Notes on

SPECTRAL THEORY

by

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1. Introduction

It is well known, and distressing to students and teachers, that although spectral theory is comparatively easy for Hermitian operators, it is comparatively hard for normal operators. The nature of the difficulties varies with the approach. For instance, in the approach via Banach algebras, the ultimate weapon is the Gelfand-Naimark representation theorem for commutative C*-algebras, a definitely advanced technique [9]. In the approach through integration theory, it is not so easy to see, through the maze of details, where the difficulties lie and how they are overcome, but it is probable that the most taxing demand of this approach is the technical facility it requires; the measure-theoretic formulation of the spectral theorem can be grasped only when the measure-theroretic techniques are at one's fingertips. The Hermitian case is straightforward enough [6], but the normal case is fraught with complications [5].

It may therefore come as a pleasant surprise to the reader, as it did to me, that the measure-theoretic approach to the spectral theorem is really no harder for normal operators than it is for Hermitian operators. It is true that one needs a few more techniques for the normal case, but these techniques are no deeper than for the Hermitian case. Nor does one need any deeper analytical tools in the normal case; the classical Weierstrass approximation theorem for a closed interval of the line, and the Riesz theorem on representing positive linear forms in the same context, are all that is required, and we do not even need to know that the complex numbers are algebraically complete [7]. Briefly, it is possible to deduce the spectral theorem for a normal operator from the spectral theorem for a Hermitian operator, using only elementary measure-theoretic techniques. My principal aim in this little book is to give such an exposition of spectral theory. At the same time, I feel that the exposition is clarified by placing some of the underlying techniques in a slightly more (though not the most) general setting. For example, it is instructive to see the "spectral mapping theorem" formulated in the context of a locally compact space (see Sec. 7). The extra generality should not be a barrier to the spectral theorem itself, since the reader may avoid general topology altogether by substituting "reals" or "complexes" for "locally compact space".

I have also included a number of results which, though not actually needed for the spectral theorem, are natural and instructive complements to the development; such results are marked with an asterisk, and may be omitted. For the benefit of the reader who is seeing the spectral theory for the first time, I have written out most of the proofs in detail. The reader who has been through Chapter II of [5] will find most of the novelty of the present exposition concentrated in the proofs of the theorems in Sections 8 and 12; the reader who has mastered Chapter II of [5] may prefer to start with the skeletons of these proofs, and fill in the details for himself.

2. Prerequisites

Our basic references for measure theory and Hilbert space are [4] and [5], respectively. In particular, the term "Borel set" refers to the concept defined in [4], contrary to its usage in [5] (cf. [5, p. 111]). We write $\mathfrak{S}(\mathscr{E})$ for the σ-ring generated by a class of sets $\mathscr{E}$, and $(x|y)$ for the inner product of vectors x and y. Some specific results that will be needed are as follows.

Operator theory. We take for granted Sections 1–34 of [5]. At several points it is convenient to have available the square root of a positive operator: if A is a positive operator, that is if $(Ax|x) \geqslant 0$ for all vectors x, then there exists a unique positive operator B such that $B^2 = A$; moreover, $B \in \{A\}''$ (the double commutant of A), that is, B commutes with every operator T that commutes with A (more concisely, $T \leftrightarrow A$ implies $B \leftrightarrow T$). An elementary proof, independent of spectral theory, is given in [8, Sec. 104].

Weierstrass approximation theorem. If f is a continuous real (i.e., real-valued) function on a closed interval $[a,b]$, there exists a sequence of real polynomials p_n such that $p_n \to f$ uniformly on $[a,b]$. (The two-dimensional analogue of this theorem will not be needed.)

Riesz-Markoff theorem [4, Sec. 56]. If X is a locally compact space, $\mathscr{L}(X)$ is the real linear space of all continuous real functions on X with compact support, and if L is a positive linear form on $\mathscr{L}(X)$, then there exists one and only one Baire measure ν on X such that

$$L(f) = \int f\, d\nu$$

for all f in $\mathscr{L}(X)$. For the spectral theorem, we actually need this result only for the case that $X = \mathbf{R}$, the space of reals.

Regularity of Baire measures [4, p. 228]. Every Baire measure ν on a locally compact space X is regular in the sense that

$$\nu(M) = \mathrm{LUB}\,\{\nu(C):\ C \subset M\,,\ C \text{ compact } G_\delta\}\,,$$
$$\nu(M) = \mathrm{GLB}\,\{\nu(U):\ U \supset M\,,\ U \text{ open Baire set}\}\,,$$

for every Baire set M. For the spectral theorem, we need this result only for the case that $X = \mathbf{R}$ or $X = \mathbf{C}$ (the space of complexes, in other words, topologically, the space $\mathbf{R} \times \mathbf{R}$). (In fact, granted an elementary result on regularity in product spaces, the case $X = \mathbf{R}$ would be sufficient; [2, p. 199ff].)

Baire sets multiply [4, p. 222]. If $\mathscr{B}_0(X)$ denotes the class of Baire sets in a locally compact space X, then

$$\mathscr{B}_0(X_1 \times X_2) = \mathscr{B}_0(X_1) \times \mathscr{B}_0(X_2)\,, \tag{1}$$

that is, the Baire sets of $X_1 \times X_2$ may be described as the σ-ring generated by the rectangles $M_1 \times M_2$, where M_k is a Baire set in X_k $(k = 1, 2)$. For the spectral theorem, we need this result only for the case that $X_1 = X_2 = \mathbf{R}$.

We take "measure" to be the concept defined in [4, p. 30], thus a measure is extended real valued and positive. All actual integration will be performed in the context of measure spaces as defined in [4, p. 73] (in fact, finite measure spaces are enough). In particular, "integrals" with respect to "complex measures" will be treated as a convenient but purely formal notation.

To stress the elementary level of what we need in the way of measure and integration, we mention explicitly that neither the Radon-Nikodym theorem nor Fubini's theorem will be needed. Though we shall consider certain measures defined on the Cartesian product of two σ-rings, the concept of product measure will not be presupposed.

3. Positive operator valued measures

Let us fix, once and for all, a nonzero complex Hilbert space, whose elements we shall call *vectors*; the letters x, y, z (no others will be needed) are reserved for vectors. All operators to be considered are defined on this fixed Hilbert space.

DEFINITION 1. A *positive operator valued measure* (briefly, PO-measure) is a triple $(X, \mathscr{R}, E)$, where X is a set, $\mathscr{R}$ is a ring of subsets of X, and E is an operator valued set function on $\mathscr{R}$ with the following properties:

(PO1) E is *positive*, that is, $E(M) \geqslant 0$ for each M in $\mathscr{R}$.

(PO2) E is *additive*, that is, $E(M \cup N) = E(M) + E(N)$ whenever M and N are disjoint sets in $\mathscr{R}$.

(PO3) E is *continuous* in the sense that $E(M) = \mathrm{LUB}\, E(M_n)$ whenever M_n is an increasing sequence of sets in $\mathscr{R}$ whose union M is also in $\mathscr{R}$.

