RATE OF DECOHERENCE FOR AN ELECTRON WEAKLY COUPLED TO A PHONON GAS

RICCARDO ADAMI AND LÁSZLÓ ERDŐS

Abstract. We study the dynamics of an electron weakly coupled to a phonon gas. The initial state of the electron is the superposition of two spatially localized distant bumps moving towards each other, and the phonons are in a thermal state. We investigate the dynamics of the system in the kinetic regime and show that the time evolution makes the non-diagonal terms of the density matrix of the electron decay, destroying the interference between the two bumps. We show that such a damping effect is exponential in time, and the related decay rate is proportional to the total scattering cross section of the electron-phonon interaction.

1. Introduction and result

Quantum interference of a state can be destroyed via interactions with an external environment. This phenomenon is called decoherence and it is recognized to be responsible for the transition from quantum superpositions to classical probability measures. This feature makes it relevant in the field of quantum information, where the superposition of states allows for parallel computation and therefore it must be preserved.

Rigorous results on decoherence have so far concerned systems with a discrete energy spectrum or decoherence induced by practically instantaneous scattering at time zero. For the first model, we mention that the decoherence of an electron trapped in a harmonic oscillator and interacting with an electromagnetic field in the dipole approximation was studied in [DS]. For the second situation, in [DFT, AFFT1, AFFT2] it was showed that decoherence is the most relevant effect in the dynamics of many heavy particles in an environment of light particles, when the mass ratio is small. For an extensive physical introduction to decoherence we refer to [GJKKSZ, O].

Here we investigate the occurrence of decoherence in the dynamics of an electron weakly coupled to a phonon field. The energy spectrum of the electron is continuous and the phonons constantly bombard the electron along its time evolution. We will study the problem in the kinetic regime with a weak coupling. We introduce a scale parameter ε and analyze the time evolution at large space and time scales of order ε^{-1}. If (x,t) denote the original (microscopic) space and time coordinates, then we introduce the macroscopic space and time coordinates (X,T) by

$$T := \varepsilon t, \quad X := \varepsilon x.$$

The coupling constant between the electron and phonon field will be given by $\lambda = \sqrt{\varepsilon}$.

On the macroscopic space and time scales, the dynamics of the electron in such a limit is described by a linear Boltzmann equation, as proven in [E]. In this paper we prove that, in addition to such behavior, on the microscopic scale the quantum interference remains detectable and decays exponentially in time with a rate growing with the strength of the electron-phonon interaction.

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To be more specific, we choose a special initial state that exhibits a strong interference pattern without phonon interaction. We assume that initially the electron lies in the superposition of two distant macroscopic wave packets ("bumps") in $\mathbb{R}^d$ ($d \geq 3$) moving towards each other, namely

$$\begin{aligned} \psi_0^\varepsilon(x) &:= \psi_{0,+}^\varepsilon(x) + \psi_{0,-}^\varepsilon(x) \\ \psi_{0,+}^\varepsilon(x) &:= \varepsilon^{d/2} f(\varepsilon x + Q) e^{iP\cdot x} \\ \psi_{0,-}^\varepsilon(x) &:= \varepsilon^{d/2} f(\varepsilon x - Q) e^{-iP\cdot x} \end{aligned} \tag{1.1}$$

where f is L^2-normalized and satisfies the following hypothesis of regularity

$$f \in H^{\frac{3d}{2}+6}(\mathbb{R}^d). \tag{1.2}$$

The vectors $P, Q \in \mathbb{R}^d$ are parallel, i.e. $P \cdot Q = |P||Q|$ and assume $P \neq 0$. The wave packets $\psi_{0,-}^\varepsilon$ and $\psi_{0,+}^\varepsilon$ are localized around the macroscopic points Q and $-Q$ and they move towards each other by a momentum P. The function f describes the macroscopic envelope of the bumps. We remark that ψ_0^ε is not normalized in L^2, but it is easily seen that $\lim_{\varepsilon\to 0} \|\psi_0^\varepsilon\| = \sqrt{2}$, so we will always consider ε sufficiently small to guarantee $1 \leq \|\psi_0^\varepsilon\|_2 \leq 2$.

To explain our result, we preliminarily examine what happens if the interaction between electron and field is absent. In this case the state of the electron $\psi_t^\varepsilon = \psi_{t,\mathrm{free}}^\varepsilon$ evolves according to the free Schrödinger equation $i\partial_t \psi_t = -\frac{1}{2}\Delta\psi_t$. The two wave packets maximally overlap at time $\bar{t} = \varepsilon^{-1}|Q|/|P|$, when the probability density of finding the electron at x, apart from the normalization factor, is given by

$$|\psi_{\bar{t},\mathrm{free}}^\varepsilon(x)|^2 = 2(1+\cos(2P\cdot x)) \left| \varepsilon^{d/2} \left(e^{i\varepsilon^2 \bar{t} \frac{\Delta}{2}} f \right)(\varepsilon x) \right|^2 .$$

The shape of the bumps has undergone a semiclassical change and their supports are shifted to the origin. If the state were classical, the probability densities of the bumps should add up, yielding the factor of 2. The cosine term carries the typical interference fringes and therefore the information that the electron lies in a quantum superposition of two wave packets.

Let us remark the presence of two spatial scales in $|\psi_{\bar{t},\mathrm{free}}^\varepsilon(x)|^2$: the envelope of the wave function is spread on a length of order ε^{-1}, while the interference pattern manifests itself on a length scale of order 1. The size of the interference fringes can be quantified evaluating the Fourier transform of $|\psi_{\bar{t},\mathrm{free}}^\varepsilon|^2$ and then performing the limit $\varepsilon \to 0$. One easily sees

$$\lim_{\varepsilon\to 0} \int dx\, e^{2i\widetilde{P}\cdot x} |\psi_{\bar{t},\mathrm{free}}^\varepsilon(x)|^2 = \begin{cases} 2 & \text{if } \widetilde{P} = 0 \\ 1 & \text{if } \widetilde{P} = \pm P \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

If, instead of being isolated, the electron interacts from the beginning with a phonon field that initially lies in a thermal state, then the interference fringes are expected to be damped by a factor exponentially decreasing in the collision time in macroscopic units, i.e. with $|Q|/|P|$.

Indeed, we prove that

$$\lim_{\varepsilon\to 0} \int dx\, e^{\pm 2iP\cdot x} \rho_{\bar{t}}^\varepsilon(x) = e^{-\frac{\sigma_P |Q|}{|P|}}$$

where σ_P, given in (7.4), is the total scattering cross section for an electron with velocity P interacting with the phonon field, and $\rho_{\bar{t}}^\varepsilon(x)$ is the probability density of finding the interacting electron at x at time $\bar{t}$. The interacting Hamiltonian will be defined in Section 4. Our method easily extends to more

general initial states whose free evolution exhibits interference fringes. For simplicity we discuss only the particular case given by (1.1).

To state our result in a more precise way, we use the Wigner function formalism instead of the wave-function description of the state of the electron. At time zero, the Wigner function of the electron reads

$$W_0(x,v) \;:=\; \int_{\mathbb{R}^d} \frac{dy}{(2\pi)^d}\, e^{-iv\cdot y}\psi_0^\varepsilon(x+y/2)\overline{\psi_0^\varepsilon(x-y/2)}\;. \tag{1.4}$$

According to the decomposition of ψ_0^ε given in (1.1), formula (1.4) yields a decomposition in four terms for the Wigner function of the initial state

$$W_0 \;=\; \sum_{\alpha,\alpha'\in\{\pm\}} W_{0,\alpha\alpha'}, \qquad W_{0,\alpha\alpha'}(x,v) \;:=\; \int_{\mathbb{R}^d} \frac{dy}{(2\pi)^d}\, e^{-iv\cdot y}\psi_{0,\alpha}^\varepsilon(x+y/2)\overline{\psi_{0,\alpha'}^\varepsilon(x-y/2)}\;,$$

where the so-called diagonal terms $W_{0,++}$ and $W_{0,--}$ represent an electron in the state $\psi_{0,+}^\varepsilon$ and $\psi_{0,-}^\varepsilon$ respectively, and the non-diagonal terms $W_{0,+-}$ and $W_{0,-+}$ represent the interference between the states $\psi_{0,+}^\varepsilon$ and $\psi_{0,-}^\varepsilon$.

Since the time evolution of the Wigner function is linear, one can study separately the evolution of diagonal terms $W_{++}(t)$ and $W_{--}(t)$ and of non-diagonal ones $W_{+-}(t)$ and $W_{-+}(t)$. We will use the notations $W_{\pm,\pm,\mathrm{free}}(t)$ and $W_{\pm,\pm}(t)$ for the corresponding components of the Wigner transform of the state at time t under the free and the interacting evolutions, respectively. The interacting evolution will be defined in Section 4. We remark that after tracing out the phonon degrees of freedom, the electron will be in a mixed state and thus $W_{\pm,\pm}(t)$ will be Wigner functions of density matrices.

In order to easily define the phonon field operators in a rigorous way we consider the system confined in a d-dimensional box $\Lambda_L = [-L/2, L/2]^d$ and then perform the thermodynamic limit $L\to\infty$. Such a limit is taken after the phonon trace but before the scaling limit $\varepsilon\to 0$, and the result is uniform in L. The parameter L will be suppressed in the notation of $W_{\pm,\pm}(t)$. Before performing the thermodynamic limit $L\to\infty$, any integral in the space variables will be thought of as one over the volume Λ, and any integral in the momentum variable is actually a summation over $\Lambda^\star = (L^{-1}\mathbb{Z})^d$. For the computation of the thermodynamic limit we refer to section 4.5 in [E]. The free evolution will always be considered in the infinite volume to avoid taking the unnecessary thermodynamic limit in $W_{\pm,\pm,\mathrm{free}}(t)$.

We will test the evolved Wigner functions against observables, $J(x,v) = J_\varepsilon(x,v)$ that may scale with ε to detect the interference fringes. Let $\langle J, W\rangle = \int \overline{J(x,v)}W(x,v)dxdv$ denote the expectation value of the observable J on a state given by the Wigner function W.

We will always assume that

$$\|J\| := \sup_\varepsilon \int \sup_v |\widehat{J}_\varepsilon(\xi,v)|d\xi < \infty\;, \tag{1.5}$$

where we use the convention that the hat on functions defined on the electronic phase space denotes the Fourier transform in the space variable only, i.e.

$$\widehat{J}_\varepsilon(\xi,v) = \frac{1}{(2\pi)^{d/2}}\int e^{-i\xi\cdot x}J_\varepsilon(x,v)dx\;.$$

In the following Theorem 1.1 we compare the kinetic limit for $W_{+-}(t)$ and $W_{-+}(t)$ with the corresponding terms under the free evolution. With these notations, we have the following main theorem:

Theorem 1.1. *Let the initial state of the electron be given by (1.1) with $P \neq 0$ and fix a macroscopic time T. At any time t, let $W_{+-,\mathrm{free}}(t)$ be a non-diagonal component of the Wigner function of a non-interacting ("isolated") electron evolved under the free evolution. Let $W_{+-}(t)$ be the analogous non-diagonal component of the Wigner function describing the electron at time t interacting with a phonon field through the interaction Hamiltonian (4.1). Then, for any observable J_ε satisfying* (1.5)*, we have*

$$\lim_{\varepsilon\to 0}\left|\lim_{L\to\infty}\langle J_\varepsilon, W_{+-}(\varepsilon^{-1}T)\rangle \;-\; e^{-T\sigma_P}\langle J_\varepsilon, W_{+-,\mathrm{free}}(\varepsilon^{-1}T)\rangle\right| \;=\; 0 \tag{1.6}$$

where σ_P, given in (7.4)*, is the total cross section for an electron with momentum P in the phonon field.*

To detect the destruction of the interference fringes, the two-scale structure of $W_{+-}(t)$ and $W_{-+}(t)$ obliges us to test $W_{+-}(t)$ and $W_{-+}(t)$ against observables endowed with the same two-scale structure. A possible class of such observables J_ε will be given in Section 4.4. In particular, for these observables one easily obtains that $\lim_{\varepsilon\to 0}\langle J_\varepsilon, W_{+-,\mathrm{free}}(\varepsilon^{-1}T)\rangle$ exists. The precise statement is formulated in Corollary 4.1.

The evolution of the diagonal terms, W_{++} and W_{--}, on the macroscopic scales is given by the linear Boltzmann equation,

$$(\partial_T + \nabla e(V)\cdot\nabla_X)F_T(X,V) = \int \sigma(V,U)F_T(X,U)dU - \Big(\int \sigma(U,V)dU\Big)F_T(X,V) \tag{1.7}$$

with collision kernel $\sigma(U,V)$ and free dispersion relation $e(V) = \frac{1}{2}V^2$. The two terms on the right hand side are called the gain and loss terms, respectively. For the precise statement, see [E].

Theorem 1.1 describes the evolution of the off-diagonal terms. The physical scenario can be explained as follows: in the kinetic limit, the two initial bumps evolve independently as classical phase space probability densities obeying the linear Boltzmann equation (1.7). In addition to that, there is an interference term that equals the corresponding term for the free case, except for a damping factor, exponential in time. In other words, the interference term evolves according to a Boltzmann equation without gain term.

From the technical point of view, we expand the time evolution into a Duhamel sum, trace out the degrees of freedom of the phonon and lastly perform the scaling limit. This is conveniently made using Feynman graph expansions, similarly to [EY] and [E]. The novelty in our model is that the relevant graphs are no longer given by the ladder terms: these ones vanish in the limit and the resulting dynamics is given by the renormalized free evolution.

