact, let us consider an arbitrary locally compact space X and let $\mathcal{H}$ be a Hilbert X-module, i.e. $\mathcal{H}$ is a Hilbert space and a nondegenerate $*$-morphism $\phi \mapsto \phi(Q)$ of $\mathcal{C}_\mathrm{o}(X)$ into $B(\mathcal{H})$ is given. For example, one may take $\mathcal{H} = L^2(X, \mu)$ for some Radon measure μ. Then we have the following simple compactness criterion: *if R is a bounded self-adjoint operator on $\mathcal{H}$ such that (i) if $\phi \in \mathcal{C}_\mathrm{o}(X)$ then $\phi(Q)R$ is a compact operator, (ii) one has $\pm R \leq \theta(Q)$ for some $\theta \in \mathcal{C}_\mathrm{o}(X)$, then R is a compact operator*. Indeed, note first that the operator $R\phi \equiv R\phi(Q)$ will also be compact for all $\phi \in \mathcal{C}_\mathrm{o}(X)$. Let $\varepsilon > 0$ and let us choose ϕ such that $0 \leq \phi \leq 1$ and $\theta\phi^\perp \leq \varepsilon$, where $\phi^\perp = 1 - \phi$. Then $\pm\phi^\perp R\phi^\perp \leq \phi^\perp\theta\phi^\perp \leq \varepsilon$ which implies $\|\phi^\perp R\phi^\perp\| \leq \varepsilon$. So we have $\|R - \phi R - \phi^\perp R\phi\| \leq \varepsilon$ and $\phi R + \phi^\perp R\phi$ is a compact operator. Now let us say that a self-adjoint operator H on $\mathcal{H}$ is *locally compact* if $\phi(Q)(H+i)^{-1}$ is compact for all $\phi \in \mathcal{C}_\mathrm{o}(X)$. Then we get: *If H is a locally compact self-adjoint operator on $\mathcal{H}$ and if there is a continuous function $\Theta : X \to \mathbb{R}$ such that $\lim_{x\to\infty} \Theta(x) = +\infty$ and $H \geq \Theta(Q)$, then the spectrum of H is purely discrete* (the nondegeneracy of the morphism is needed for the definition of $\Theta(Q)$ for unbounded Θ).

2. On the other hand, Proposition 1 can be significantly generalized. For example, Δ may be replaced by a higher order operator with matrix valued coefficients and V does not have to be a function. These results are consequences of the following "abstract" fact. We fix a locally compact abelian group X, choose a finite dimensional Hilbert space E, and define $\mathcal{H} = L^2(X) \otimes E$. For $a \in X$ and $k \in X^*$ (the dual locally compact abelian group) we denote U_a and V_k the unitary operators on $\mathcal{H}$ given by

$$(U_a f)(x) = f(x + a) \ \text{ and } \ (V_k f)(x) = k(x) f(x).$$

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We denote additively the operations both in X and in X^* and denote 0 their neutral elements.

Theorem 4. *Let H be a self-adjoint operator on $\mathcal{H}$ such that for some (hence for all) $z \in \mathbb{C}$ not in the spectrum of H the operator $R = (H - z)^{-1}$ satisfies*

$$\lim_{k\to 0} \|V_k R V_k^* - R\| = 0, \quad \lim_{a\to 0} \|(U_a - 1)R\| = 0. \tag{2}$$

Then H has purely discrete spectrum if and only if $\text{w-lim}_{a\to\infty} U_a R U_a^* = 0$.

Proof: If the spectrum of H is purely discrete then R is compact so $\text{w-lim}_{a\to\infty} U_a R U_a^* = 0$. The reciprocal assertion is a consequence of Theorem 1.2 from [GI]. Indeed, with the terminology used there, all the localizations at infinity of H will be equal to ∞ hence the essential spectrum of H will be zero. □

Some notations: if ϕ is a $B(E)$-valued Borel function on X then $\phi(Q)$ is the operator of multiplication by ϕ on $\mathcal{H}$; if ψ is a similar function on X^* then $\psi(P) = \mathcal{F}^{-1} M_\psi \mathcal{F}$, where $\mathcal{F}$ is the Fourier transformation and M_ψ is the operator of multiplication by ψ on $L^2(X^*) \otimes E$. Note that $V_k \psi(P) V_k^* = \psi(P + k)$.