It is often convenient to say that E is a PO-measure "on X" or "on $\mathscr{R}$", although strictly speaking a PO-measure is a triple (really a quadruple, counting the Hilbert space). It is clear from positivity and additivity that a PO-measure E is *monotone*, that is, $M \subset N$ implies $E(M) \leqslant E(N)$, where Hermitian operators are partially ordered in the usual way [5, p. 42]. Thus if M_n is an increasing sequence of sets in $\mathscr{R}$ whose union M is also in $\mathscr{R}$, then $E(M_n)$ is an increasing sequence of Hermitian operators, and $E(M_n) \leqslant E(M)$ for all n; the fact that the LUB in question exists is covered by the following well-known proposition, which will also be needed later on:

PROPOSITION 1. *If (A_j) is an increasingly directed family of Hermitian operators, and if the family is bounded above in the sense that there exists a Hermitian operator B such that $A_j \leqslant B$ for all j, then* LUB A_j *exists. Writing $A = \mathrm{LUB}\, A_j$, we have $A_j \to A$ weakly, that is, $(A_j x|y) \to (Ax|y)$ for each pair of vectors x, y. In particular,*

$$(A_j x|x) \uparrow (Ax|x)$$

for each vector x, and it follows from this that $A_j \to A$ strongly, that is,

$$\|A_j x - Ax\| \to 0$$

for each vector x. We write, briefly, $A_j \uparrow A$.

Proof. We are assuming that for each pair of indices i and j, there is an index k such that $A_i \leqslant A_k$ and $A_j \leqslant A_k$.

Fix an index m. It is clearly sufficient to find a LUB for the family of those A_j for which $A_j \geqslant A_m$. Thus, after a change of notation, we may assume (without loss of generality for any of the conclusions) that the family (A_j) has a first element A_m.

For each vector x, the family of real numbers $(A_j x|x)$ is increasingly directed, bounded above by $(Bx|x)$, and is therefore convergent. By the polarization identity [5, p. 13] we may define

$$\varphi(x, y) = \lim (A_j x|y)$$

for each ordered pair of vectors x, y. The functional φ is evidently sesquilinear, and for each vector x we have

$$(A_j x|x) \uparrow \varphi(x,x)\,,$$
$$(A_m x|x) \leqslant \varphi(x,x) \leqslant (Bx|x)\,.$$

It follows that

$$|\varphi(x,x)| \leqslant \|x\|^2 \max\{\|A_m\|, \|B\|\}\,,$$

thus φ is bounded [5, p. 33], and there exists an operator A such that $\varphi(x,y) = (Ax|y)$ for all x and y [5, p. 39]. Evidently A is Hermitian, $A_j \leqslant A$ for all j, and if C is any Hermitian operator such that $A_j \leqslant C$ for all j, necessarily $A \leqslant C$; briefly, $A = \mathrm{LUB}\, A_j$.

To prove that $A_j \to A$ strongly, we observe that $A_m \leqslant A_j \leqslant A$ for all j, and consequently $\|A_j\|$ is bounded. Thus the operators $B_j = A - A_j$ are positive, $\|B_j\|$ is bounded, and $(B_j x|x) \downarrow 0$ for each vector x. The fact that $\|B_j x\| \to 0$ follows from the generalized Schwarz inequality

$$|(B_j x|y)|^2 \leqslant (B_j x|x)(B_j y|y)$$

in the usual way [8, Sec. 104]. ◊

Thus the continuity condition (PO3) may be expressed concisely as follows: $M_n \uparrow M$ implies $E(M_n) \uparrow E(M)$, and consequently $E(M_n) \to E(M)$ strongly.

Incidentally, it is clear from the proof of Proposition 1 that if (A_j) is an increasingly directed family of Hermitian operators, and if A is a Hermitian operator such that $(A_j x|x) \to (Ax|x)$ for every vector x, then $A = \mathrm{LUB}\, A_j$, and consequently $A_j \to A$ strongly.

Rather than work out the theory of PO-measures from scratch, it is convenient to reduce matters to the numerical case by means of the following characterization of PO-measures (cf. [5,p. 59]):

THEOREM 1. *Consider a triple* $(X, \mathscr{R}, E)$, *where* $\mathscr{R}$ *is a ring of subsets of* X, *and* E *is a set function on* $\mathscr{R}$ *whose values are positive operators.*

In order that E *be a* PO-*measure, it is necessary and sufficient that for each vector* x, *the set function* μ_x *on* $\mathscr{R}$ *defined by the formula*

$$\mu_x(M) = (E(M)x|x) \tag{2}$$

be a measure.

Proof. The details are elementary, and are left to the reader. The essential ingredients of the proof are Proposition 1, the remarks following it, and the fact that a non-negative real valued set function on $\mathscr{R}$ is countably additive if and only if it is finitely additive and continuous from below [4, p. 39]. ♢

In accordance with Theorem 1 we may associate, with each PO-measure E, a family of finite measures μ_x defined by (2). It is occasionally useful to know when a family of finite measures, indexed by the vectors of the underlying Hilbert space, can be generated by a suitable PO-measure:

THEOREM 2. *Let $\mathscr{R}$ be a ring of subsets of a set X, and suppose that for each vector x there is given a finite measure μ_x on $\mathscr{R}$.*

In order that there exists a PO-*measure E on $\mathscr{R}$ such that*

$$\mu_x(M) = (E(M)x\big|x) \tag{2}$$

for all vectors x and all M in $\mathscr{R}$, it is necessary and sufficient that

$$[\mu_{x+y}(M)]^{1/2} \leqslant [\mu_x(M)]^{1/2} + [\mu_y(M)]^{1/2}\,, \tag{3}$$

$$\mu_{cx}(M) = |c|^2\mu_x(M)\,, \tag{4}$$

$$\mu_{x+y}(M) + \mu_{x-y}(M) = 2\mu_x(M) + 2\mu_y(M)\,, \tag{5}$$

for all vectors x, y, all complex numbers c, and all M in $\mathscr{R}$, and that there exist, for each M in $\mathscr{R}$, a constant k_M such that

$$\mu_x(M) \leqslant k_M\|x\|^2 \tag{6}$$

for all vectors x.

In this case, the formula (2) *determines E uniquely.*

Proof. By a classical and elementary argument of J. von Neumann and P. Jordan [12, p. 124] the relations (3)–(5) are equivalent to the assertion that for each M, the functional

$$\varphi_M(x,y) = \frac{1}{4}\{\mu_{x+y}(M) - \mu_{x-y}(M) + i\,\mu_{x+iy}(M) - i\,\mu_{x-iy}(M)\}$$

is a Hermitian and positive sesquilinear form such that $\varphi_M(x,x) = \mu_x(M)$, and the boundedness of this sesquilinear form is equivalent to (6).* Thus it is clear that the validity of (3)–(6) is equivalent to the existence of a positive operator valued set function E on $\mathscr{R}$ satisfying (2), and such an E is necessarily a PO-measure by Theorem 1. ♢

* See also [15, p. 176, Th. 41.6]; the assumptions (3) and (6) assure the continuity of the mapping $x \mapsto \mu_x(M)^{1/2}$ (needed for proving sesquilinearity of φ_M) and hence of φ_M, whence $\varphi_M(x,y) = (E(M)x|y)$ for a suitable bounded operator $E(M)$ on the given Hilbert space [5, p. 38, Th. 1 of §22].