The paper is organized as follows. In Section 2 we fix some basic notation and conventions; in Section 3 we state regularity assumptions on the dispersion relations and on the form factor of the interaction, and give a lemma on some estimates that will be used repeatedly along the paper. In Section 4 we define the model and the scaling, and specify the assumptions on the initial state and the observables. In Section 5 we explain the link with the result provided in [E]. In Section 6 we prove that the contribution of the ladder graphs is negligible in the kinetic limit, and in Section 7 we compute the contribution due to renormalized free propagators. The paper ends by an appendix containing the proof of the lemma stated in Section 3.

2. Notation and conventions

We model the electron as a spinless particle, so the state space for the electron is $\mathcal{H}_{\rm e} := L^2_{\rm per}(\Lambda_L)$. The electron dispersion relation is denoted by $e(k)$, $k \in \mathbb{R}^d$, and the Hamiltonian $H_{\rm e}$ of a free electron acts on $\mathcal{H}_{\rm e}$ as follows

$$H_{\rm e} \;=\; e(-i\nabla). \tag{2.1}$$

One can think of the classical dispersion relation $e(p) = \frac{|p|^2}{2M}$ or the pseudo-relativistic one $e(p) = \sqrt{|p|^2 + M^2}$, where M is the mass of the electron.

Let γ be the density operator representing the state of the electron; γ is a positive operator on $L^2(\mathbb{R}^d)$ and its operator kernel is denoted by $\gamma(x,y)$. Let $\widehat{\gamma}$ be the Fourier transform of γ as an operator, i.e. its kernel is given by

$$\widehat{\gamma}(p,u) \;:=\; \int \frac{dxdy}{(2\pi)^d} e^{-ip\cdot x + iu\cdot y}\gamma(x,y).$$

The Wigner transform of γ is defined as

$$W_\gamma(x,v) \;=\; \int e^{-iv\cdot y}\gamma\left(x+\frac{y}{2}, x-\frac{y}{2}\right)\frac{dy}{(2\pi)^d}. \tag{2.2}$$

W_γ is a real function defined on the classical phase space of the electron. We recall the convention that the hat on functions defined on the phase space of the electron denotes Fourier transform in the space variable only, i.e.

$$\widehat{W_\gamma}(\xi,v) \;=\; \int \frac{dx}{(2\pi)^{\frac{d}{2}}} e^{-i\xi\cdot x}\, W_\gamma(x,v) \;=\; (2\pi)^{-\frac{d}{2}}\widehat{\gamma}\left(v+\frac{\xi}{2}, v-\frac{\xi}{2}\right). \tag{2.3}$$

The rescaled (macroscopic) Wigner function is defined as follows

$$W^\varepsilon_\gamma(X,V) \;=\; \varepsilon^{-d} W_\gamma\left(\frac{X}{\varepsilon}, V\right).$$

If the density matrix γ is the orthogonal projection on the space spanned by the funtion φ (i.e. the electron lies in a pure state), then we denote the associated Wigner function by W_φ. We will sometimes use symbols like $W(t)$, without explicitly referring to a density matrix γ. In such cases, we will always mean the Wigner function of the electron at time t, i.e. the Wigner transform of the reduced density matrix for the electron $\gamma_{{\rm e},t}$ to be defined in (4.6).

The pure states of the phonon field are represented by vectors in the bosonic Fock space $\mathcal{H}_{\rm ph} = \oplus_{n=0}^\infty \left[L^2(\Lambda)\right]^{\otimes_s n}$, where $\otimes_s n$ is the n-fold symmetrized tensor product. We introduce the phonon creation and annihilation operators, $a_k^\dagger$, a_k, with momentum k, satisfying the usual commutation relations

$$[a_k, a_{k'}^\dagger] = \delta(k-k') \;.$$

The number operator of the phonons in mode k is

$$\mathcal{N}_k \;:=\; a_k^\dagger a_k.$$

The Hamiltonian of a free phonon field reads

$$H_{\rm ph} \;=\; \int dk\, \omega(k)\, \mathcal{N}_k. \tag{2.4}$$

Here $\omega(k)$ is the dispersion relation for phonons. The state of thermal equilibrium for a phonon field at inverse temperature β and chemical potential μ is given by the density operator

$$\gamma_{\rm ph} := Z^{-1}\Big[e^{-\beta H_{\rm ph}+\mu\int N_k dk}\Big], \qquad \text{with} \quad Z := \mathrm{Tr}_{\mathcal{H}_{\rm ph}}\left(e^{-\beta H_{\rm ph}+\mu\int N_k dk}\right). \tag{2.5}$$

The expected number of phonons in the mode k reads

$$\mathcal{N}(k) := \mathrm{Tr}_{\mathcal{H}_{\rm ph}}(\gamma_{\rm ph} N_k) = \frac{e^{-\beta\omega(k)+\mu}}{1-e^{-\beta\omega(k)+\mu}}.$$

The symbol C will denote various positive constants arising in estimates.

3. Assumptions

In the following we will use extensively the notation $\langle x\rangle := (x^2+1)^{\frac{1}{2}}$ and the estimates

$$\begin{aligned} \langle x+y\rangle &\le C\langle x\rangle\langle y\rangle \\ \langle x+y\rangle^{-1} &\le C\langle x\rangle^{-1}\langle y\rangle. \end{aligned} \tag{3.1}$$

The dispersion relations $e(\cdot)$, $\omega(\cdot)$ for electrons and phonons are assumed to be:

- spherically symmetric;
- decaying for large k up to the $2d^{\rm th}$ derivative:

$$\begin{aligned} |\nabla^l e(k)| &\le C(1+\langle k\rangle^{2-l}), \ \ l=0,1,2,\dots,2d; \\ |\nabla^l \omega(k)| &\le C(1+\langle k\rangle^{2-l}), \ \ l=0,1,2,\dots,2d. \end{aligned} \tag{3.2}$$

We will often consider the functions

$$\Phi_\pm(p,k) := e(k+p) \pm \omega(k). \tag{3.3}$$

We assume that the following relations hold

$$\lim_{k\to\infty} \Phi_\pm(p,k) = \infty \tag{3.4}$$

$$\exists\, C_1, C_2 \in \mathbb{R} \quad \text{s.t.} \quad 0 < C_1 \le \mathrm{Hess}_k \Phi_\pm(p,k) \le C_2. \tag{3.5}$$

Let us consider the "thick level sets"

$$E_\pm(p,\theta,\delta) := \{k : \ |\Phi_\pm(p,k)-\theta| \le \delta\}$$

for the functions $\Phi_\pm$. From (3.2), (3.4), (3.5) one can conclude that there exist $\tilde C$, $\tilde\rho$ such that for any $\delta,\rho \le \tilde\rho$

$$\sup_{p,q,\theta} |E_\pm(p,\theta,\delta)\cap B(q,\rho)| \le \tilde C \delta \rho^{d-1}, \tag{3.6}$$

where $|\cdot|$ denotes the Lebesgue measure, and $B(q,\rho)$ is the ball of radius ρ centered in $q\in\mathbb{R}^d$. We require a further condition on the intersection of two such sets: there exist $\tilde\rho, C_3 > 0$ such that for any $\delta_1,\delta_2,\rho\le\tilde\rho$, $p_1,p_2\in\mathbb{R}^d$, $\theta_1,\theta_2\in\mathbb{R}$

$$\sup_q |E_\pm(p_1,\theta_1,\delta_1)\cap E_\pm(p_2,\theta_2,\delta_2)\cap B(q,\rho)| \le \frac{C_3\delta_1\delta_2\rho^{d-2}}{|p_1-p_2|}. \tag{3.7}$$

The listed conditions are fulfilled for a classical or a pseudo-relativistic electron if $\|\nabla^2\omega\|_\infty$ is sufficiently small.

To ensure that the density operator for the phonon field is trace class we assume

$$\inf_k \omega(k) - \mu\beta^{-1} \geq C > 0.$$

The electron and the phonon field will be coupled via an interaction form factor $F(k) : \mathbb{R}^d \to \mathbb{R}$. It is assumed to be real and symmetric, namely

$$F(k) = F(-k) = \overline{F(k)}$$

and it has a fast decay up to $2d$-derivatives

$$\max_{l=0,\ldots,2d} \left|\nabla_k^l F(k)\right| \ \leq \ C\langle k\rangle^{-2d-12}. \tag{3.8}$$

For $\sigma = \pm 1$ we define the functions

$$L(k,\sigma) := |F(k)|^2 \left(\mathcal{N}(k) + \frac{\sigma+1}{2}\right)$$

and

$$\mathcal{L}(k) := L(k,1) = |F(k)|^2 \left(\mathcal{N}(k)+1\right). \tag{3.9}$$

Obviously, $L(k,\pm 1) \leq \mathcal{L}(k)$, and $\mathcal{L}$ shares with F the decay estimate (3.8).

These conditions on $e(k)$, $\omega(k)$ and $F(k)$ ensure the following key estimates that we will subsequently use:

Lemma 3.1. *For the functions $\mathcal{L}$ defined in (3.9), and $\Phi_\pm$ defined in (3.3), the following estimates hold*

$$\sup_{p,\theta} \int_{\mathbb{R}^d} \frac{\mathcal{L}(k)\,dk}{|\theta - \Phi_\pm(p,k) + i\eta|^{m+1}} \ \leq \ C\eta^{-m}, \quad m > 0; \tag{3.10}$$

$$\int_{\mathbb{R}^d} \frac{\mathcal{L}(k)\,dk}{|\theta - \Phi_\pm(p,k) + i\eta|\,|\widetilde{\theta} - \Phi_\pm(u,k) - i\eta|\langle p\rangle\langle u\rangle} \ \leq \ \frac{C(\log^\star \eta)^2}{|p-u|_\star \langle\theta\rangle^{\frac{1}{2}}\langle\widetilde{\theta}\rangle^{\frac{1}{2}}}; \tag{3.11}$$

$$\sup_v \int_{\mathbb{R}} \frac{d\alpha}{|\alpha - e(v_2+\xi/2) + i\eta|\,\langle\alpha\rangle} \ \leq \ C\log^\star \eta; \tag{3.12}$$

where $\eta > 0$ and we used the notation $f^\star = \max(1,|f|)$, $f_\star = \min(1,|f|+\eta)$.

The new estimate compared with [E] is (3.11), whose proof is given in the appendix.

In the proof of Theorem 1.1 we will use the following inequality too:

$$\sup_v \left| \sum_{\sigma\in\{\pm\}} \int e^{-is\Phi_\sigma(p,k)} L(k,\sigma)\,dk \right| \ \leq \ \frac{C}{\langle s\rangle^{\frac{d}{2}}} \tag{3.13}$$

The proof of (3.13) is in Lemma 4.1 of [E].

4. The model

4.1. **Hamiltonian.** The dynamics of the system consisting in an electron interacting with a phonon field is generated by the Hamiltonian

$$H \;=\; H_{\rm e}\otimes \mathbb{I}_{\mathcal{H}_{\rm ph}} + \mathbb{I}_{\mathcal{H}_{\rm e}}\otimes H_{\rm ph} + H_I.$$

In any tensor product the first factor acts on $\mathcal{H}_{\rm e}$ and the second on $\mathcal{H}_{\rm ph}$. The free Hamiltonians $H_{\rm e}$ and $H_{\rm ph}$ have been defined in (2.1) and (2.4). The interaction Hamiltonian H_I is defined by

$$H_I \;=\; i\lambda \int dk\, F(k)\,(e^{-ik\cdot x}a_k^\dagger - e^{ik\cdot x}a_k). \tag{4.1}$$

As stated in Section 1 we will always assume $\lambda = \sqrt{\varepsilon}$.

We assume that initially the electron and the phonons are independent and their initial state is represented by the density operator

$$\Gamma_0 \;:=\; \gamma_{\rm e,0}\otimes\gamma_{\rm ph}$$

where $\gamma_{\rm e,0}$ is an electronic density matrix and $\gamma_{\rm ph}$ is the phonon thermal state defined in (2.5). The time evolution of Γ_t is described by

$$i\partial_t\Gamma_t \;=\; [H,\Gamma_t]$$

with Γ_0 as initial data, or, equivalently

$$\Gamma_t \;=\; e^{-iHt}\Gamma_0 e^{iHt}.$$

4.2. **Initial state of the electron.** The initial state of the electron is represented by the wave function (1.1). The density operator corresponding to the wave function ψ_0^ε is the one-dimensional projection

$$\gamma_{\rm e,0} \;=\; |\psi_0^\varepsilon\rangle\langle\psi_0^\varepsilon|$$

with integral kernel

$$\gamma_{\rm e,0}(x,y) \;=\; \psi_0^\varepsilon(x)\overline{\psi_0^\varepsilon(y)}.$$

According to (1.1), the initial density operator for the electron can be split in four terms

$$\gamma_{\rm e,0} \;= \gamma_{\rm e,0,++} + \gamma_{\rm e,0,--} + \gamma_{\rm e,0,+-} + \gamma_{\rm e,0,-+} \tag{4.2}$$

where

$$\gamma_{\rm e,0,\alpha\alpha'} \;=\; |\psi_{0,\alpha}^\varepsilon\rangle\langle\psi_{0,\alpha'}^\varepsilon| \tag{4.3}$$

with $\alpha,\alpha'\in\{+,-\}$. We stress once again that, whereas $\gamma_{\rm e,0,++}$ and $\gamma_{\rm e,0,--}$ describe an electron in the state $\psi_{0,+}^\varepsilon$ and $\psi_{0,-}^\varepsilon$, respectively, the terms $\gamma_{\rm e,0,+-}$ and $\gamma_{\rm e,0,-+}$ do not represent any physical state: in particular, as operators they are not positive. They represent the interference between the states $\psi_{0,+}^\varepsilon$ and $\psi_{0,-}^\varepsilon$.