If $\phi \in L^\infty(X)$ and $\phi \geq 0$ then it is easy to check that $\text{w-lim}_{a\to\infty} U_a \phi(Q) U_a^* = 0$ if and only if $\text{s-lim}_{a\to\infty} \phi(Q) U_a = 0$ and also if and only if there is a compact neighborhood of the origin W such that $\lim_{a\to\infty} \int_{a+W} \phi \mathrm{d}x = 0$. Then we say that ϕ is *weakly vanishing (at infinity)*. See Section 6 in [GG] for further properties of this class of functions. Below W is a compact neighborhood of the origin, $W_a = a + W$, and we denote $|M|$ the Haar measure of a set M.

Lemma 5. *A positive function $\phi \in L^\infty(X)$ is weakly vanishing if and only if for any number $\lambda > 0$ the set $\Omega^\lambda = \{x \mid \phi(x) > \lambda\}$ has the property* $\lim_{a\to\infty} |W_a \cap \Omega^\lambda| = 0$.

This follows from the estimates

$$\lambda |W_a \cap \Omega^\lambda| \leq \int_{W_a} \phi \,\mathrm{d}x \leq \|\phi\|_{L^\infty} |W_a \cap \Omega^\lambda| + \lambda |W|.$$

Proposition 6. *Let H be an invertible self-adjoint operator satisfying* (2) *and such that $\pm H^{-1} \leq \phi(Q)$ for some weakly vanishing function ϕ. Then H has purely discrete spectrum.*

Indeed, we may take $R = H^{-1}$ and then for any $f \in \mathcal{H}$ we have $|\langle f | U_a R U_a^* \rangle| \leq |\langle f | U_a \phi(Q) U_a^* \rangle|$.

Proof of Proposition 1: Here $X = \mathbb{R}^n$ and we identify as usual X with its dual by setting $k(x) = \mathrm{e}^{ikx}$ for $x, k \in X$. Then if $P_j = -i\partial_j$ and $P = (P_1, \ldots, P_n)$ we get $V_k P V_k^* = P + k$. To simplify notations we write H for $\Delta + V + 1 + \nu$, so that $H \geq (1 - \mu)\Delta + V_+ + 1 \geq V_+ + 1 \geq 1$. Then observe that the form domain of H is $\mathcal{G} \equiv D(H^{1/2}) = \{f \in \mathcal{H}^1 \mid V_+^{1/2} f \in L^2\}$ where $\mathcal{H}^1$ is the first order Sobolev space. Thus $R = H^{-1} : L^2 \to \mathcal{H}^1$ is continuous and this implies the second part of condition (2). On the other hand, H extends to a continuous bijective operator $\mathcal{G} \to \mathcal{G}^*$ whose inverse is an extension of R to a continuous map $\mathcal{G}^* \to \mathcal{G}$. We keep the notations H, R for these extensions. Clearly V_k leaves invariant $\mathcal{G}$ hence extends to a continuous operator on $\mathcal{G}^*$ and the groups of operators $\{V_k\}$ are of class C_0 in both spaces. Now $H_k := V_k H V_k^* = (P + k)^2 + V = H + 2kP + k^2$ in $B(\mathcal{G}, \mathcal{G}^*)$ so if $R_k := V_k R V_k^*$ then

$$R_k - R = R_k (H - H_k) R = -R_k (2kP + k^2) R$$

in $B(\mathcal{G}^*, \mathcal{G})$. Now clearly the first part of (2) is fulfilled. Finally, it suffices to show that $H^{-1} \leq \phi(Q)$ for a weakly vanishing function ϕ. But $H \geq 1 + V_+$ and we may take $\phi = (1 + V_+)^{-1}$ due to (1). □

Remark 3 is a consequence of the next result.

Lemma 7. *Let $\Omega \subset \mathbb{R}^n$ be a Borel set and let $\omega : \mathbb{R}^n \to \mathbb{R}$ be defined by $\omega(a) = |B_a \cap \Omega|$. If ω^p is integrable on Ω for some $p > 0$ then $\omega(a) \to 0$ as $a \to \infty$.*