DEFINITION 2. A *spectral measure* is a PO-measure $(X, \mathscr{R}, E)$ such that the values of E are projections ($=$ Hermitian idempotents).

We remark that the definition of "spectral measure" given in [5, p. 58] requires in addition that $\mathscr{R}$ be a σ-algebra and that $E(X) = I$. We prefer to defer these restrictions until they are absolutely necessary; meanwhile, the term "spectral measure" is handier than "projection-valued PO-measure".

Of course $0 \leqslant P \leqslant I$ for any projection P. In connection with Definitions 1 and 2, it is important to note that if (P_j) is an increasingly directed family of projections, then $\mathrm{LUB}\, P_j$, as calculated in Proposition 1, is also a projection:

PROPOSITION 2. *If (P_j) is an increasingly directed family of projections, and if $P = \mathrm{LUB}\, P_j$ in the sense of Proposition 1, then P is also a projection. Indeed, P is the projection whose range is the smallest closed linear subspace containing the ranges of the P_j.*

Proof. Let Q be the projection whose range is the closed linear span of the ranges of the P_j; thus Q is the least upper bound of the P_j in the set of all projections [5, p. 49]. Since $P_j \leqslant Q$ for all j [5, p. 48], we have $P \leqslant Q$ by Proposition 1. On the other hand, $P_j \leqslant P$ for all j; to show that $Q \leqslant P$, it will therefore suffice to show that P is a projection.

Thus we are reduced to showing that $P^2 = P$. Since $\|P_j\| \leqslant 1$ and $P_j^2 = P_j$ for all j, we have, for any vector x,

$$\begin{aligned}\|P^2x - Px\| &\leqslant \|P(Px) - P_j(Px)\| + \|P_j(Px - P_jx)\| + \|P_jx - Px\| \\ &\leqslant \|P(Px) - P_j(Px)\| + \|Px - P_jx\| + \|P_jx - Px\|,\end{aligned}$$

and $P^2x - Px = 0$ results from the fact that $\|P_jy - Py\| \to 0$ for every vector y (Proposition 1). $\diamondsuit$

Spectral measures are characterized among PO-measures by the property of being *multiplicative*; the easy proof is written out in [5, p. 58]:

THEOREM 3. *In order that a PO-measure $(X, \mathscr{R}, E)$ be a spectral measure, it is necessary and sufficient that*

$$E(M \cap N) = E(M)E(N)$$

for all M, N in $\mathscr{R}$. In this case,

$$E(M) \leftrightarrow E(N)$$

for all M and N, and the projections $E(M)$ and $E(N)$ are orthogonal when M and N are disjoint.

Another characterization, in terms of numerical measures, is as follows:

*THEOREM 4. *In order that a* PO-*measure* $(X, \mathscr{R}, E)$ *be a spectral measure, it is necessary and sufficient that*

$$(\mu_x)_M = \mu_{E(M)x} \tag{7}$$

for all vectors x *and all* M *in* $\mathscr{R}$, *where the measures* μ_x *are defined as in* (2), *and* $(\mu_x)_M$ *is the contraction of* μ_x *by* M.

Proof. The set function $(\mu_x)_M$ is defined by the formula

$$(\mu_x)_M(N) = \mu_x(M \cap N),$$

and is itself a finite measure on $\mathscr{R}$ [2, p. 12].

For all M, N in $\mathscr{R}$, and all vectors x, we have

$$(\mu_x)_M(N) = \mu_x(M \cap N) = (E(M \cap N)x \big| x),$$
$$\mu_{E(M)x}(N) = (E(N)E(M)x \big| E(M)x) = (E(M)E(N)E(M)x \big| x).$$

Thus the relation (7) holds identically in M and x if and only if

$$E(M \cap N) = E(M)E(N)E(M) \tag{7$'$}$$

for all M and N.

If E is projection-valued, the relation (7′) follows at once from Theorem 3. If, conversely, (7′) holds, then in particular $M = N$ yields $E(M) = E(M)^3$, and therefore $E(M)^2 = E(M)^4$; since positive square roots are unique, we conclude that $E(M) = E(M)^2$. ◊

Though it is true, by a beautiful theorem of M.A. Naimark, that (practically)† every PO-measure may be "dilated" to be projection-valued [10], it would not materially simplify our arguments to cite this result, and to prove it would be an outright digression.

4. Extensions of bounded PO-measures

It is vital to be able to extend PO-measures from rings to σ-rings. The next theorem shows that an extra hypothesis will be needed to perform such extensions:

† The necessary and sufficient condition is that $\|E(M)\| \leqslant 1$ for all $M \in \mathscr{R}$ (cf. Ths. 5, 7 below, and Th. 1 of [13]).

THEOREM 5. *If E is a* PO-*measure whose domain of definition is a σ-ring $\mathscr{S}$, then*

$$\mathrm{LUB}\,\{\|E(M)\| : \ M \in \mathscr{S}\} < \infty\,,$$

that is, $\|E(M)\|$ is bounded.

Proof. Let K be the indicated LUB, and choose a sequence M_n in $\mathscr{S}$ such that $\mathrm{LUB}\,\|E(M_n\| = K$. If M is the union of the M_n, then $M \in \mathscr{S}$, and $E(M) \geqslant E(M_n) \geqslant 0$ for all n, and so $\|E(M_n)\| \leqslant \|E(M)\|$ for all n [5, p.41]. Thus $K \leqslant \|E(M)\| < \infty$. ♢

This circumstance calls for the following definition:

DEFINITION 3. A PO-measure E on a σ-ring $\mathscr{R}$ is said to be *bounded* in case $\|E(M)\|$ is bounded as M varies over $\mathscr{R}$.

It is easy to see that a PO-measure E is bounded if and only if there exists a Hermitian operator T such that $E(M) \leqslant T$ for all M. In particular, every spectral measure is bounded. The key result of the section is that every bounded PO-measure on a ring $\mathscr{R}$ may be extended uniquely to a PO-measure on the σ-ring $\mathfrak{S}(\mathscr{R})$ generated by $\mathscr{R}$. The uniqueness part of the proof is worth separating out as a useful result in its own right:

THEOREM 6. *If $\mathscr{R}$ is a ring of sets, and if E_1 and E_2 are* PO-*measures defined on $\mathfrak{S}(\mathscr{R})$ such that*

$$E_1(M) = E_2(M)$$

for all M in $\mathscr{R}$, then $E_1 = E_2$.