According to (4.2), the Wigner transform of the initial state for the electron can be written as

$$W_{\psi_0^\varepsilon} = W_{0,++} + W_{0,--} + W_{0,+-} + W_{0,-+} \tag{4.4}$$

where

$$\begin{aligned} W_{0,++}(x,v) &= W_f\left(\varepsilon x + Q, \frac{v-P}{\varepsilon}\right) \\ W_{0,--}(x,v) &= W_f\left(\varepsilon x - Q, \frac{v+P}{\varepsilon}\right) \\ W_{0,+-}(x,v) &= e^{2iP\cdot x} e^{\frac{2iv\cdot Q}{\varepsilon}} W_f\left(\varepsilon x, \frac{v}{\varepsilon}\right) \\ W_{0,-+}(x,v) &= e^{-2iP\cdot x} e^{-\frac{2iv\cdot Q}{\varepsilon}} W_f\left(\varepsilon x, \frac{v}{\varepsilon}\right) \end{aligned}$$

as one can immediately verify by (2.2).

Recalling the convention adopted in (2.3)

$$\begin{aligned} \widehat{W}_{0,+-}(\xi,v) &= \left(\sqrt{2\pi}\varepsilon\right)^{-d} e^{2i\frac{v\cdot Q}{\varepsilon}} \widehat{f}\left(\frac{v}{\varepsilon} + \frac{\xi - 2P}{2\varepsilon}\right) \overline{\widehat{f}\left(\frac{v}{\varepsilon} - \frac{\xi - 2P}{2\varepsilon}\right)} \\ \widehat{W}_{0,-+}(\xi,v) &= \left(\sqrt{2\pi}\varepsilon\right)^{-d} e^{-2i\frac{v\cdot Q}{\varepsilon}} \widehat{f}\left(\frac{v}{\varepsilon} + \frac{\xi + 2P}{2\varepsilon}\right) \overline{\widehat{f}\left(\frac{v}{\varepsilon} - \frac{\xi + 2P}{2\varepsilon}\right)}. \end{aligned} \tag{4.5}$$

All components of W_0 depend on ε, but this is omitted in notation.

4.3. **Time evolution of the electron.** The state of the electron at time t can be described by the marginal (or reduced density operator)

$$\gamma_{\mathrm{e},t} := \mathrm{Tr}_{\mathcal{H}_{\mathrm{ph}}} \Gamma_t \tag{4.6}$$

or, equivalently, by the Wigner function

$$W(t;x,v) = \int \frac{dy}{(2\pi)^d} e^{-iv\cdot y} \gamma_{\mathrm{e},t}\left(x + \frac{y}{2}, x - \frac{y}{2}\right).$$

Since the time evolution is linear, the initial decompositions (4.2) and (4.4) propagate at any time t:

$$\begin{aligned} \gamma_{\mathrm{e},t} &= \gamma_{\mathrm{e},t,++} + \gamma_{\mathrm{e},t,--} + \gamma_{\mathrm{e},t,+-} + \gamma_{\mathrm{e},t,-+} \\ W(t) &= W_{++}(t) + W_{--}(t) + W_{+-}(t) + W_{-+}(t) \end{aligned} \tag{4.7}$$

Again, the first two terms in the decomposition are called diagonal, and the other two terms non-diagonal.

4.4. **Observables.** We will use the notation $\mathcal{O}$ for the observables considered as operators and their Wigner transforms will be denoted by $J = W_{\mathcal{O}}$. The expectation value of an observable is given by

$$\mathrm{Tr}\ \mathcal{O}^\star \gamma_{\mathrm{e}} = \int \overline{J(x,v)} W(x,v) dx dv = \int \overline{\widehat{J_\varepsilon}(u,v)} \widehat{W}(u,v)\ du dv.$$

We need observables capable to resolve the size of the interference fringes. Therefore we consider observables $\mathcal{O} = \mathcal{O}_\varepsilon$ endowed with a two-scale structure in the space variables but we always assume the condition (1.5) on J_ε. This implies that

$$|\!|\!| \mathcal{O} |\!|\!|^2 := \sup_\varepsilon \|\mathcal{O}_\varepsilon^\star \mathcal{O}_\varepsilon\| < \infty \tag{4.8}$$

since the operator norm of $\mathcal{O}^\star\mathcal{O}$ is bounded by $\|J\|^2$.

In order to select observables that actually detect the exponential decay of the interference fringes, we provide sufficient conditions on J_ε for $\lim_{\varepsilon\to 0}\langle J_\varepsilon, W_{+-,\mathrm{free}}(\varepsilon^{-1}T)\rangle$ to exist. We assume that the Wigner transform of $\mathcal{O}_\varepsilon$ has the structure

$$J_\varepsilon(x,v) \;=\; A(\varepsilon x, v)b(x) \tag{4.9}$$

where A describes the macroscopic profile of the observable and b encodes the short-scale structure.

First we consider for simplicity the case of Newtonian dispersion relation, i.e. $e(v)=\frac{1}{2}v^2$. One has

$$W_{+-,\mathrm{free}}(\varepsilon^{-1}T;x,v) \;=\; e^{2iP\cdot x}e^{2i\frac{v}{\varepsilon}\cdot(Q-PT)}W_f\left(\varepsilon x - vT, \frac{v}{\varepsilon}\right)$$

Due to the regularity hypothesis (1.2), W_f belongs to $L^1(\mathbb{R}^{2d})$, and then $\|W_{+-,\mathrm{free}}(\varepsilon^{-1}T)\|_1 = \|W_f\|_1$. Besides, due to (1.5), it is clear that $J_\varepsilon \in L^\infty(\mathbb{R}^{2d})$ with a uniform bound in ε.

After an elementary change of variables, one gets

$$\langle J_\varepsilon, W_{+-,\mathrm{free}}(\varepsilon^{-1}T)\rangle \;=\; \int dx\,dv\,\overline{A}(x+\varepsilon vT,\varepsilon v)\overline{b}\left(\frac{x}{\varepsilon}+vT\right)e^{2iP\cdot\left(\frac{x}{\varepsilon}+vT\right)}e^{2iv\cdot(Q-PT)}W_f(x,v)$$

and the limit for $\varepsilon\to 0$ can be investigated using the dominated convergence theorem, assuming that A and b are bounded and continuous functions. One finds that:

- if $\lim_{|x|\to\infty} b(x) = 0$, then $\lim_{\varepsilon\to 0}\langle J_\varepsilon, W_{+-,\mathrm{free}}(\varepsilon^{-1}T)\rangle$ vanishes. In this case the observable does not resolve the interference fringes, even though the limit of the expectation value exists;
- if $\lim_{|x|\to\infty} b(x)e^{-2iP\cdot x} =: c_b \neq 0$, then

$$\lim_{\varepsilon\to 0}\langle J_\varepsilon, W_{+-,\mathrm{free}}(\varepsilon^{-1}T)\rangle = \overline{c_b}\int dx\,dv\,\overline{A(x,0)}e^{2iv\cdot(Q-PT)}W_f(x,v) \tag{4.10}$$

In general, the short-scale factor b must exhibit the periodicity of the interference fringes. We require the distributional Fourier transform $\widehat{b}(\xi)$ to be a complex measure with a finite total variation. If it has a non-trivial Dirac delta component with frequency $2P$, i.e.

$$\lim_{\tau\to 0+}\frac{1}{(2\pi)^{d/2}}\int_{|\xi-2P|\le\tau}\widehat{b}(\xi)\,d\xi =: c_b \neq 0, \tag{4.11}$$

then the interference fringes in W_{+-} can be detected by J_ε. For the other off-diagonal term, W_{-+}, one needs a non-trivial component with frequency $-2P$. Working in Fourier space requires more conditions on A, for simplicity we assume that A is a Schwartz function.

In the case of a more general dispersion relation $e(v)$ the Wigner transform of the free evolution, $W_{+-,\mathrm{free}}(t)$, is given by

$$\widehat{W}_{+-,\mathrm{free}}(t;\xi,v) = (\sqrt{2\pi}\varepsilon)^{-d}e^{-it[e(v+\xi/2)-e(v-\xi/2)]}e^{2iv\cdot\frac{Q}{\varepsilon}}\widehat{f}\left(\frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right)\overline{\widehat{f}\left(\frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right)} \tag{4.12}$$

in Fourier space. After a change of variable one gets

$$\begin{aligned}\langle J_\varepsilon, W_{+-,\mathrm{free}}(\varepsilon^{-1}T)\rangle \;=&(2\pi)^{-d}\int d\xi\,dv\,d\zeta\,\overline{\widehat{A}}\left(\xi+\frac{2P-\zeta}{\varepsilon},\varepsilon v\right)\overline{\widehat{b}}(\zeta)e^{-iT\varepsilon^{-1}[e(\varepsilon v+\varepsilon\xi/2+P)-e(\varepsilon v-\varepsilon\xi/2-P)]}\\ &\times e^{2iv\cdot Q}\widehat{f}\left(v+\frac{\xi}{2}\right)\overline{\widehat{f}\left(v-\frac{\xi}{2}\right)}.\end{aligned}$$

Again, by using dominated convergence, we have

$$\lim_{\varepsilon\to 0}\langle J_\varepsilon, W_{+-,\mathrm{free}}(\varepsilon^{-1}T)\rangle \;=\; (2\pi)^{-d/2}\overline{c_b}\int d\xi\, dv\, \overline{\widehat{A}(\xi,0)}e^{2iv\cdot(Q-T\nabla e(P))}\widehat{f}\left(v+\frac{\xi}{2}\right)\overline{\widehat{f}\left(v-\frac{\xi}{2}\right)}$$

that, recalling (2.3), generalizes (4.10) to a generic dispersion relation.

In summarizing, we have proved the following

Corollary 4.1. *Assume the hypotheses of Theorem 1.1 and let $J_\varepsilon(x,v)=A(\varepsilon x,v)b(x)$. Let either*

i) *$e(v)$ be quadratic, $A\in C_b(\mathbb{R}^{2d})$, $b\in C_b(\mathbb{R}^d)$ and $c_b:=\lim_{|x|\to\infty}b(x)e^{-2iP\cdot x}$; or*

ii) *$A\in\mathcal{S}(\mathbb{R}^{2d})$ and $\widehat{b}$ be a measure with finite total variation with c_b being the coefficient of the Dirac delta component at $2P$ of this measure (see (4.11)).*

Then,

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{L\to\infty}\langle J_\varepsilon, W_{+-}(\varepsilon^{-1}T)\rangle &= e^{-T\sigma_P}\lim_{\varepsilon\to 0}\langle J_\varepsilon, W_{+-,\mathrm{free}}(\varepsilon^{-1}T)\rangle\\ &= e^{-T\sigma_P}\overline{c_b}\int dx\, dv\, \overline{A(x,0)}e^{2iv\cdot(Q-T\nabla e(P))}W_f(x,v)\ .\end{aligned}$$

The hypotheses cover the case $J_\varepsilon(x,v)=e^{2i\widetilde{P}\cdot x}$ for the quadratic dispersion relation (see (1.3)).

5. Known results

The proof partially follows the arguments in [E]. To avoid duplicating them but still keep the present paper relatively self-contained, here we summarize some notations and results from [E].

5.1. **The main term of $\gamma_{\mathrm{e,t}}$.** We consider the density matrix of the whole system

$$\Gamma_t \;=\; e^{-iHt}\Gamma_0 e^{iHt}$$

and expand the propagator e^{-iHt} into a Duhamel sum up to the order

$$N_0 \;=\; \frac{2.2\log\varepsilon}{\log|\log\varepsilon|}.$$

After performing the phonon trace and defining the reduced density matrix of the electron as

$$\gamma_{\mathrm{e},t} \;=\; \mathrm{Tr}_{\mathrm{ph}}\left(e^{-iHt}\Gamma_0 e^{iHt}\right),$$

$\gamma_{\mathrm{e},t}$ is decomposed into the sum of a main term and an error term

$$\gamma_{\mathrm{e},t} \;=\; \gamma_K^{\mathrm{main}}(t)+\gamma_K^{\mathrm{err}}(t) \tag{5.1}$$

where $K\in\mathbb{N}$ is an arbitrary integer. For the precise definitions of $\gamma_K^{\mathrm{main}}(t)$ and $\gamma_K^{\mathrm{err}}(t)$ see formula (2.18) in [E], but their explicit form will not be needed here. We need only to know that

$$\limsup_{K\to\infty}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\left|\mathrm{Tr}\;\gamma_K^{\mathrm{err}}(\varepsilon^{-1}T)\mathcal{O}_\varepsilon^\star\right| \;=\; 0, \tag{5.2}$$

i.e. that the estimate (2.24) of [E] holds in our case as well. To arrive at this result, the argument in [E] used only assumption (1.26) in [E] on the initial state and assumption (2.21) in [E] on the observable. In our case, the first assumption is guaranteed by the hypothesis (1.2); the second one is inequality (4.8). For the same reason, Lemma 3.1, Proposition 10.1 and Lemma 10.2 from [E] also hold in our case.