Proof: The main point is the following observation due to Hans Henrik Rugh: *let ν be the minimal number of (closed) balls of radius $1/2$ needed to cover a ball of radius one; then for any a there is a Borel set $A_a \subset B_a \subset \Omega$ with $|A_a| \geq \omega(a)/\nu$ such that $\omega(x) \geq \omega(a)/\nu$ if $x \in A_a$*. Indeed, let N be a set of ν points such that $B_a \subset \cup_{b\in N} B_b(1/2)$. If $D_b = B_a \cap B_b(1/2)$ then $\omega(a) \leq \sum_b |D_b \cap \Omega|$ hence there is $b(a)$ such that $A_a = D_{b(a)} \cap \Omega$ satisfies $|A_a| \geq \omega(a)/\nu$. Since A_a has diameter smaller than one, for $x \in A_a$ we have $A_a \subset B_x \cap \Omega$ hence $\omega(x) \geq |A_a|$, which proves the remark. Now let us set $R = |a| - 1$ and denote $\Omega(R)$ the set of points $x \in \Omega$ such that $|x| \geq R$. Then we have

$$\int_{\Omega(R)} \omega^p \mathrm{d}x \geq \int_{A_a} \omega^p \mathrm{d}x \geq [\nu\omega(a)]^{p+1}$$

which clearly implies the assertion of the lemma. □

3. We present here some consequences of Proposition 6. We refer to [GI] for general classes of operators verifying condition (2) and consider here only some particular cases. We mention that *if H is a bounded from below operator satisfying* (2) *and if $\theta : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\theta(\lambda) \to +\infty$ when $\lambda \to +\infty$ the $\theta(H)$ also satisfies* (2).

If $R \in B(\mathcal{H})$ satisfies the first part of (2) we say that R is a *regular operator* (or Q-regular). The regularity of the resolvent of a differential operators on $\mathbb{R}^n$ is easy to check because $V_k P V_k^* = P + k$, cf. the proof of Proposition 1. The second part of (2) is equivalent to the existence of a factorization $R = \psi(P)S$ with $\psi \in \mathcal{C}_o(X^*)$ and $S \in B(\mathcal{H})$. If $X = \mathbb{R}^n$ then it suffices that the domain of H be included in some Sobolev space $\mathcal{H}^m$ with $m > 0$ real. We now give an extension of Proposition 1 which is proved in essentially the same way. We assume $X = \mathbb{R}^n$ and work with Sobolev spaces but a similar statement holds for an arbitrary X: it suffices to replace the function $\langle k \rangle^m$ which defines $\mathcal{H}^m$ by an arbitrary weight [GI] and the ball B_a by $a + W$ where W is a compact neighborhood of the origin.

Proposition 8. *Let H_0 be a bounded from below self-adjoint operator on $\mathcal{H}$ with form domain equal to $\mathcal{H}^m$ for some real $m > 0$ and satisfying $\lim_{k\to 0} V_k H_0 V_k^* = H_0$ in norm in $B(\mathcal{H}^m, \mathcal{H}^{-m})$. Let V be a positive locally integrable function such that $\lim_{a\to\infty} |\{x \in B_a \mid V(x) < \lambda\}| = 0$ for each $\lambda > 0$. Then the self-adjoint operator H associated to the form sum $H_0 + V$ has purely discrete spectrum.*

Let $h : X \to B(E)$ be a continuous symmetric operator valued function with $c'|p|^{2m} \leq h(p) \leq c''|p|^{2m}$ (as operators on E) for some constants $c', c'' > 0$ and all large p. Let $W : \mathcal{H}^m \to \mathcal{H}^{-m}$ be a symmetric operator such that $W \geq -\mu h(P) - \nu$ with $\mu < 1$ and such that $V_k W V_k^* \to W$ in norm in $B(\mathcal{H}^m, \mathcal{H}^{-m})$ as $k \to 0$. Then the form sum $h(P) + W$ is bounded from below and closed on $\mathcal{H}^m$ and the self-adjoint operator H_0 associated to it satisfies the conditions of Proposition 8.

Assume that $m \geq 1$ is an integer and let $L = \sum_{\alpha,\beta} P^\alpha a_{\alpha\beta}(Q) P^\beta : \mathcal{H}^m \to \mathcal{H}^{-m}$ where α, β are multi-indices of length $\leq m$ and $a_{\alpha\beta}$ are functions $X \to B(E)$ such that $a_{\alpha\beta}(Q)$ is a continuous map $\mathcal{H}^{m-|\beta|} \to \mathcal{H}^{|\alpha|-m}$. If $\langle f|Lf\rangle \geq \mu\|f\|^2_{\mathcal{H}_m} - \nu\|f\|^2_{\mathcal{H}}$ for some $\mu, \nu > 0$ then L is a closed bounded from below form on $\mathcal{H}^m$ and the self-adjoint operator H_0 associated to it verifies Proposition 8.

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