Proof. Suppose, more generally, that $E_1(M) \leqslant E_2(M)$ for all M in $\mathscr{R}$. For each vector x, consider the measures defined on $\mathfrak{S}(\mathscr{R})$ by the formulas

$$\mu_x^1(N) = (E_1(N)x\,|\,x)\,,$$
$$\mu_x^2(N) = (E_2(N)x\,|\,x)\,.$$

Thus μ_x^1 and μ_x^2 are finite measures on $\mathfrak{S}(\mathscr{R})$ such that $\mu_x^1 \leqslant \mu_x^2$ on $\mathscr{R}$, consequently $\mu_x^1 \leqslant \mu_x^2$ on $\mathfrak{S}(\mathscr{R})$ [2, p. 8]. Thus $E_1(N) \leqslant E_2(N)$ for all N in $\mathfrak{S}(\mathscr{R})$. ♢

The key extension theorem, for our purposes, is as follows:

THEOREM 7. *If F is a bounded* PO-*measure defined on a ring $\mathscr{R}$, there exists one and only one (necessarily bounded)* PO-*measure E on the σ-ring $\mathfrak{S}(\mathscr{R})$ generated by $\mathscr{R}$ such that E is an extension of F. If, moreover, F is a spectral measure, then so is E.*

Proof. For each vector x, we write $\nu_x(M) = (F(M)x|x)$ for all M in $\mathscr{R}$. Since ν_x is a finite measure on $\mathscr{R}$, there is a unique measure μ_x on $\mathfrak{S}(\mathscr{R})$ which extends ν_x [4, p. 54]. Suppose

$$\|F(M)\| \leqslant K < \infty$$

for all M in $\mathscr{R}$. Then

$$\mu_x(M) = \nu_x(M) \leqslant K\,\|x\|^2$$

for all M in $\mathscr{R}$; since the class of all sets N in $\mathfrak{S}(\mathscr{R})$ which satisfy the relation

$$\mu_x(N) \leqslant K\,\|x\|^2 \tag{8}$$

is evidently a monotone class containing $\mathscr{R}$, we conclude that (8) holds for every N in $\mathfrak{S}(\mathscr{R})$ [4, p. 27]. In particular, the measures μ_x are finite.

To obtain the PO-measure E, we propose to apply Theorem 2 to the family of finite measures μ_x. Thus we must verify the criteria (3)–(6) of that theorem. The proofs of (3)–(5) are similar; for instance, the validity of the relation

$$(\mu_{x+y})^{1/2} \leqslant (\mu_x)^{1/2} + (\mu_y)^{1/2}$$

on $\mathfrak{S}(\mathscr{R})$ is deduced from its validity on $\mathscr{R}$, using the same type of argument as was used in establishing (8) (cf. [2, p. 8]). The relation (6) follows at once from (8); indeed, we may take $k_N = K$ for every N in $\mathfrak{S}(\mathscr{R})$.

Thus, by Theorem 2, there exists a PO-measure E on $\mathfrak{S}(\mathscr{R})$ such that

$$\mu_x(N) = (E(N)x|x)$$

for all vectors x, and all N in $\mathfrak{S}(\mathscr{R})$. In particular, for M in $\mathscr{R}$ we have

$$(E(M)x|x) = \mu_x(M) = \nu_x(M) = (F(M)x|x)\,,$$

thus E is an extension of F. Such an extension is unique by Theorem 6.

Finally, assuming F is projection-valued, it is to be shown that E is also projection-valued. Fix a vector x, and consider the class $\mathscr{N}$ of all N in $\mathfrak{S}(\mathscr{R})$ such that

$$(E(N)^2x|x) = (E(N)x|x)\,,$$

that is,

$$(E(N)x|E(N)x) = (E(N)x|x)\,. \tag{9}$$

Since E extends F, and since $F(M)^2 = F(M)$ for all M in $\mathscr{R}$, we have $\mathscr{R} \subset \mathscr{N}$. To prove, as we wish to do, that $\mathfrak{S}(\mathscr{R}) \subset \mathscr{N}$, it will suffice to show that $\mathscr{N}$ is a monotone class. Suppose for example, that N_n is an increasing sequence of sets in $\mathscr{N}$, with union N. Thus

$$(E(N_n)x \big| E(N_n)x) = (E(N_n)x \big| x)$$

for each n; since $E(N_n) \to E(N)$ strongly by Proposition 1, it follows from the continuity of the inner product that N satisfies the relation (9). $\diamondsuit$

The following extension theorem, though less important than the preceding one, still has a place in a systematic exposition:

*THEOREM 8. *If E is a bounded* PO-*measure on a ring $\mathscr{R}$ of subsets of X, and if $\mathscr{A}$ is the class of all sets $A \subset X$ such that $A \cap M \in \mathscr{R}$ for all M in $\mathscr{R}$, then the formula*

$$G(A) = \mathrm{LUB}\,\{E(M) :\ M \subset A,\ M \in \mathscr{R}\} \tag{10}$$

defines a bounded PO-*measure G on the algebra $\mathscr{A}$, and G is an extension of E. If, moreover, E is a spectral measure, then so is G.*

Proof. If $A \in \mathscr{A}$, then the class of sets $\{M \in \mathscr{R} :\ M \subset A\}$ is increasingly directed by inclusion; since E is monotone and bounded, it follows from Proposition 1 that the formula (10) may be used to define a positive operator $G(A)$. For each vector x, write

$$\begin{aligned} \mu_x(M) &= (E(M)x \big| x) \qquad && (M \in \mathscr{R}), \\ \rho_x(M) &= (G(A)x \big| x) \qquad && (A \in \mathscr{A}). \end{aligned}$$

By Proposition 1,

$$\rho_x(A) = \mathrm{LUB}\,\{\mu_x(M) :\ M \subset A,\ M \in \mathscr{R}\}$$

for each A in $\mathscr{A}$, hence ρ_x is a (finite) measure on $\mathscr{A}$ [2, p. 33]. It follows at once from Theorem 1 that G is a PO-measure. Since ρ_x extends μ_x, G extends E.

Finally, if E is projection-valued, then so is G, by Proposition 2. $\diamondsuit$

Incidentally, Theorem 8 will be cited in the proof of one of the starred theorems in Section 6.

*EXERCISE. A spectral measure on a σ-ring $\mathscr{S}$ is uniquely determined by its values on any system of generators for $\mathscr{S}$.

5. Operator valued integrals

All PO-measures to be considered from now on will be defined on some σ-ring. We remind the reader that such a PO-measure is automatically bounded (Theorem 5); although the word "bounded" will be omitted from the statements of definitions and theorems, the boundedness is there implicitly.

Our eventual objective (attained in Section 12) is to associate with each normal operator a suitable spectral measure; by "suitable" we mean that the spectral measure should be uniquely determined by the normal operator, the normal operator should in turn be uniquely determined by the spectral measure, and the spectral measure should facilitate the detailed study of the operator. These criteria for suitability are effectively met by requiring that the normal operator be retrievable from the spectral measure by means of a certain integration process. These vague remarks will be rendered precise in Section 12. In the present section, and in the next two, we develop the necessary theory of integration with respect to (why not be more general if it is painless) a PO-measure.

Fix a set X, a σ-ring $\mathscr{S}$ of subsets of X, and a PO-measure E defined on $\mathscr{S}$. This is the context for the entire section. For each vector x of the underlying Hilbert space, we write μ_x for the measure on $\mathscr{S}$ defined by

$$\mu_x(M) = (E(M)x \big| x)\,. \tag{11}$$

According to Theorem 5, there exists a non negative real number K such that

$$\mu_x(M) \leqslant K\,\|x\|^2 \tag{12}$$

for all M and x.