To evaluate $\gamma_K^{\rm main}(t)$ we use Proposition 10.1 and Lemma 10.2 in [E]. Results therein can be resumed as follows

$$\limsup_{\varepsilon\to 0}\Bigg|\lim_{L\to\infty}\langle J_\varepsilon, W_{\gamma_K^{\rm main}(\varepsilon^{-1}T)}\rangle - \sum_{N,\widetilde{N}=0}^{K-1}\sum_{n=0}^{\min(N,\widetilde{N})}\sum_{\substack{\underline{m}=(m_0,\dots,m_n)\in\mathbb{N}^{n+1}\\ N=n+2\sum_{j=0}^n m_j}}\sum_{\substack{\widetilde{\underline{m}}=(\widetilde{m}_0,\dots,\widetilde{m}_n)\in\mathbb{N}^{n+1}\\ \widetilde{N}=n+2\sum_{j=0}^n \widetilde{m}_j}} C^*_{\underline{m},\widetilde{\underline{m}},\mathrm{id}}(\varepsilon^{-1}T)\Bigg| = 0 \tag{5.3}$$

where, for any $\underline{m},\widetilde{\underline{m}}\in\mathbb{N}^{n+1}$, $t>0$ we defined

$$\begin{aligned} C^*_{\underline{m},\widetilde{\underline{m}},\mathrm{id}}(t) \;:=\; & (2\pi)^{-2}\,\lambda^{2n+2|\underline{m}|+2|\widetilde{\underline{m}}|}\sum_{\substack{\sigma_j\in\{\pm\}\\ j=1,\dots,n}}\int d\xi\,dv\,\left(\prod_{j=1}^n dk_j\right)\overline{\widehat{J}_\varepsilon\left(\xi, v+\sum_{l=1}^n k_l\right)}\\ & \times\widehat{W}_{\psi_0^\varepsilon}(\xi,v)\left(\prod_{j=1}^n L(k_j,\sigma_j)\right)e^{2t\eta}\int_{\mathbb{R}} d\alpha\, e^{-it\alpha}\left(\prod_{j=0}^n R_j^{m_j+1}\Upsilon_j^{m_j}\right)\\ & \times\int_{\mathbb{R}} d\widetilde{\alpha}\, e^{it\widetilde{\alpha}}\left(\prod_{j=0}^n \widetilde{R}_j^{\widetilde{m}_j+1}\widetilde{\Upsilon}_j^{\widetilde{m}_j}\right) \end{aligned} \tag{5.4}$$

Here $\eta>0$, $\underline{k}=(k_1,\dots,k_n)\in\mathbb{R}^{nd}$, $|\underline{m}|=\sum_{j=0}^n m_j$, $|\widetilde{\underline{m}}|=\sum_{j=0}^n\widetilde{m}_j$ and to define R_j and Υ_j we first introduce

$$\begin{aligned} R(\alpha,v,z) &= \frac{1}{\alpha-e(v)-z}\\ \Upsilon_\eta(\alpha,v) &= \sum_{\sigma=\pm}\int\frac{L(k,\sigma)dk}{\alpha-e(v+k)-\sigma\omega(k)+i\eta} \end{aligned} \tag{5.5}$$

for $\alpha\in\mathbb{R}$, $v\in\mathbb{R}^d$, $z\in\mathbb{C}$, and $\eta>0$, and then we set

$$\begin{aligned} R_j &:= R\left(\alpha, v+\sum_{l=j+1}^n k_l+\frac{\xi}{2}, \sum_{l=j+1}^n\sigma_l\omega(k_l)-i\eta\right)\\ \widetilde{R}_j &:= R\left(\widetilde{\alpha}, v+\sum_{l=j+1}^n k_l-\frac{\xi}{2}, \sum_{l=j+1}^n\sigma_l\omega(k_l)+i\eta\right)\\ \Upsilon_j &:= \Upsilon_\eta\left(\alpha-\sum_{l=j+1}^n\sigma_l\omega(k_l), v+\sum_{l=j+1}^n k_l+\frac{\xi}{2}\right)\\ \widetilde{\Upsilon}_j &:= \overline{\Upsilon}_\eta\left(\widetilde{\alpha}-\sum_{l=j+1}^n\sigma_l\omega(k_l), v+\sum_{l=j+1}^n k_l-\frac{\xi}{2}\right) \end{aligned}$$

Formula (5.4) holds for any $\eta > 0$; later we will choose $\eta = \varepsilon$.

The following estimates will be used in the next section

$$\begin{aligned} &\sup_{p,\alpha,\eta} |\Upsilon_\eta(\alpha,p)| \le C, \quad |R_j|^{m_j+1} \le \eta^{-m_j}|R_j|, \\ &\sup_{p,\alpha,\eta} \left(|\nabla_p \Upsilon_\eta(\alpha,p)| + |\partial_\alpha \Upsilon_\eta(\alpha,p)| + |\partial_\eta \Upsilon_\eta(\alpha,p)|\right) \ \le \ C\eta^{-1/2} \end{aligned} \tag{5.6}$$

The first and the third inequalities in (5.6) are proven in [E] (Lemma 4.1), the second is trivial.

5.2. **Ladder Wigner functions and related Feynman graphs.** We introduce the family of "ladder Wigner functions" $W^{\rm ladder}_{n,\underline{m},\underline{\widetilde{m}}}(t)$, defined implicitly by the relation

$$C^*_{\underline{m},\underline{\widetilde{m}},{\rm id}}(t) \ = \ \langle J_\varepsilon, W^{\rm ladder}_{n,\underline{m},\underline{\widetilde{m}}}(t)\rangle. \tag{5.7}$$

The explicit expression of $W^{\rm ladder}_{n,\underline{m},\underline{\widetilde{m}}}(t)$ in the Fourier space can be easily obtained comparing (5.7) with (5.4). The content of (5.3) can be expressed saying that, in the thermodynamic limit, the expectation value of $\mathcal{O}_\varepsilon$ on $\gamma^{\rm main}_K(t)$ can be decomposed in the sum of the expectation values of J_ε on the states represented by $W^{\rm ladder}_{n,\underline{m},\underline{\widetilde{m}}}(t)$.

The function $\widehat{W}^{\rm ladder}_{n,\underline{m},\underline{\widetilde{m}}}(t;\xi,v)$ is linear in $\widehat{W}_{\psi_0^\varepsilon}$, thus it contains two evolved copies of the electron wavefunction: one copy initially lies in the state represented by ψ_0^ε, the other in $\overline{\psi_0^\varepsilon}$. In the following we will call them, for short, the first and the second copy. In the time interval $(0,t)$ each copy of the electron emits and absorbs phonons. An important feature of the evolution described by $\widehat{W}^{\rm ladder}_{n,\underline{m},\underline{\widetilde{m}}}(t;\xi,v)$ is that any emitted phonon must be reabsorbed and vice versa, but a phonon emitted by one copy of the electron can be reabsorbed by the other.

This fact permits us to distinguish two different processes: the exchange of phonons between the two copies of the electron, and the recollision of the same copy with the same phonon. In the evolution described by $\widehat{W}^{\rm ladder}_{n,\underline{m},\underline{\widetilde{m}}}(t;\xi,v)$ all possible recollisions are immediate, i.e., there are no further events in the time interval between the emission of a phonon by a copy of the electron and its reabsorption by the same copy. Moreover, exchanged phonons appear in the same order in the time evolution of the two copies of the wavefunction of the electron.

In (5.4) there are n exchanged phonons and their momenta are labeled $k_1,\dots,k_n$. The electron propagator between consecutive phonon collisions is represented by the factors R_j (for the first copy) and $\widetilde{R}_j$ (for the second). Between the jth and the $(j+1)$st exchanges of phonons, the first copy of the electron undergoes m_j immediate recollisions, each of them embodied in a factor Υ_j in (5.4). Analogously, the second copy of the electron suffers $\widetilde{m}_j$ immediate recollisions, each of them represented in (5.4) by a factor $\widetilde{\Upsilon}_j$.

The expectation value of an observable $\mathcal{O}_\varepsilon$ on the state represented by $\widehat{W}^{\rm ladder}_{n,\underline{m},\underline{\widetilde{m}}}(t;\xi,v)$ can be represented as a Feynman graph (see Fig. 1).

5.3. **Non-diagonal terms.** Following (4.7), the expectation of any ladder Wigner function can be decomposed in diagonal and non diagonal terms

$$C^*_{\underline{m},\underline{\widetilde{m}},{\rm id}}(t) = \sum_{\alpha,\alpha'\in\{+,-\}} C^*_{\alpha\alpha',\underline{m},\underline{\widetilde{m}},{\rm id}}(t)$$

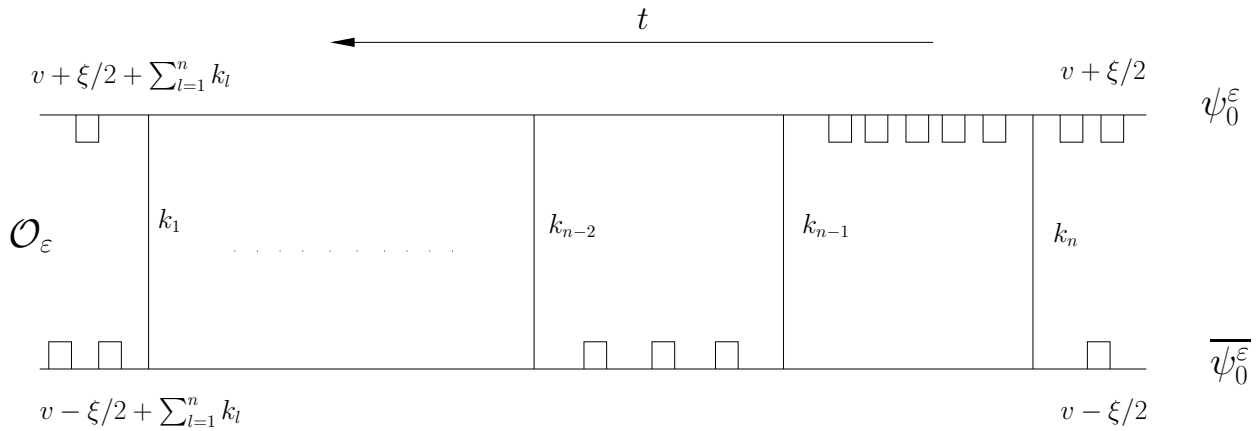

FIGURE 1. A ladder Feynman graph. The two horizontal lines represent the time evolution of the two copies of the electron, and the vertical lines joining the two horizontal ones represent the exchanges of phonons between them. They are labeled by the momenta of the exchanged phonons. By convention time goes from the right to the left and the labeling of the exchanged phonons follows the opposite ordering. The square loops on the higher horizontal line symbolize the recollisions in the evolution of the first copy of the electron. Such loops have no momentum labeling because in the related factors Υ_j an internal loop momentum integral has been performed. An analogous interpretation holds for the square loops in the lower horizontal line. Here $m_0 = 1, m_{n-2} = 0, m_{n-1} = 5, m_n = 2$ and $\widetilde{m}_0 = 2, \widetilde{m}_{n-2} = 3, \widetilde{m}_{n-1} = 0, \widetilde{m}_n = 1$.

where $C^*_{\alpha\alpha',\underline{m},\underline{\widetilde{m}},\mathrm{id}}(t)$ is defined from (5.4) after replacing $\widehat{W}_{\psi_0^\varepsilon}$ by $\widehat{W}_{0,\alpha\alpha'}$. The corresponding Wigner transforms, $W^{\mathrm{ladder}}_{\alpha\alpha',n,\underline{m},\underline{\widetilde{m}}}(t)$, are defined implicitly via

$$C^*_{+-,\underline{m},\underline{\widetilde{m}},\mathrm{id}}(t) = \langle J_\varepsilon, W^{\mathrm{ladder}}_{\alpha\alpha',n,\underline{m},\underline{\widetilde{m}}}(t)\rangle \tag{5.8}$$

and of course,

$$W^{\mathrm{ladder}}_{n,\underline{m},\underline{\widetilde{m}}}(t) \;=\; \sum_{\alpha,\alpha'\in\{+,-\}} W^{\mathrm{ladder}}_{\alpha\alpha',n,\underline{m},\underline{\widetilde{m}}}(t)$$

Since both the evolution of density matrices and the definition of $\gamma_K^{\mathrm{main}}(t)$ and $\gamma_K^{\mathrm{err}}(t)$ are linear in the initial data (see (2.18) and (2.19) in [E]), all results listed in Section 5.1 hold separately for diagonal and non-diagonal terms, according to decompositions (4.7). In particular, formulas (5.1), (5.2), (5.3) remain valid after replacing $\gamma_{\mathrm{e},t}$, $\gamma_K^{\mathrm{main}}(t)$ and $\gamma_K^{\mathrm{err}}(t)$ with the corresponding non-diagonal component $\gamma_{\mathrm{e},t,+-}$, $\gamma^{\mathrm{main}}_{K,+-}(t)$ and $\gamma^{\mathrm{err}}_{K,+-}(t)$, respectively.

Then, formulas (5.2), (5.3), applied to non-diagonal terms give

$$\limsup_{K\to\infty}\limsup_{\varepsilon\to 0}\Bigg|\lim_{L\to\infty}\langle J_\varepsilon, W_{+-}(\varepsilon^{-1}T)\rangle - \sum_{N,\widetilde{N}=0}^{K-1}\sum_{n=0}^{\min(N,\widetilde{N})}\sum_{\substack{\underline{m}=(m_0,\dots,m_n)\in\mathbb{N}^{n+1}\\ N=n+2\sum_{j=0}^n m_j}}\sum_{\substack{\underline{\widetilde{m}}=(\widetilde{m}_0,\dots,\widetilde{m}_n)\in\mathbb{N}^{n+1}\\ \widetilde{N}=n+2\sum_{j=0}^n \widetilde{m}_j}} C^*_{+-,\underline{m},\underline{\widetilde{m}},\mathrm{id}}(\varepsilon^{-1}T)\Bigg| = 0.$$

In [E] it is proved that the limit

$$\lim_{K\to\infty}\lim_{\varepsilon\to 0}\lim_{L\to\infty}\left[\mathrm{Tr}\;\gamma_K^{\mathrm{main}}(\varepsilon^{-1}T)\mathcal{O}_\varepsilon^\star\right]$$

equals the expectation value of $\mathcal{O}_\varepsilon$ on a distribution on the phase space that solves a linear Boltzmann equation, provided that $\mathcal{O}_\varepsilon$ has no short-scale structure (i.e. $b=1$ in (4.9)).

In the rest of the paper we study the same limit of the non-diagonal part for a two-scale observable (4.9). In the following section we prove that ladder terms with at least one rung do not contribute. In contrast, diagrams with no rungs and pure recollisions give a non vanishing contribution, characterized by the exponential damping described in Section 1. This is the content of Section 7.

6. Non-diagonal n-rung ladder term with immediate recollisions: estimate

We estimate the contribution of the generic non-diagonal ladder term $W^{\mathrm{ladder}}_{+-,n,\underline{m},\underline{\widetilde{m}}}(t)$ to the expectation value of a two-scale observable J_ε.

Lemma 6.1. *Given $n\geq 1$, $\underline{m},\underline{\widetilde{m}}\in\mathbb{N}^{n+1}$, $P\neq 0$, let $\widehat{W}^{\mathrm{ladder}}_{+-,n,\underline{m},\underline{\widetilde{m}}}$ be defined as in* (5.8)*, and J_ε be a two-scale observable satisfying estimate* (1.5)*. Then for any fixed $T>0$*

$$\left|\langle J_\varepsilon, W^{\mathrm{ladder}}_{+-,n,\underline{m},\underline{\widetilde{m}}}(\varepsilon^{-1}T)\rangle\right| \;\leq\; C^{n+|\underline{m}|+|\underline{\widetilde{m}}|}\|J\|\frac{\langle P\rangle^2}{|P|_\star}(\log^\star\varepsilon)^4\varepsilon\|f\|^2_{H^2(\mathbb{R}^d)}, \tag{6.1}$$

where $\|J\|$ is defined in (1.5)*, $|P|_\star=\min(1,|P|+\varepsilon)$, $\log^\star\varepsilon=\max(1,|\log\varepsilon|)$ and the constant C depends on T.*

Proof. From (4.5), (5.4) and (5.8) we have

$$\begin{aligned}
&\langle J_\varepsilon, W^{\mathrm{ladder}}_{+-,n,\underline{m},\underline{\widetilde{m}}}(t)\rangle\\
=\;&(2\pi)^{-2-\frac{d}{2}}\,\varepsilon^{-d}\lambda^{2n+2|\underline{m}|+2|\underline{\widetilde{m}}|}\sum_{\substack{\sigma_j\in\{\pm\}\\ j=1,\dots,n}}\int d\xi\,dv\,\left(\prod_{j=1}^n dk_j\right)\overline{\widehat{J}_\varepsilon\left(\xi,v+\sum_{l=1}^n k_l\right)}\,e^{2i\frac{v\cdot Q}{\varepsilon}}\\
&\times\widehat{f}\left(\frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right)\overline{\widehat{f}\left(\frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right)}\left(\prod_{j=1}^n L(k_j,\sigma_j)\right)e^{2t\eta}\int_{\mathbb{R}}d\alpha\,e^{-it\alpha}\left(\prod_{j=1}^n R_j^{m_j+1}\Upsilon_j^{m_j}\right)\\
&\times\int_{\mathbb{R}}d\widetilde{\alpha}\,e^{it\widetilde{\alpha}}\left(\prod_{j=1}^n\widetilde{R}_j^{\widetilde{m}_j+1}\widetilde{\Upsilon}_j^{\widetilde{m}_j}\right)\,.
\end{aligned}$$

To estimate $|\langle J_\varepsilon, W^{\rm ladder}_{+-,n,\underline{m},\underline{\widetilde{m}}}(\varepsilon^{-1}T)\rangle|$ we use the first two inequalities in (5.6), with $\eta=\varepsilon$, and definition (3.9). Then

$$\begin{aligned}&\left|\langle J_\varepsilon, W^{\rm ladder}_{+-,n,\underline{m},\underline{\widetilde{m}}}(\varepsilon^{-1}T)\rangle\right|\\ \leq\ & \varepsilon^{-d}\lambda^{2n}(C\lambda^2\varepsilon^{-1})^{|\underline{m}|+|\underline{\widetilde{m}}|}\int d\xi\, dv\, \left\|\widehat{J}_\varepsilon\,(\xi,\cdot)\right\|_\infty \left|\widehat{f}\left(\frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right)\right|\left|\widehat{f}\left(\frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right)\right|\\ &\times\sum_{\substack{\sigma_j\in\{\pm\}\\ j=1,\dots,n}}\left(\prod_{j=1}^n\int\mathcal{L}(k_j)\,dk_j\right)\int_{\mathbb{R}}d\alpha\left(\prod_{j=1}^n|R_j|\right)\int_{\mathbb{R}}d\widetilde{\alpha}\left(\prod_{j=1}^n|\widetilde{R}_j|\right)\,.\end{aligned}\tag{6.2}$$

Let us focus on the integral in $dk_1\dots dk_n d\alpha d\widetilde{\alpha}$.

$$\begin{aligned}&\left(\prod_{j=1}^n\int\mathcal{L}(k_j)\,dk_j\right)\int_{\mathbb{R}}d\alpha\left(\prod_{j=1}^n|R_j|\right)\int_{\mathbb{R}}d\widetilde{\alpha}\left(\prod_{j=1}^n|\widetilde{R}_j|\right)\\ =\ &\int d\alpha d\widetilde{\alpha}|R_n||\widetilde{R}_n|\int dk_n\,\mathcal{L}(k_n)|R_{n-1}||\widetilde{R}_{n-1}|\dots\int dk_1\,\mathcal{L}(k_1)|R_0||\widetilde{R}_0|\end{aligned}\tag{6.3}$$

Indeed, the variable k_l appears in R_j and $\widetilde{R}_j$ only if $j\leq l-1$. Therefore, we can integrate in (6.3) following the order $k_1,\dots,k_n$. The integrals in k_j with $1\leq j\leq n-1$ are estimated by standard Cauchy-Schwarz inequality

$$\int\mathcal{L}(k_j)|R_{j-1}||\widetilde{R}_{j-1}|\,dk_j\ \leq\ C\int\mathcal{L}(k_j)|R_{j-1}|^2\,dk_j\ +\ C\int\mathcal{L}(k_j)|\widetilde{R}_{j-1}|^2\,dk_j\ \ \leq\ \ C\varepsilon^{-1}\tag{6.4}$$

where we exploited estimate (3.10), with $\eta=\varepsilon$ and $m\geq 1$. We finally obtain the factor $C^{n-1}\varepsilon^{-n+1}$.

The integral in k_n is estimated using (3.11) with $\theta=\alpha$, $\widetilde{\theta}=\widetilde{\alpha}$, $p=v+\xi/2$, $\widetilde{p}=v-\xi/2$, $\eta=\varepsilon$. We obtain

$$\begin{aligned}&\int\frac{\mathcal{L}(k_n)\,dk_n}{|\alpha-e(v+\xi/2+k_n)-\sigma_n\omega(k_n)+i\varepsilon||\widetilde{\alpha}-e(v-\xi/2+k_n)-\sigma_n\omega(k_n)-i\varepsilon|}\\ &\qquad\leq\ C\frac{(\log^\star\varepsilon)^2\langle v+\xi/2\rangle\langle v-\xi/2\rangle}{|\xi|_\star\langle\alpha\rangle^{1/2}\langle\widetilde{\alpha}\rangle^{1/2}}\end{aligned}\tag{6.5}$$

where, according to the notation used in Lemma 3.1, we defined $|\xi|_\star=\min(1,|\xi|+\varepsilon)$.

Finally, by (3.12) both integrals in $d\alpha$ and $d\widetilde{\alpha}$ give a factor $\log^\star \varepsilon$. Then, by (6.2), (6.4) and (6.5) we have

$$\begin{aligned}
&\left|\langle J_\varepsilon, W^{\text{ladder}}_{+-,n,\underline{m},\widetilde{\underline{m}}}(\varepsilon^{-1}T)\rangle\right| \\
\leq\ & \varepsilon^{-d}(C\lambda^2\varepsilon^{-1})^{n+|\underline{m}|+|\widetilde{\underline{m}}|}\ (\log^\star \varepsilon)^4\,\varepsilon \int d\xi\, dv\ \left\|\widehat{J}_\varepsilon\,(\xi,\cdot)\right\|_\infty \left|\widehat{f}\left(\frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right)\right| \\
&\times\left|\widehat{f}\left(\frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right)\right| \frac{\langle v+\xi/2\rangle\langle v-\xi/2\rangle}{|\xi|_\star}\,.
\end{aligned} \tag{6.6}$$

We focus on the integral factor. Using (1.5) we get

$$\begin{aligned}
&\int d\xi\, dv\ \left\|\widehat{J}_\varepsilon\,(\xi,\cdot)\right\|_\infty \left|\widehat{f}\left(\frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right)\right| \left|\widehat{f}\left(\frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right)\right| \frac{\langle v+\xi/2\rangle\langle v-\xi/2\rangle}{|\xi|_\star} \\
\leq\ & \|J\| \sup_\xi \left[\frac{1}{|\xi|_\star}\int dv\ \left|\widehat{f}\left(\frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right)\right| \left|\widehat{f}\left(\frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right)\right| \langle v+\xi/2\rangle\langle v-\xi/2\rangle\right] \\
\leq\ & \varepsilon^d \|J\| \|f\|^2_{H^2(\mathbb{R}^d)} \sup_\xi \left[\frac{1}{|\xi|_\star}\left(\sup_v \frac{\langle v+\xi/2\rangle\langle v-\xi/2\rangle}{\left\langle \frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right\rangle^2 \left\langle \frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right\rangle^2}\right)\right]\,.
\end{aligned} \tag{6.7}$$

By (3.1) one has

$$\frac{\langle v+\xi/2\rangle\langle v-\xi/2\rangle}{\left\langle \frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right\rangle \left\langle \frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right\rangle} \ \leq\ C\langle P\rangle^2 \tag{6.8}$$

and

$$\left\langle \frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right\rangle \left\langle \frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right\rangle \ \geq\ \left\langle \frac{\xi-2P}{2\varepsilon}\right\rangle \tag{6.9}$$

Elementary calculations show

$$\sup_\xi \frac{1}{\left\langle \frac{\xi-2P}{2\varepsilon}\right\rangle |\xi|_\star} \leq \frac{C}{|P|_\star}. \tag{6.10}$$

Then, plugging (6.8), (6.9) and (6.10) into (6.7) we obtain

$$\begin{aligned}
&\int d\xi\, dv\ \left\|\widehat{J}_\varepsilon\,(\xi,\cdot)\right\|_\infty \left|\widehat{f}\left(\frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right)\right| \left|\widehat{f}\left(\frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right)\right| \frac{\langle v+\xi/2\rangle\langle v-\xi/2\rangle}{|\xi|_\star} \\
&\leq C\varepsilon^d \|J\| \frac{\langle P\rangle^2}{|P|_\star} \|f\|^2_{H^2(\mathbb{R}^d)}.
\end{aligned}$$

By (6.6), recalling that $\lambda = \sqrt{\varepsilon}$ we conclude the proof. □

Remark 6.2. For $n \geq 2$ one can gain another ε factor in estimate (6.1). This is easily accomplished using inequality

$$\sup_{\theta,\widetilde{\theta}} \int_{\mathbb{R}^d} \frac{L(k)\,dk}{|\theta-\Phi_\pm(p,k)+i\eta|\,|\widetilde{\theta}-\Phi_\pm(u,k)-i\eta|} \leq \frac{C(\log^\star \eta)^2}{|p-u|_\star}$$

for estimating the integral in k_1 in (6.3).

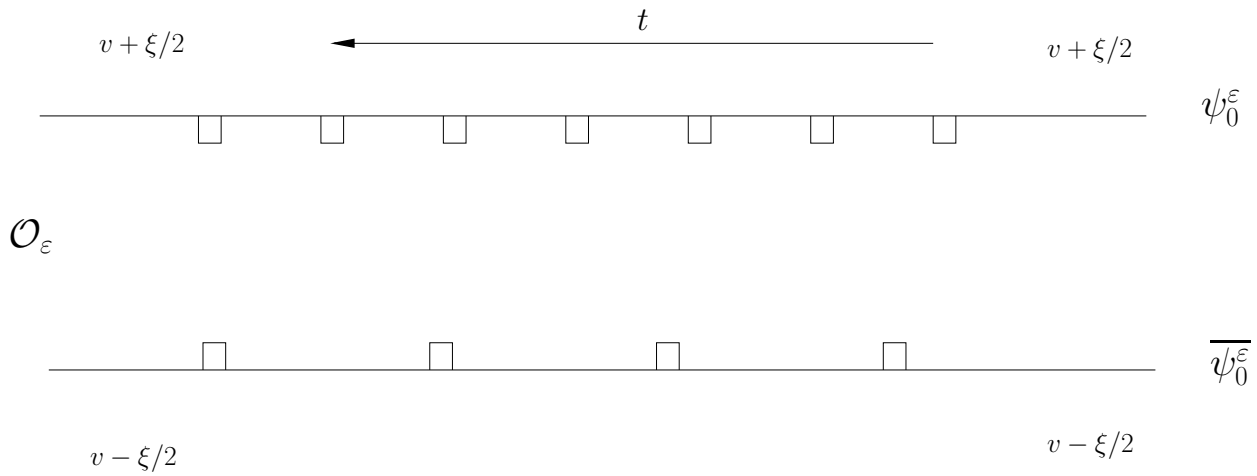

FIGURE 2. An example of Feynman graph associated to a pure recollision term. Here $m = 7$, $\widetilde{m} = 4$.

7. Terms with immediate recollisions only

In this section we prove Theorem 1.1, i.e. we compute the contribution of terms consisting of immediate recollisions only, namely terms with $n = 0$ in (5.3). The corresponding Feynman diagrams are illustrated in Fig. 2. We stress that, according to the results in the previous sections, these are the sole non vanishing terms.