If f is a complex (i.e., complex-valued) function on X, say $f = g + i\,h$ where g and h are real functions, we say that f is *measurable* (with respect to $\mathscr{S}$) in case g and h are measurable with respect to $\mathscr{S}$ in the sense of [4, p. 77]. If, moreover, μ is a measure on $\mathscr{S}$, and if g and h are integrable with respect to μ, we say that f is μ*-integrable*, and we define

$$\int f\,d\mu = \int g\,d\mu + i \int h\,d\mu\,.$$

A measurable complex function f is μ-integrable if and only if $|f|$ is μ-integrable, and in this case

$$\left| \int f \, d\mu \right| \leqslant \int |f| \, d\mu \, . \tag{13}$$

(Incidentally, it is essential for (13) that μ be positive; the inequality is obviously false for signed measures.)

DEFINITION 4. A measurable complex function f on X will be called *E-integrable* in case it is μ_x-integrable for each vector x. We write $\mathscr{I}(E)$, or briefly $\mathscr{I}$, for the class of all E-integrable functions.

Evidently $\mathscr{I}$ is a complex linear space containing every bounded measurable function, and in particular every simple function. If f is a measurable function, then $f \in \mathscr{I}$ if and only if $|f| \in \mathscr{I}$. If $f \in \mathscr{I}$, and g is a bounded measurable function, then $gf \in \mathscr{I}$.

DEFINITION 5. If f is an E-integrable function, then for each ordered pair of vectors x, y, we define

$$\begin{aligned} \int f \, d\mu_{x,y} = & \frac{1}{4} \Big\{ \int f \, d\mu_{x+y} - \int f \, d\mu_{x-y} \\ & + i \int f \, d\mu_{x+iy} - i \int f \, d\mu_{x-iy} \Big\}. \end{aligned} \tag{14}$$

We are not concerned here with the theory of integration with respect to "complex measures"; we are simply using the symbol

$$\int f \, d\mu_{x,y}$$

as a suggestive abbreviation for the expression on the right side of (14). Our objective in this section is to develop an operator valued correspondence, to be denoted

$$f \mapsto \int f \, dE \, ,$$

for a suitable class of functions f (yet to be delineated), having the formal properties of a "functional representation". Deferring for the moment the question of which functions are eligible, we announce that the operators $\int f \, dE$ will be produced by invoking the usual representation theorem for bounded sesquilinear forms; this is the motivation for the next group of lemmas.

LEMMA 1. *For each pair of vectors* x, y, *the functional*

$$f \mapsto \int f \, d\mu_{x,y} \qquad (f \in \mathscr{I}) \tag{15}$$

is a linear form on the complex linear space $\mathscr{I}$. *Moreover,*

$$\int \chi_M \, d\mu_{x,y} = (E(M)x \big| y) \tag{16}$$

for all M *in* $\mathscr{S}$.*

Proof. The linearity of the functional (15) is evident from the defining formula (14), and the linearity of integration. For each M in $\mathscr{S}$, and each vector z, we have

$$\int \chi_M \, d\mu_z = \mu_z(M) = (E(M)z \big| z),$$

hence (16) follows from (14) and the polarization identity. ♢

It follows trivially from the defining formula (14) that the linear forms described in Lemma 1 have the following "continuity" property:

LEMMA 2. *If* f *and* f_n $(n = 1, 2, 3, \dots)$ *are functions in* $\mathscr{I}$ *such that*

$$\int f_n \, d\mu_z \to \int f \, d\mu_z$$

for each vector z, *then*

$$\int f_n \, d\mu_{x,y} \to \int f \, d\mu_{x,y}$$

for each pair of vectors x, y.

The next two definitions provide some convenient abbreviations:

DEFINITION 6. If $f \in \mathscr{I}$ and x, y are vectors, we write

$$L_{x,y}(f) = \int f \, d\mu_{x,y} \, . \tag{17}$$

Thus $L_{x,y}$ is a linear form on $\mathscr{I}$, for each pair of vectors x, y (Lemma 1).

* χ_M denotes the characteristic function of M.

DEFINITION 7. We say that a linear form L on the complex linear space $\mathscr{I}$ is *quasicontinuous* in case: if f and f_n $(n = 1, 2, 3, \ldots)$ are functions in $\mathscr{I}$ such that $0 \leqslant f_n \uparrow f$, then $L(f_n) \to L(f)$.

We remark that any linear combination of quasicontinuous forms is quasicontinuous. The next lemma is a useful characterization of the linear forms $L_{x,y}$:

LEMMA 3. *Let L be a linear form on $\mathscr{I}$, and let x, y be a pair of vectors. In order that*

$$L(f) = \int f \, d\mu_{x,y} \tag{18}$$

for all f in $\mathscr{I}$, it is necessary and sufficient that (i) *L be quasicontinuous, and* (ii) *$L(\chi_M) = (E(M)x \big| y)$ for all M in $\mathscr{S}$.*

Proof. Suppose (18) holds, in other words, $L = L_{x,y}$. If f and f_n $(n = 1, 2, 3, \ldots)$ are functions in $\mathscr{I}$ such that $0 \leqslant f_n \uparrow f$, then for each vector z,

$$\int f_n \, d\mu_z \uparrow \int f \, d\mu_z$$

by the monotone convergence theorem, hence

$$L_{x,y}(f_n) \to L_{x,y}(f)$$

by Lemma 2. Thus $L_{x,y}$ is quasicontinuous. Moreover, $L_{x,y}(\chi_M) = (E(M)x \big| y)$ by (16).

Suppose, conversely, that L satisfies the conditions (i) and (ii). Thus L is quasicontinuous, and

$$L(f) = L_{x,y}(f) \tag{$*$}$$

whenever $f = \chi_M$. It follows from linearity that (*) holds whenever f is simple. To establish (*) for an arbitrary f in $\mathscr{I}$, we may clearly suppose, by linearity, that $f \geqslant 0$. Choose a sequence of simple functions f_n such that $0 \leqslant f_n \uparrow f$. Since (*) holds for each f_n, and since both L and $L_{x,y}$ are quasicontinuous, we conclude that (*) holds also for f. $\diamondsuit$

The formal expression $\int f \, d\mu_{x,y}$ can be interpreted as an actual integral when $x = y$:

LEMMA 4. *For each vector x,*

$$\int f \, d\mu_{x,x} = \int f \, d\mu_x \tag{19}$$

for all f in $\mathscr{I}$.

Proof. The formula $L(f) = \int f\,d\mu_x$ defines a linear form on $\mathscr{I}$ which is quasicontinuous by the monotone convergence theorem; since, moreover,

$$L(\chi_M) = \int \chi_M\,d\mu_x = \mu_x(M) = (E(M)x|x)$$

for all M in $\mathscr{S}$, we conclude from Lemma 3 that $L = L_{x,x}$. $\diamondsuit$

The expression $\int f\,d\mu_{x,y}$ was shown in Lemma 1 to be linear in f, for each fixed pair of vectors x, y; in the next lemma we show that it is sesquilinear in x and y, for each fixed f:

LEMMA 5. *For each f in $\mathscr{I}$, the functional*

$$(x,y) \mapsto \int f\,d\mu_{x,y}$$

is sesquilinear, that is:

$$\int f\,d\mu_{x+x',y} = \int f\,d\mu_{x,y} + \int f\,d\mu_{x',y} \tag{20}$$

$$\int f\,d\mu_{cx,y} = c\int f\,d\mu_{x,y} \tag{21}$$

$$\int f\,d\mu_{x,y+y'} = \int f\,d\mu_{x,y} + \int f\,d\mu_{x,y'} \tag{22}$$

$$\int f\,d\mu_{x,cy} = \overline{c}\int f\,d\mu_{x,y}\,. \tag{23}$$

Moreover,

$$\int f\,d\mu_{y,x} = \overline{\int \overline{f}\,d\mu_{x,y}}\,. \tag{24}$$

Proof. Fix vectors x, x', y; proving (20) amounts to showing that

$$L_{x+x',y} = L_{x,y} + L_{x',y}\,.$$

Defining $L = L_{x,y}+L_{x',y}$, we see from Lemma 3 that L is a quasicontinuous lilnear form on $\mathscr{I}$; since

$$\begin{aligned} L(\chi_M) &= L_{x,y}(\chi_M) + L_{x',y}(\chi_M) \\ &= (E(M)x|y) + (E(M)x'|y) \\ &= (E(M)(x+x')|y) \\ &= L_{x+x',y}(\chi_M) \end{aligned}$$

for all M in $\mathscr{S}$, we conclude from Lemma 3 that $L = L_{x+x',y}$. The relations (21)–(23) are proved similarly.

To prove (24), consider the functional

$$L(f) = \overline{L_{x,y}(\overline{f})} \qquad (f \in \mathscr{I});$$

clearly L is a quasicontinuous linear form on $\mathscr{I}$, and

$$L(\chi_M) = \overline{(E(M)x\big|y)} = (E(M)y\big|x)$$

for all M in $\mathscr{S}$, hence $L = L_{y,x}$ by Lemma 3. $\diamondsuit$

To proceed further without running into unbounded linear transformations, it is necessary to narrow the class of functions. The following class is adequate for our purposes, and has the advantage of being independent of the particular PO-measure E:

Definition 8. — We denote by $\mathscr{M}$ the class of all bounded measurable complex functions f on X. For such a function, we write

$$\|f\|_\infty = \mathrm{LUB}\,\{|f(\lambda)| : \ \lambda \in X\}.$$

Evidently $\mathscr{M}$ is a complex algebra, and $f \in \mathscr{M}$ implies $\overline{f} \in \mathscr{M}$. Since each of the measures μ_x is finite, every f in $\mathscr{M}$ is E-integrable, that is, $\mathscr{M} \subset \mathscr{I}$. (It is a fact, but one we shall not need, that $\mathscr{M}$ is a Banach algebra with respect to the norm of Definition 8; moreover, $\mathscr{I}$ is an $\mathscr{M}$-module in the sense that the relations $f \in \mathscr{M}$ and $g \in \mathscr{I}$ imply $fg \in \mathscr{I}$.) It is the functions in $\mathscr{M}$ for which we shall obtain an operatorial representation:

Theorem 9. *If $f \in \mathscr{M}$, there exists one and only one operator T such that*

$$(Tx\big|x) = \int f\,d\mu_x$$

for all vectors x. Moreover,

$$(Tx\big|y) = \int f\,d\mu_{x,y}$$

for all vectors x and y.

Proof. The functional $(x,y) \to \int f\,d\mu_{x,y}$ is a sesquilinear form by Lemma 5, and this form is bounded since

$$\Big|\int f\,d\mu_{x,x}\Big| = \Big|\int f\,d\mu_x\Big| \leqslant \int |f|\,d\mu_x \leqslant \|f\|_\infty K\,\|x\|^2$$

(cf. Lemma 4, and the formulas (12), (13)). $\diamondsuit$

DEFINITION 9. With notation as in Theorem 9 (in particular $f \in \mathscr{M}$), we write $T = \int f\,dE$. Thus $\int f\,dE$ is the unique operator such that

$$((\textstyle\int f\,dE)x|x) = \int f\,d\mu_x \tag{25}$$

for all vectors x; one has, moreover,

$$((\textstyle\int f\,dE)x|y) = \int f\,d\mu_{x,y} \tag{26}$$

for all vectors x, y.

We observe in the next theorem that the correspondence $f \mapsto \int f\,dE$ has many of the properties of a "functional representation." The multiplicative property $\int fg = \bigl(\int f\bigr)\bigl(\int g\bigr)$ is conspicuously missing; indeed, it holds only when E is a spectral measure, as we shall see later in the section.

THEOREM 10. *The correspondence*

$$f \mapsto \int f\,dE \qquad (f \in \mathscr{M})$$

has the following properties:

$$\int (f+g)\,dE = \int f\,dE + \int g\,dE \tag{27}$$

$$\int (cf)\,dE = c\int f\,dE \tag{28}$$

$$\int \overline{f}\,dE = \Bigl(\int f\,dE\Bigr)^* \tag{29}$$

$$\int \chi_M\,dE = E(M)\,. \tag{30}$$

If f is real-valued, then $\int f\,dE$ is Hermitian, and

$$\Bigl\|\int f\,dE\Bigr\| \leqslant K\,\|f\|_\infty\,;$$

if, moreover, $f \geqslant 0$, then $\int f\,dE \geqslant 0$. For any f in $\mathscr{M}$,

$$\Bigl\|\int f\,dE\Bigr\| \leqslant 2K\,\|f\|_\infty\,. \tag{31}$$

Proof. Given $f, g \in \mathscr{M}$, define $A = \int f\,dE$, $B = \int g\,dE$. Citing Definition 9 and Lemma 1, we have

$$\begin{aligned}((A+B)x\big|y) &= (Ax\big|y) + (Bx\big|y)\\ &= \int f\,d\mu_{x,y} + \int g\,d\mu_{x,y}\\ &= \int (f+g)\,d\mu_{x,y}\\ &= \Big(\big(\textstyle\int (f+g)\,dE\big)x\Big|y\Big),\end{aligned}$$

thus $\int (f+g)\,dE = A + B$. The proof of (28) is similar. Citing (24), we have

$$\begin{aligned}(A^*x\big|y) = \overline{(Ay\big|x)} &= \overline{\int f\,d\mu_{y,x}}\\ &= \int \overline{f}\,d\mu_{x,y}\\ &= \Big(\big(\textstyle\int \overline{f}\,dE\big)x\Big|y\Big),\end{aligned}$$

thus (29) is verified. In particular, if $\overline{f} = f$, then $A^* = A$; since

$$(Ax\big|x) = \int f\,d\mu_{x,x} = \int f\,d\mu_x$$

by Lemma 4, we have

$$|(Ax\big|x)| \leqslant \int |f|\,d\mu_x \leqslant \|f\|_\infty K\,\|x\|^2\,,$$

hence $\|A\| \leqslant K\,\|f\|_\infty$ [5, p. 41]. If, moreover, $f \geqslant 0$, then

$$(Ax\big|x) = \int f\,d\mu_x \geqslant 0$$

for all vectors x, thus $A \geqslant 0$.