Proof. Since the vectors $\underline{m}$ and $\underline{\widetilde{m}}$ are one-dimensional, we can simply denote by m both $\underline{m}$ and its only component m_0, and by $\widetilde{m}$ both $\underline{\widetilde{m}}$ and $\widetilde{m}_0$.

We preliminarily observe that, according to (5.4), (5.7), (4.5), and the definition of non-diagonal terms given in Section 5.3, for any $t > 0$ and any $\eta > 0$ the function $W^{\mathrm{ladder}}_{+-,0,m,\widetilde{m}}(t)$ can be expressed as follows:

$$\begin{aligned}
&\widehat{W}^{\mathrm{ladder}}_{+-,0,m,\widetilde{m}}(t;\xi,v)\\
=\;& \lambda^{2m+2\widetilde{m}}\,(2\pi)^{-2-d/2}\,e^{2t\eta}\varepsilon^{-d}e^{2i\frac{v\cdot Q}{\varepsilon}}\,\widehat{f}\left(\frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right)\overline{\widehat{f}\left(\frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right)}\\
&\times\int_{\mathbb{R}} d\alpha\, e^{-it\alpha}\,R^{m+1}(\alpha,v+\xi/2,-i\eta)\Upsilon^m_\eta(\alpha,v+\xi/2)\;\int_{\mathbb{R}} d\widetilde{\alpha}\, e^{it\widetilde{\alpha}}\,R^{\widetilde{m}+1}(\widetilde{\alpha},v-\xi/2,i\eta)\\
&\times\overline{\Upsilon}^{\widetilde{m}}_\eta(\widetilde{\alpha},v-\xi/2)
\end{aligned}\tag{7.1}$$

$$
\begin{aligned}
&= \lambda^{2m+2\widetilde{m}} \left(\sqrt{2\pi}\right)^{-d} \varepsilon^{-d} e^{2i\frac{v\cdot Q}{\varepsilon}} \widehat{f}\left(\frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right) \overline{\widehat{f}\left(\frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right)} \\
&\quad \times \int_{[0,t]^{2m+1}} \left(\prod_{j=0}^{2m} ds_j\right) \delta\left(t-\sum_{j=0}^{2m} s_j\right) \left(\prod_{j=0}^{m} e^{-is_{2j}e(v+\xi/2)}\right) \\
&\quad \times \left[\prod_{j=1}^{m}\left(\sum_{\sigma\in\{\pm\}} \int dk\, L(k,\sigma)\, e^{-is_{2j-1}[e(v+\xi/2+k)+\sigma\omega(k)]}\right)\right] \\
&\quad \times \int_{[0,t]^{2\widetilde{m}+1}} \left(\prod_{j=0}^{2\widetilde{m}} d\widetilde{s}_j\right) \delta\left(t-\sum_{j=0}^{2\widetilde{m}} \widetilde{s}_j\right) \left(\prod_{j=0}^{\widetilde{m}} e^{-i\widetilde{s}_{2j}e(v-\xi/2)}\right) \\
&\quad \times \left[\prod_{j=1}^{\widetilde{m}}\left(\sum_{\sigma\in\{\pm\}} \int dk\, L(k,\sigma)\, e^{-i\widetilde{s}_{2j-1}[e(v-\xi/2+k)+\sigma\omega(k)]}\right)\right]
\end{aligned}
\tag{7.2}
$$

where the functions R and Υ_η were defined in (5.5). The expression at the r.h.s of (7.1) is said the resolvent form, the one in (7.2) is said the propagator form of the ladder Wigner function. One can easily obtain the first from the second one using the identity

$$
\delta\left(t-\sum_{j=0}^{2m} s_j\right) \;=\; \frac{e^{\eta t}}{2\pi}\int_{\mathbb{R}} e^{-i\alpha t} e^{i(\alpha+i\eta)\sum_{j=0}^{2m} s_j}\, d\alpha,
$$

integrating in the variables s_j, and proceeding analogously for the variables $\widetilde{s}_j$.

Moreover, we introduce the following quantity:

$$
\Phi_P \;:=\; \lim_{\eta\to 0+} \sum_{\sigma=\pm} \int \frac{L(k,\sigma)\, dk}{e(P)-e(P+k)-\sigma\omega(k)+i\eta} = \Upsilon_{0+}(e(P),P). \tag{7.3}
$$

The existence of the limit has been proven in Lemma 4.1 of [E].

Besides, from the definition of the Boltzmann collision kernel (see (1.27) in [E]), the total cross section σ_P for an electron with momentum P in the phonon field reads

$$
\sigma_P \;=\; 2\pi \sum_{\sigma\in\{\pm\}} \int dU\, L(P-U)\delta(e(P)-e(U)-\sigma\omega(P-U)) \tag{7.4}
$$

and identity (4.12) in [E] yields

$$
\text{Im}\, \Phi_P \;=\; -\frac{\sigma_P}{2} \tag{7.5}
$$

We organize the proof into four steps.

Step 1. Here we prove

$$
\left|\langle J_\varepsilon, W^{\text{ladder}}_{+-,0,m,,\widetilde{m}}(t)\rangle\right| \le \frac{(C\lambda^2 t)^{m+\widetilde{m}}}{m!\,\widetilde{m}!}\,. \tag{7.6}
$$

Let us mention that this estimate is stronger than (10.4) in [E], but applies only to the ladder terms.

We use the exponential representation of the propagator (7.2). Integrating the delta function with respect to the variable s_{2m} one gets

$$\begin{aligned}
&\int_{[0,t]^{2m+1}} \left(\prod_{j=0}^{2m} ds_j\right) \delta\left(t - \sum_{j=0}^{2m} s_j\right) \left(\prod_{j=0}^{m} e^{-is_{2j}e(v+\xi/2)}\right) \\
&\times \left[\prod_{j=1}^{m} \left(\sum_{\sigma\in\{\pm\}} \int dk\, L(k,\sigma)\, e^{-is_{2j-1}[e(v+k+\xi/2)+\sigma\omega(k)]}\right)\right] \\
= \;& e^{-ite(v+\xi/2)} \int_0^t ds_0 \int_0^{t-s_0} ds_2 \ldots \int_0^{t-\sum_{j=0}^{m-2} s_{2j}} ds_{2m-2} \\
&\times \int_0^{t-\sum_{j=0}^{m-1} s_{2j}} ds_1 \left(\sum_{\sigma\in\{\pm\}} \int dk\, L(k,\sigma)\, e^{-is_1[e(v+\xi/2+k)+\sigma\omega(k)-e(v+\xi/2)]}\right) \ldots \\
&\cdots \times \int_0^{t-\sum_{j=0}^{m-1} s_{2j}-\sum_{j=0}^{m-2} s_{2j+1}} ds_{2m-1} \left(\sum_{\sigma\in\{\pm\}} \int dk\, L(k,\sigma)\, e^{-is_{2m-1}[e(v+\xi/2+k)+\sigma\omega(k)-e(v+\xi/2)]}\right)
\end{aligned}$$

Now we can use inequality (3.13) and obtain

$$\begin{aligned}
&\left| \int_{[0,t]^{2m+1}} \left(\prod_{j=0}^{2m} ds_j\right) \delta\left(t - \sum_{j=0}^{2m} s_j\right) \left(\prod_{j=0}^{m} e^{-is_{2j}e(v+\xi/2)}\right) \right. \\
&\left. \times \left[\prod_{j=1}^{m} \left(\sum_{\sigma\in\{\pm\}} \int dk\, L(k,\sigma)\, e^{-is_{2j-1}[e(v+\xi/2+k)+\sigma\omega(k)]}\right)\right] \right| \\
\leq \;& \int_0^t ds_0 \int_0^{t-s_0} ds_2 \ldots \int_0^{t-\sum_{j=0}^{m-2} s_{2j}} ds_{2m-2} \left(\prod_{j=1}^{m} \int_0^{+\infty} ds_{2j-1} \frac{C}{\langle s_{2j-1}\rangle^{d/2}}\right)
\end{aligned}$$

Since the space dimension is at least three, the quantity at the r.h.s. can be estimated by

$$C^m \int_0^t ds_0 \int_0^{t-s_0} ds_2 \ldots \int_0^{t-\sum_{j=0}^{m-2} s_{2j}} ds_{2m-2} \;=\; \frac{C^m t^m}{m!}$$

An analogous estimate holds for the factor in (7.2) involving the variables $\widetilde{s}_0, \ldots, \widetilde{s}_{\widetilde{m}}$.

$$\begin{aligned}
\left|\langle J_\varepsilon, W^{\text{ladder}}_{+-,0,m,,\widetilde{m}}(t)\rangle\right| &\leq \frac{(C\lambda^2 t)^{m+\widetilde{m}}}{m!\widetilde{m}!} \varepsilon^{-d} \int d\xi\, dv\, |\widehat{J}_\varepsilon(\xi,v)| \left|\widehat{f}\left(\frac{v}{\varepsilon} + \frac{\xi-2P}{2\varepsilon}\right)\right| \left|\widehat{f}\left(\frac{v}{\varepsilon} - \frac{\xi-2P}{2\varepsilon}\right)\right| \\
&\leq \frac{(C\lambda^2 t)^{m+\widetilde{m}}}{m!\widetilde{m}!} \int d\xi \sup_v \left|\widehat{J}_\varepsilon(\xi,v)\right| \int dv \left|\widehat{f}\left(v + \frac{\xi-2P}{2\varepsilon}\right)\right| \left|\widehat{f}\left(v - \frac{\xi-2P}{2\varepsilon}\right)\right|
\end{aligned}$$

where we performed the change of variables $v \longrightarrow \varepsilon v$. Estimating the integral in v by Cauchy-Schwarz inequality and using (1.5) we prove (7.6).

Step 2. Here we prove

$$\lim_{\varepsilon\to 0}\Big|\langle J_\varepsilon, W^{\rm ladder}_{+-,0,m,,\widetilde m}(\varepsilon^{-1}T)\rangle \;-\; \langle J_\varepsilon, \mathcal{W}^{\rm ladder}_{+-,0,m,\widetilde m}(\varepsilon^{-1}T)\rangle\Big| \;=\; 0 \tag{7.7}$$

where

$$\begin{aligned}\mathcal{W}^{\rm ladder}_{+-,0,m,\widetilde m}(t;\xi,v) \;:=\; & (2\pi)^{-2-d/2}\,\varepsilon^{-d}\lambda^{2m+2\widetilde m}\Phi_P^m\overline{\Phi_P^{\widetilde m}}e^{2i\frac{v\cdot Q}{\varepsilon}}\,\widehat f\left(\frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right)\overline{\widehat f\left(\frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right)}\\ & \times e^{2t\eta}\int_{\mathbb{R}} d\alpha\, e^{-it\alpha}\,R^{m+1}(\alpha,v+\xi/2,-i\eta)\int_{\mathbb{R}} d\widetilde\alpha\, e^{it\widetilde\alpha}\,R^{\widetilde m+1}(\widetilde\alpha,v-\xi/2,i\eta)\end{aligned} \tag{7.8}$$

and Φ_P was defined in (7.3). Comparing formula (7.8) with (7.1), one easily notices that $\mathcal{W}^{\rm ladder}_{+-,0,m,\widetilde m}$ equals $W^{\rm ladder}_{+-,0,m,\widetilde m}$ apart from the replacement of Υ_η by Φ_P. Then, in the case $m=\widetilde m=0$ there is nothing to prove. We give the proof in detail for the generic case with both m and $\widetilde m$ greater than zero, and leave to the reader the case in which either m or $\widetilde m$ vanishes. We have

$$\begin{aligned}& \langle J_\varepsilon, W^{\rm ladder}_{+-,0,m,\widetilde m}(t)\rangle - \langle J_\varepsilon, \mathcal{W}^{\rm ladder}_{+-,0,m,\widetilde m}(t)\rangle\\ = \;& (2\pi)^{-2-d/2}\,\varepsilon^{-d}\lambda^{2m+2\widetilde m}e^{2\eta t}\int d\xi\,dv\,\overline{\widehat{J_\varepsilon}\,(\xi,v)}\,e^{2i\frac{v\cdot Q}{\varepsilon}}\,\widehat f\left(\frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right)\overline{\widehat f\left(\frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right)}\\ & \times\int d\alpha\,d\widetilde\alpha\,e^{-it\alpha}\,e^{it\widetilde\alpha}\,R^{m+1}\,\widetilde R^{\widetilde m+1}\left[\Upsilon_\eta^m\widetilde\Upsilon_\eta^{\widetilde m}-\Phi_P^m\overline{\Phi_P^{\widetilde m}}\right]\end{aligned} \tag{7.9}$$

where for brevity we used the notations

$$\begin{aligned} R &= R(\alpha,v+\xi/2,-i\eta), \qquad & \Upsilon_\eta &= \Upsilon_\eta(\alpha,v+\xi/2)\\ \widetilde R &= R(\widetilde\alpha,v-\xi/2,i\eta), \qquad & \widetilde\Upsilon_\eta &= \overline{\Upsilon}_\eta(\widetilde\alpha,v-\xi/2)\,.\end{aligned}$$