If f is an arbitrary function in $\mathscr{M}$, say $f = g + ih$ with g and h real-valued, then

$$\begin{aligned}\Big\|\int f\,dE\Big\| &= \Big\|\int g\,dE + i\int h\,dE\Big\|\\ &\leqslant \Big\|\int g\,dE\Big\| + \Big\|\int h\,dE\Big\|\\ &\leqslant K\,\|g\|_\infty + K\,\|h\|_\infty \leqslant 2K\,\|f\|_\infty\,.\end{aligned}$$

Finally, we see from (16) that

$$\Big((\int \chi_M \, dE)x\big|y\Big) = \int \chi_M \, d\mu_{x,y} = (E(M)x\big|y)$$

for all x, y, thus $\int \chi_M \, dE = E(M)$. $\diamondsuit$

The correspondence $f \mapsto \int f \, dE$ also has some very valuable "continuity" properties:

THEOREM 11. *Let f and f_n $(n = 1, 2, 3, \ldots)$ be functions in $\mathscr{M}$.*

(i) *If $f_n \to f$ uniformly on X, then $\int f_n \, dE \to \int f \, dE$ uniformly, that is,*

$$\Big\| \int f_n \, dE - \int f \, dE \Big\| \to 0 \, .$$

(ii) *If $f_n \to f$ pointwise on X, and $\|f_n\|_\infty$ is bounded, then $\int f_n \, dE \to \int f \, dE$ weakly, that is,*

$$\big((\int f_n \, dE)x\big|y\big) \to \big((\int f \, dE)x\big|y\big)$$

for each pair of vectors x, y.

(iii) *If $0 \leqslant f_n \uparrow f$, then $\int f_n \, dE \uparrow \int f \, dE$ in the sense of Proposition* 1, *and consequently $\int f_n \, dE \to \int f \, dE$ strongly, that is,*

$$\big\|(\int f_n \, dE)x - (\int f \, dE)x\big\| \to 0$$

for each vector x.

Proof. Let $A_n = \int f_n \, dE$, $A = \int f \, dE$. Since

$$\|A_n - A\| \leqslant 2K \, \|f_n - f\|_\infty$$

by Theorem 10, the assertion concerning uniform convergence is clear.

Suppose the hypotheses of (ii) are fulfilled. For each vector x,

$$\int f_n \, d\mu_x \to \int f \, d\mu_x$$

by the bounded convergence theorem, hence

$$\int f_n \, d\mu_{x,y} \to \int f \, d\mu_{x,y}$$

for each pair of vectors x, y by Lemma 2; this means, in view of (26), that

$$(A_n x\big|y) \to (Ax\big|y)$$

for all x, y.

Suppose, finally, that the hypotheses of (iii) are fulfilled. It follows at once from Lemma 3 that $(A_n x|x) \uparrow (Ax|x)$ for each vector x. Then $A = \mathrm{LUB}\, A_n$ by the remarks following Proposition 1, and consequently $A_n \to A$ strongly. ♢

Definition 10. We say that an operator T *commutes* with the PO-measure E, and we write $T \leftrightarrow E$, in case

$$T \leftrightarrow E(M)$$

for each M in $\mathscr{S}$.

Theorem 12. *For an operator T, the following conditions are equivalent*:

$$T \leftrightarrow E \tag{32a}$$

$$\int \chi_M \, d\mu_{Tx,y} = \int \chi_M \, d\mu_{x,T^*\mathrm{y}} \tag{32b}$$

$$\int f \, d\mu_{Tx,y} = \int f \, d\mu_{x,T^*\mathrm{y}}\,, \tag{32c}$$

where it is understood that (32b) *is to hold for all vectors x, y, and all M in $\mathscr{S}$, whereas* (32c) *is to hold for all vectors x, y, and all E-integrable functions f.*

Proof. If $L_{x,y}$ is the linear form on $\mathscr{I}$ described in Definition 6, it is clear from Lemma 3 that

$$L_{Tx,y} = L_{x,T^*y}$$

if and only if

$$L_{Tx,y}(\chi_M) = L_{x,T^*\mathrm{y}}(\chi_M)$$

for all M. Thus (32b) and (32c) are equivalent. Since

$$(TE(M)x|y) = (E(M)x|T^*y) = \int \chi_M \, d\mu_{x,T^*\mathrm{y}}\,,$$
$$(E(M)Tx|y) = \int \chi_M \, d\mu_{Tx,y}\,,$$

clearly (32a) and (32b) are equivalent. ♢

Definition 11. If $\mathscr{E}$ is any set of operators, we write $\mathscr{E}'$ for the *commutant* of $\mathscr{E}$; this is the set of all operators T such that $T \leftrightarrow S$ for every S in $\mathscr{E}$. If in particular $\mathscr{E} = \{E(M) : M \in \mathscr{S}\}$, we write $\{E\}'$ instead of $\mathscr{E}'$.

Thus $T \in \{E\}'$ if and only if $T \leftrightarrow E$ in the sense of Definition 10. The following general properties of commutants are elementary and well known:

(i) $\mathscr{E} \subset \mathscr{E}''$, (ii) $\mathscr{E}_1 \subset \mathscr{E}_2$ implies $\mathscr{E}_1' \supset \mathscr{E}_2'$, (iii) $\mathscr{E}''' = \mathscr{E}'$, (iv) $\mathscr{E}'$ is a subalgebra of the algebra of all bounded operators, (v) $\mathscr{E} \subset \mathscr{E}'$ if and only if $\mathscr{E}$ is a commutative set of operators. If $\mathscr{E}$ is a self-adjoint set of operators (that is, if $S \in \mathscr{E}$ implies $S^* \in \mathscr{E}$), then $\mathscr{E}'$ is a self-adjoint algebra of operators.

In particular, since $\{E(M) : M \in \mathscr{S}\}$ is obviously a self-adjoint set of operators, it follows that $\{E\}'$ is a self-adjoint algebra of operators, hence so is $\{E\}''$.

THEOREM 13. *For all f in $\mathscr{M}$, we have*

$$\int f\,dE \in \{E\}''. \tag{33}$$

Proof. Let $f \in \mathscr{M}$, $A = \int f\,dE$. Assuming $T \leftrightarrow E(M)$ for all M in $\mathscr{S}$, it is to be shown that $A \leftrightarrow T$. Indeed, by Theorem 12,

$$\begin{aligned}(ATx|y) &= \int f\,d\mu_{Tx,y} = \int f\,d\mu_{x,T^*y}\\ &= (Ax|T^*y) = (TAx|y)\end{aligned}$$

for all vectors x, y. ◊

The foregoing results can be sharpened considerably when the operators $E(M)$ commute among themselves, and still sharper results are obtained when E is a spectral measure, as we shall see in the next three theorems.

DEFINITION 12. We say that the PO-measure E is *commutative* in case $E(M_1) \leftrightarrow E(M_2)$ for all M_1 and M_2 in $\mathscr{S}$, in other words, $\{E(M) : M \in \mathscr{S}\}$ is a commutative set of operators.