By the first estimate in (5.6) we have

$$\begin{aligned}\left|\Upsilon_\eta^m\widetilde\Upsilon_\eta^{\widetilde m}-\Phi_P^m\overline{\Phi_P^{\widetilde m}}\right| \;&=\; \left|\Upsilon_\eta^m\left(\widetilde\Upsilon_\eta-\overline{\Phi}_P\right)\sum_{j=0}^{\widetilde m-1}\widetilde\Upsilon_\eta^j\,\overline{\Phi}_P^{\widetilde m-j-1}+\overline{\Phi}_P^{\widetilde m}\,(\Upsilon_\eta-\Phi_P)\sum_{j=0}^{m-1}\Upsilon_\eta^j\,\Phi_P^{m-j-1}\right|\\ &\le\; C^{m+\widetilde m}\left(|\Upsilon_\eta-\Phi_P|+\left|\widetilde\Upsilon_\eta-\overline{\Phi}_P\right|\right)\,.\end{aligned} \tag{7.10}$$

Now we estimate (7.10). From the third formula in (5.6) one has

$$\begin{aligned}|\Upsilon_\eta-\Phi_P| \;&\le\; C\eta^{-1/2}\,(|\alpha-e(P)|+|v+\xi/2-P|+\eta)\\ &\le\; C\eta^{-1/2}\,(|\alpha-e(v+\xi/2)|+|e(v+\xi/2)-e(P)|+|v+\xi/2-P|+\eta)\end{aligned} \tag{7.11}$$

Moreover, the first estimate in (5.6) shows that

$$|\Upsilon_\eta-\Phi_P| \;\le\; |\Upsilon_\eta|+|\Phi_P| \;\le\; C \tag{7.12}$$

From (7.11) and (7.12) we obtain

$$|\Upsilon_\eta-\Phi_P| \;\le\; C\eta^{-1/2}\Big(|e(v+\xi/2)-e(P)|+|v+\xi/2-P|+\min\left(|\alpha-e(v+\xi/2)|,1\right)\Big) \tag{7.13}$$

where we used $\eta = \varepsilon < 1$. In order to obtain an analogous estimate for $\left|\widetilde{\Upsilon}_\eta - \overline{\Phi}_P\right|$ we first observe that, due to the symmetry of the functions $e(\cdot)$, $\omega(\cdot)$, $L(\cdot,\sigma)$, one has

$$\overline{\Phi}_P = \overline{\Upsilon}_{0+}(e(P), P) = \overline{\Upsilon}_{0+}(e(P), -P).$$

Hence, mimicking (7.11), (7.12), (7.13), one obtains

$$\left|\widetilde{\Upsilon}_\eta - \overline{\Phi}_P\right| \;\leq\; C\eta^{-1/2}\Big(|e(v-\xi/2)-e(P)| + |v-\xi/2+P| + \min\left(|\alpha - e(v-\xi/2)|, 1\right)\Big) \tag{7.14}$$

From (7.9), (7.10), (7.13), (7.14) we obtain

$$\begin{aligned}
&\left|\langle J_\varepsilon, W^{\text{ladder}}_{+-,0,m,\widetilde{m}}(\varepsilon^{-1}T)\rangle - \langle J_\varepsilon, \mathcal{W}^{\text{ladder}}_{+-,0,m,\widetilde{m}}(\varepsilon^{-1}T)\rangle\right| \\
\leq\;& \varepsilon^{-d}\eta^{-1/2}(C\lambda^2)^{m+\widetilde{m}} \int d\xi\, dv \left|\widehat{J}_\varepsilon\,(\xi, v)\right| \left|\widehat{f}\left(\frac{v}{\varepsilon} + \frac{\xi - 2P}{2\varepsilon}\right)\right| \left|\widehat{f}\left(\frac{v}{\varepsilon} - \frac{\xi - 2P}{2\varepsilon}\right)\right| \\
&\times \int_{\mathbb{R}^2} d\alpha\, d\widetilde{\alpha}\, |R|^{m+1}\, |\widetilde{R}|^{\widetilde{m}+1} \Big[|e(v+\xi/2)-e(P)| + |e(v-\xi/2)-e(P)| \\
&+|v+\xi/2-P| + |v-\xi/2-P| + \min\left(|\alpha - e(v+\xi/2)|, 1\right) + \min\left(|\widetilde{\alpha} - e(v-\xi/2)|, 1\right)\Big] \\
=\;& (I) + (I') + (II) + (II') + (III) + (III')
\end{aligned} \tag{7.15}$$

where the last decomposition is made according to the terms in square parentheses in the r.h.s. of (7.15).

The first term is estimated as follows

$$\begin{aligned}
(I) \;\leq\;& \varepsilon^{-d}\eta^{-1/2}(C\lambda^2)^{m+\widetilde{m}} \int d\xi \sup_v |\widehat{J}_\varepsilon(\xi, v)| \int dv \left|\widehat{f}\left(\frac{v}{\varepsilon} + \frac{\xi - 2P}{2\varepsilon}\right)\right| \left|\widehat{f}\left(\frac{v}{\varepsilon} - \frac{\xi - 2P}{2\varepsilon}\right)\right| \\
&\times |e(v+\xi/2)-e(P)| \int_{\mathbb{R}^2} d\alpha\, d\widetilde{\alpha}\, |R|^{m+1}\, |\widetilde{R}|^{\widetilde{m}+1}\,.
\end{aligned}$$

Recalling that $m \geq 1$, we have

$$\int \frac{d\alpha}{|\alpha - e(v+\xi/2) + i\eta|^{m+1}} \;\leq\; C\eta^{-m}$$

and analogously for the integral in $\widetilde{\alpha}$.

By Cauchy-Schwarz the integral in v gives

$$\begin{aligned}
&\int dv \left|\widehat{f}\left(\frac{v}{\varepsilon} + \frac{\xi - 2P}{2\varepsilon}\right)\right| \left|\widehat{f}\left(\frac{v}{\varepsilon} - \frac{\xi - 2P}{2\varepsilon}\right)\right| |e(v+\xi/2)-e(P)| \\
\leq\;& \varepsilon^{d/2}\|\widehat{f}\| \left[\int dv \left|\widehat{f}\left(\frac{v}{\varepsilon} + \frac{\xi - 2P}{2\varepsilon}\right)\right|^2 |e(v+\xi/2)-e(P)|^2\right]^{\frac{1}{2}}
\end{aligned}$$

Changing variable to $u := \frac{v}{\varepsilon} + \frac{\xi}{2\varepsilon} - \frac{P}{\varepsilon}$ one gains a further factor $\varepsilon^{d/2}$. Besides, using hypothesis (3.2) one gets

$$|e(v+\xi/2)-e(P)| \;=\; |e(\varepsilon u + P)-e(P)| \;\leq\; C\varepsilon\langle u\rangle^2\langle P\rangle$$

Then

$$\int dv \left|\widehat{f}\left(\frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right)\right|\left|\widehat{f}\left(\frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right)\right| |e(v+\xi/2)-e(P)| \;\leq\; C\varepsilon^{d+1}\langle P\rangle \|f\|_{H^2(\mathbb{R}^d)}$$

After getting rid of the integral in the variable ξ by (1.5) we obtain

$$(I) \;\leq\; \varepsilon\eta^{-1/2}(C\lambda^2\eta^{-1})^{m+\widetilde{m}}\langle P\rangle\|f\|_{H^2(\mathbb{R}^d)} \;\leq\; C^{m+\widetilde{m}}\varepsilon^{1/2}\langle P\rangle\|f\|_{H^2(\mathbb{R}^d)}$$

where we chose $\eta=\varepsilon$ and used $\lambda=\sqrt{\varepsilon}$.

Term (II) in (7.15) has the same structure as (I), apart from the replacement of $e(v+\xi/2)-e(P)$ with $v+\xi/2-P$.

For estimating (III) we proceed as in lemma 10.3 in [E]. Using the notation $\theta(s)=\min(|s|,1)$, we observe that

$$\frac{\theta(\alpha-e(v+\xi/2))}{|\alpha-e(v+\xi/2)+i\eta|} \;\leq\; \frac{C}{\langle\alpha-e(v+\xi/2)\rangle}$$

Then, by assumption (1.5), hypothesis $m\geq 1$, Cauchy-Schwarz inequality and the following two estimates, easily derived from (3.10),

$$\begin{aligned}\int \frac{d\alpha}{\langle\alpha-e(v+\xi/2)\rangle|\alpha-e(v+\xi/2)+i\eta|^m} &\leq C^m\eta^{1-m}\\ \int \frac{d\widetilde{\alpha}}{|\widetilde{\alpha}-e(v-\xi/2)+i\eta|^{\widetilde{m}+1}} &\leq C^{\widetilde{m}}\eta^{-\widetilde{m}}\end{aligned}$$

we conclude

$$\begin{aligned}(III) \;&\leq\; \varepsilon^{-d}(C\lambda^2)^{m+\widetilde{m}}\eta^{-1/2}\int d\xi\, dv\, \left|\widehat{J}(\xi,v)\right|\left|\widehat{f}\left(\frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right)\right|\left|\widehat{f}\left(\frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right)\right|\\ &\quad\times\int\frac{d\alpha\, d\widetilde{\alpha}}{\langle\alpha-e(v+\xi/2)\rangle|\alpha-e(v+\xi/2)+i\eta|^m|\widetilde{\alpha}-e(v-\xi/2)+i\eta|^{\widetilde{m}+1}}\\ &\leq\; \varepsilon^{\frac{1}{2}}C^{m+\widetilde{m}}\end{aligned}$$

For terms $(I'),(II'),(III')$ in (7.15) we have the same estimates found for the corresponding terms without prime. Then,

$$\left|\langle J_\varepsilon, W^{\rm ladder}_{+-,0,m,,\widetilde{m}}(\varepsilon^{-1}T)\rangle \;-\; \langle J_\varepsilon, \mathcal{W}^{\rm ladder}_{+-,0,m,\widetilde{m}}(\varepsilon^{-1}T)\rangle\right| \;\leq\; C^{m+\widetilde{m}}\sqrt{\varepsilon}\langle P\rangle\|f\|_{H^2(\mathbb{R}^d)}$$

and (7.7) is proven.

Step 3. Here we prove

$$\langle J_\varepsilon, \mathcal{W}^{\rm ladder}_{+-,0,m,\widetilde{m}}(\varepsilon^{-1}T)\rangle \;=\; \frac{(-iT\Phi_P)^m}{m!}\frac{(iT\overline{\Phi_P})^{\widetilde{m}}}{\widetilde{m}!}\langle J_\varepsilon, W_{+-,\rm free}(\varepsilon^{-1}T)\rangle \tag{7.16}$$

First, notice that the integrals in $d\alpha$ and $d\widetilde{\alpha}$ in the definiton (7.8) of $\mathcal{W}^{\rm ladder}_{+-,0,m,\widetilde{m}}(t)$ can be explicitly computed using residue theorem. One obtains

$$\int d\alpha\,\frac{e^{-it\alpha}}{[\alpha-e(v+\xi/2)+i\eta]^{m+1}} \;=\; -2\pi i e^{-\eta t}e^{-ite(v+\xi/2)}\frac{(-it)^m}{m!}$$

and analogously for the integral in $d\widetilde{\alpha}$. Therefore, choosing $\eta = \varepsilon$, $t = \varepsilon^{-1}T$, $\lambda = \sqrt{\varepsilon}$, from (7.8) one obtains

$$\begin{aligned}\langle J_\varepsilon, \mathcal{W}^{\text{ladder}}_{+-,0,m,\widetilde{m}}(\varepsilon^{-1}T)\rangle \;&=\; \varepsilon^{-d}\,(2\pi)^{-d/2}\,\frac{(-iT\Phi_P)^m}{m!}\frac{(iT\overline{\Phi_P})^{\widetilde{m}}}{\widetilde{m}!}\int d\xi\,dv\;e^{-i\frac{T}{\varepsilon}[e(v+\xi/2)-e(v-\xi/2)]}\\ &\quad\times\overline{\widehat{J_\varepsilon}\,(\xi,v)}\,e^{2i\frac{v\cdot Q}{\varepsilon}}\,\widehat{f}\left(\frac{v}{\varepsilon}+\frac{\xi-2P}{2\varepsilon}\right)\overline{\widehat{f}\left(\frac{v}{\varepsilon}-\frac{\xi-2P}{2\varepsilon}\right)}.\end{aligned}$$

(7.17)

To compute $W_{+-,\text{free}}(\varepsilon^{-1}T)$ we write

$$\gamma_{\text{e},\varepsilon^{-1}T,+-,\text{free}} \;=\; e^{-i\frac{T}{\varepsilon}H_\text{e}}|\psi^\varepsilon_{0,+}\rangle\langle\psi^\varepsilon_{0,-}|e^{i\frac{T}{\varepsilon}H_\text{e}}$$

or equivalently, in the Fourier space

$$\widehat{\gamma}_{\text{e},\varepsilon^{-1}T,+-,\text{free}}(p,u) \;=\; \varepsilon^{-d}e^{-i\frac{T}{\varepsilon}[e(p)-e(u)]}e^{i(p+u)\cdot\frac{Q}{\varepsilon}}\,\widehat{f}\left(\frac{p-P}{\varepsilon}\right)\overline{\widehat{f}\left(\frac{u+P}{\varepsilon}\right)}.$$

Finally, (7.16) immediately follows from (7.17) by using (4.12).

Step 4. Here we prove (1.6). By formulas (5.2), (5.3), (5.4), (5.7) for non-diagonal terms, and eliminating terms with $n > 0$ by Lemma 6.1, one gets

$$\begin{aligned}&\lim_{\varepsilon\to0}\left|\lim_{L\to\infty}\langle J_\varepsilon, W_{+-}(\varepsilon^{-1}T)\rangle \;-\; e^{-T\sigma_P}\langle J_\varepsilon, W_{+-,\text{free}}(\varepsilon^{-1}T)\rangle\right|\\ =\;&\lim_{K\to\infty}\lim_{\varepsilon\to0}\left|\sum_{m,\widetilde{m}=0}^{[\frac{K-1}{2}]}\langle J_\varepsilon, W^{\text{ladder}}_{+-,0,m,\widetilde{m}}(\varepsilon^{-1}T)\rangle \;-\; e^{-T\sigma_P}\langle J_\varepsilon, W_{+-,\text{free}}(\varepsilon^{-1}T)\rangle\right|,\end{aligned}$$

where we used (7.6) to exchange the limits in K and in ε. Then, from (7.7) and (7.16)

$$\begin{aligned}&\lim_{\varepsilon\to0}\left|\lim_{L\to\infty}\langle J_\varepsilon, W_{+-}(\varepsilon^{-1}T)\rangle \;-\; e^{-T\sigma_P}\langle J_\varepsilon, W_{+-,\text{free}}(\varepsilon^{-1}T)\rangle\right|\\ =\;&\lim_{K\to\infty}\lim_{\varepsilon\to0}\left|\sum_{m,\widetilde{m}=0}^{[\frac{K-1}{2}]}\frac{(-iT\Phi_P)^m}{m!}\frac{(iT\overline{\Phi_P})^{\widetilde{m}}}{\widetilde{m}!}\langle J_\varepsilon, W_{+-,\text{free}}(\varepsilon^{-1}T)\rangle \;-\; e^{-T\sigma_P}\langle J_\varepsilon, W_{+-,\text{free}}(\varepsilon^{-1}T)\rangle\right|\\ \le\;&\limsup_{K\to\infty}\left|e^{-T\sigma_P} \;-\; \sum_{m,\widetilde{m}=0}^{[\frac{K-1}{2}]}\frac{(-iT\Phi_P)^m}{m!}\frac{(iT\overline{\Phi_P})^{\widetilde{m}}}{\widetilde{m}!}\right|\limsup_{\varepsilon\to0}\left|\langle J_\varepsilon, W_{+-,\text{free}}(\varepsilon^{-1}T)\rangle\right|\end{aligned}$$

The first factor in the r.h.s. vanishes due to (7.5), while the second factor is bounded since

$$|\langle J_\varepsilon, W_{+-,\text{free}}(t)\rangle| = \Big|\;\text{Tr}\;\mathcal{O}^\star_\varepsilon|e^{-itH_\text{e}}\psi^\varepsilon_{0,+}\rangle\langle e^{-itH_\text{e}}\psi^\varepsilon_{0,-}|\;\Big| \le \|\mathcal{O}_\varepsilon\|.$$

The proof is complete. □

8. Appendix

Here we prove Lemma 3.1.

Proof. Inequality (3.10) corresponds to formula (5.19) in [E], and we refer to that paper for the proof. We prove (3.11). We treat the case of Φ_+, since for Φ_- the proof is the same.

Consider a tiling of the k-space in cubes $\{Q_i\}_{i\in\mathbb{N}}$ of size thinner than $\widetilde{\rho}$ defined in (3.6), (3.7). Fix $l_0 := [\log^\star(\widetilde{\rho}\eta^{-1})]$, and define the sets

$$\begin{aligned} S_l &:= \{k\in\mathbb{R}^d : e^{-l-1}\widetilde{\rho} \le |\theta-\Phi_+(p,k)| \le e^{-l}\widetilde{\rho}\}, \quad 1\le l\le l_0-1 \\ S_{l_0} &:= \{k\in\mathbb{R}^d : |\theta-\Phi_+(p,k)| \le e^{-l_0}\widetilde{\rho}\} \\ S_0 &:= \{k\in\mathbb{R}^d : |\theta-\Phi_+(p,k)| \ge \widetilde{\rho}/e\} \end{aligned}$$

and analogously the sets $\widetilde{S}_{\widetilde{l}}$, $0\le\widetilde{l}\le l_0$, where $\widetilde{\theta}$ replaces θ in the definitions above.
Then,

$$\int_{\mathbb{R}^3} \frac{\mathcal{L}(k)\,dk}{|\theta-\Phi_+(p,k)+i\eta|\,|\widetilde{\theta}-\Phi_+(u,k)-i\eta|\langle p\rangle\langle u\rangle} \;\le\; \sum_i \sum_{l,\widetilde{l}=0}^{l_0} U_{i,l,\widetilde{l}} \left| Q_i\cap S_l\cap\widetilde{S}_{\widetilde{l}}\right| \tag{8.1}$$

where by $|\cdot|$ we denoted the Lebesgue measure in $\mathbb{R}^d$ and introduced the notation

$$U_{i,l,\widetilde{l}} := \sup_{k\in Q_i\cap S_l\cap\widetilde{S}_{\widetilde{l}}} \left(\frac{\mathcal{L}(k)}{|\theta-\Phi_+(p,k)+i\eta|\,|\widetilde{\theta}-\Phi_+(u,k)-i\eta|\langle p\rangle\langle u\rangle}\right)$$

Since the actual value of $\widetilde{\rho}$ does not play any role, in what follows we will absorb it in the constant C.

We prove the following estimate:

$$U_{i,l,\widetilde{l}} \;\le\; C\frac{e^{l+\widetilde{l}}}{\langle\theta\rangle^{\frac12}\langle\widetilde{\theta}\rangle^{\frac12}} \sup_k \left[\langle k\rangle^4|\mathcal{L}(k)|\right], \qquad 0\le l,\widetilde{l}\le l_0. \tag{8.2}$$

First, we treat the case $1\le l,\widetilde{l}\le l_0$. We have

$$U_{i,l,\widetilde{l}} \le \sup_{k\in Q_i\cap S_l\cap\widetilde{S}_{\widetilde{l}}} \left(\frac{1}{|\theta-\Phi_+(p,k)+i\eta|\,|\widetilde{\theta}-\Phi_+(u,k)-i\eta|}\right) \sup_{k\in Q_i\cap S_l\cap\widetilde{S}_{\widetilde{l}}} \left(\frac{|\mathcal{L}(k)|}{\langle p\rangle\langle u\rangle}\right)$$

For the factor $|\theta-\Phi_+(p,k)+i\eta|^{-1}$ we proceed as follows

- if $1\le l<l_0$ then we apply $|\theta-\Phi_+(p,k)+i\eta|^{-1}\le Ce^l$;
- if $l=l_0$ then we use

$$|\theta-\Phi_+(p,k)+i\eta|^{-1} \;\le\; |\eta|^{-1} \;\le\; Ce^{l_0}.$$

Analogously,

$$|\widetilde{\theta}-\Phi_+(u,k)-i\eta|^{-1} \;\le\; Ce^{\widetilde{l}}, \qquad 1\le\widetilde{l}\le\widetilde{l}_0. \tag{8.3}$$

We finally obtain

$$\sup_{k\in Q_i\cap S_l\cap\widetilde{S}_{\widetilde{l}}} \left(\frac{1}{|\theta-\Phi_+(p,k)+i\eta|\,|\widetilde{\theta}-\Phi_+(u,k)-i\eta|}\right) \;\le\; Ce^{l+\widetilde{l}}, \qquad 1\le l,\widetilde{l}\le l_0. \tag{8.4}$$

Now we estimate $\sup_{k\in Q_i\cap S_l\cap\widetilde{S}_{\widetilde{l}}}\left(\frac{|L(k)|}{\langle p\rangle\langle u\rangle}\right)$. Using inequalities (3.1), (3.2) we have

$$\langle p\rangle^{-2} \;\le\; C\frac{\langle k\rangle^2}{\langle e(p+k)\rangle} \;\le\; C\frac{\langle k\rangle^2}{\langle\theta-\Phi_+(p,k)-\theta+\omega(k)\rangle} \;\le\; C\frac{\langle k\rangle^4\langle\theta-\Phi_+(p,k)\rangle}{\langle\theta\rangle}. \tag{8.5}$$

Notice that in S_l, $1\le l\le l_0$ one has $\langle\theta-\Phi_+(p,k)\rangle\le 1+e^{-l}\widetilde{\rho}$, therefore such a quantity can be estimated by a constant and we end up with

$$\langle p\rangle^{-1} \;\le\; C\frac{\langle k\rangle^2}{\langle\theta\rangle^{1/2}}, \qquad k\in S_l,\;\; 1\le l\le l_0. \tag{8.6}$$

Analogously,

$$\langle u\rangle^{-1} \;\le\; C\frac{\langle k\rangle^2}{\langle\widetilde{\theta}\rangle^{1/2}}, \qquad k\in\widetilde{S}_{\widetilde{l}},\, 1\le\widetilde{l}\le l_0. \tag{8.7}$$

Therefore, from (8.4), (8.6), (8.7) one obtains (8.2) for $l,\widetilde{l}\neq 0$.

To estimate $U_{i,0,0}$ we observe that, proceeding like in (8.5) and using

$$\frac{\langle\theta-\Phi_+(p,k)\rangle}{|\theta-\Phi_+(p,k)+i\eta|} \;\le\; C, \qquad k\in S_0,$$

one gets

$$\frac{1}{|\theta-\Phi_+(p,k)+i\eta|\,\langle p\rangle} \;\le\; C\frac{\langle k\rangle^2}{\langle\theta\rangle^{1/2}}, \qquad k\in S_0. \tag{8.8}$$

Analogously,

$$\frac{1}{\left|\widetilde{\theta}-\Phi_+(u,k)-i\eta\right|\,\langle u\rangle} \;\le\; C\frac{\langle k\rangle^2}{\langle\widetilde{\theta}\rangle^{1/2}}, \qquad k\in\widetilde{S}_0. \tag{8.9}$$

Then, from (8.8) and (8.9) one obtains (8.2) for $l,\widetilde{l}=0$. Finally, to obtain (8.2) for $l=0$, $\widetilde{l}\neq 0$ it is sufficient to use inequalities (8.7), (8.8) and (8.3), and analogously for the case $l\neq 0$, $\widetilde{l}=0$.

The volume factor $|Q_i\cap S_l\cap\widetilde{S}_{\widetilde{l}}|$ in (8.1) is estimated as follows:

- if both l and $\widetilde{l}$ are greater than zero, then we use inequality (3.7) and estimate the volume of integration by
$$\left|Q_i\cap S_l\cap\widetilde{S}_{\widetilde{l}}\right| \;\le\; C\frac{e^{-l-\widetilde{l}}}{|p-u|};$$
- if $l=0$ and $\widetilde{l}\neq 0$, then we use inequality (3.6) and obtain
$$\left|Q_i\cap S_0\cap\widetilde{S}_{\widetilde{l}}\right| \;\le\; Ce^{-\widetilde{l}};$$
vice versa, if $l>0,\widetilde{l}=0$, then $\left|Q_i\cap S_l\cap\widetilde{S}_0\right|$ is estimated by Ce^{-l}.
- If both l and $\widetilde{l}$ equal zero, then the volume $\left|Q_i\cap S_0\cap\widetilde{S}_0\right|$ is estimated by C.

Then,

$$\sum_i \sum_{l,\widetilde{l}=0}^{l_0} \int_{Q_i\cap S_l\cap \widetilde{S}_{\widetilde{l}}} \frac{\mathcal{L}(k)\,dk}{|\theta-\Phi_+(p,k)+i\eta|\,|\widetilde{\theta}-\Phi_+(u,k)-i\eta|\langle p\rangle\langle u\rangle}$$
$$\leq \; C\,\frac{1+2l_0+l_0^2/|p-u|}{\langle\theta\rangle^{\frac12}\langle\widetilde{\theta}\rangle^{\frac12}}\sum_i \|\langle\cdot\rangle^4\mathcal{L}\|_{L^\infty(Q_i)} \;\leq\; C\frac{(\log^\star\eta)^2}{\langle\theta\rangle^{\frac12}\langle\widetilde{\theta}\rangle^{\frac12}\min(1,|p-u|)}\sum_i \|\langle\cdot\rangle^4\mathcal{L}\|_{L^\infty(Q_i)}$$

The decay property (3.8) of $\mathcal{L}$ guarantee that the sum in i is finite. Then,

$$\int_{\mathbb{R}^d} \frac{\mathcal{L}(k)\,dk}{|\theta-\Phi_+(p,k)+i\eta|\,|\widetilde{\theta}-\Phi_+(u,k)-i\eta|\langle p\rangle\langle u\rangle} \;\leq\; C\frac{(\log^\star\eta)^2}{\langle\theta\rangle^{\frac12}\langle\widetilde{\theta}\rangle^{\frac12}\min(1,|p-u|)} \tag{8.10}$$

Inequality (8.10) can be improved in the case $|p-u|<\eta$. Indeed, in this case, we can estimate terms with $l,\widetilde{l}>0$ by means of inequality (3.6) in spite of (3.7). We obtain

$$\begin{aligned}&\int_{\mathbb{R}^d} \frac{\mathcal{L}(k)\,dk}{|\theta-\Phi_+(p,k)+i\eta|\,|\widetilde{\theta}-\Phi_+(u,k)-i\eta|\langle p\rangle\langle u\rangle}\\ \leq& C\,\frac{1+2l_0+l_0e^{l_0}}{\langle\theta\rangle^{\frac12}\langle\widetilde{\theta}\rangle^{\frac12}}\sum_i \|\langle\cdot\rangle^4\mathcal{L}\|_{L^\infty(Q_i)} \;\leq\; C\frac{\log^\star\eta}{\langle\theta\rangle^{\frac12}\langle\widetilde{\theta}\rangle^{\frac12}\eta}\end{aligned} \tag{8.11}$$

Combining (8.10) and (8.11) we arrive at (3.11).

Estimate (3.12) is trivial.

□

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R. Adami: Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca.
Current address: Via R. Cozzi 53, 20125 Milano, Italy.
E-mail address: riccardo.adami@unimib.it

L. Erdős: Mathematisches Institut der Ludwig-Maximilians-Universität München.
Current address: 39 Theresienstr., D-80333, München, Germany.
E-mail address: lerdos@mathematik.uni-muenchen.de