For example, every spectral measure is commutative (Theorem 3).

THEOREM 14. *If the* PO-*measure E is commutative* (*in particular, if E is a spectral measure*), *then the self-adjoint algebra of operators $\{E\}''$ is commutative. It follows that if $f, g \in \mathscr{M}$, then*

$$\int f\,dE \leftrightarrow \int g\,dE \tag{34}$$

$$\int f\,dE \ \textit{ is normal.} \tag{35}$$

Proof. The fact that $\{E\}''$ is commutative is an elementary calculation with commutants: writing $\mathscr{E} = \{E(M) : M \in \mathscr{S}\}$, we have $\mathscr{E} \subset \mathscr{E}'$, hence $\mathscr{E}'' \subset \mathscr{E}' = \mathscr{E}''' = (\mathscr{E}'')'$. Then (34) follows at once from (33), and (35) results from (34) on taking $g = \overline{f}$ (cf. (29)). ♢

(We remark that for a commutative PO-measure, the relation $\left\| \int f\, dE \right\| \leqslant K \|f\|_\infty$ holds for any f in $\mathscr{M}$; indeed, since $\int f\, dE$ is normal, the proof given in Theorem 10 for Hermitian A may be copied verbatim (cf. [3]). In particular, when E is a spectral measure, we may take $K = 1$, and so $\left\| \int f\, dE \right\| \leqslant \|f\|_\infty$ for every f in $\mathscr{M}$; we shall arrive at this inequality via another route in Theorem 16.)

We now specialize to the case that E is a spectral measure. The purpose of the next two theorems is to exploit the multiplicative property

$$E(M \cap N) = E(M)E(N) \tag{36}$$

known to hold from Theorem 3. These key results are very near the surface:

THEOREM 15. *If E is a spectral measure, then*

$$\int fg\, dE = \Big(\int f\, dE \Big) \Big(\int g\, dE \Big) \tag{37}$$

for all f, g in $\mathscr{M}$.

Proof. If $f = \chi_M$ and $g = \chi_N$, then (37) is precisely (36) (cf. Theorem 10), and the case that f and g are simple follows from the fact that both sides of (37) are bilinear in f, g. If f and g are arbitrary functions in $\mathscr{M}$, choose sequences of simple functions f_n and g_n such that $\|f_n - f\|_\infty \to 0$ and $\|g_n - g\|_\infty \to 0$. Then also $\|f_n g_n - fg\|_\infty \to 0$. Writing $A_n = \int f_n\, dE$, $A = \int f\, dE$, $B_n = \int g_n dE$, $B = \int g\, dE$, and $C_n = \int f_n g_n\, dE$, $C = \int fg\, dE$, we have

$$\|A_n - A\| \to 0, \quad \|B_n - B\| \to 0, \quad \|C_n - C\| \to 0$$

by Theorem 11. Since $C_n = A_n B_n$ by the preceding case, and since

$$\begin{aligned} \|C - AB\| &\leqslant \|C - A_n B_n\| + \|A_n(B_n - B)\| + \|(A_n - A)B\| \\ &\leqslant \|C - C_n\| + \|A_n\|\, \|B_n - B\| + \|A_n - A\|\, \|B\|, \end{aligned}$$

it follows that $\|C - AB\| = 0$. ♢

THEOREM 16. *If E is a spectral measure, then*

$$\Big(\int f dE \Big)^* \Big(\int f\, dE \Big) = \int |f|^2\, dE, \tag{38}$$

$$\Big\| \int f\, dE \Big\| \leqslant \|f\|_\infty \tag{39}$$

for all f in $\mathscr{M}$.

Proof. Let $A = \int f\,dE$. By (29) and (37), we have

$$A^*A = \Big(\int \overline{f}\,dE\Big)\Big(\int f\,dE\Big) = \int \overline{f}\,f\,dE = \int |f|^2\,dE\,.$$

Since

$$\Big\|\int |f|^2\,dE\Big\| \leqslant \|\,|f|^2\,\|_\infty = (\|f\|_\infty)^2$$

by Theorem 10 (recall that $\|E(M)\| \leqslant 1$ for all M), we have

$$\|A\|^2 = \|A^*A\| = \Big\|\int |f|^2\,dE\Big\| \leqslant (\|f\|_\infty)^2\,. \ \diamondsuit$$

We remark that for a spectral measure, part (ii) of Theorem 11 can be strengthened as follows: if $f_n \to f$ pointwise on X, and $\|f_n\|_\infty$ is bounded, then $\int f_n\,dE \to \int f\,dE$ *strongly.* For, writing $A_n = \int f_n\,dE$ and $A = \int f\,dE$, we have, by Theorem 16,

$$\|A_n x - Ax\|^2 = \big((A_n - A)^*(A_n - A)x \big| x\big) = \int |f_n - f|^2\,d\mu_x$$

for each vector x, and the integral tends to zero by the bounded convergence theorem.

*Exercise. Let T be any operator, and define $F(M) = T^*E(M)T$ for all measurable sets M. Then F is also a PO-measure, and

$$\int f\,dF = T^*\Big(\int f\,dE\Big)T$$

for every bounded measurable function f.

6. PO-measures on locally compact spaces

Throughout this section, X denotes a locally compact Hausdorff space. We write $\mathscr{B}_0(X)$, $\mathscr{B}(X)$, and $\mathscr{B}_{\mathrm{w}}(X)$ for the class of all Baire sets, all Borel sets, and all weakly Borel sets, respectively, in X (cf. [4, Sec. 51] and [2, p. 181]). Thus $\mathscr{B}_0$, $\mathscr{B}$, and $\mathscr{B}_{\mathrm{w}}$ are the σ-rings generated by the class of all compact G_δ's, compact sets, and closed sets, respectively. Of course $\mathscr{B}_{\mathrm{w}}$ is a σ-algebra, and is also the σ-ring generated by the class of all open sets.

Ultimately we shall be concerned only with the case that $X = \mathbf{R}$ or $\mathbf{C}$, in which case $\mathscr{B}_0 = \mathscr{B} = \mathscr{B}_{\mathrm{w}}$; indeed, these three σ-rings coincide whenever X is σ-compact and metrizable. Some of our results are most conveniently stated for spaces of this type; for instance, if X_1 and X_2 are σ-compact metrizable locally compact spaces, then so is $X_1 \times X_2$. We remark that there exist locally compact spaces X such that $\mathscr{B}_0 = \mathscr{B} = \mathscr{B}_{\mathrm{w}}$ for X but not for $X \times X$ (cf. [2, p. 183]). Despite the eventual specialization to $X = \mathbf{R}$ or $\mathbf{C}$, it is instructive to keep our arguments as general as possible, as long as possible; consequently we shall generally assume that X is an arbitrary locally compact space, adding special hypotheses from theorem to theorem when they are explicitly wanted. Incidentally, when $\mathscr{B}_0 = \mathscr{B} = \mathscr{B}_{\mathrm{w}}$ (for instance when $X = \mathbf{R}$ or $\mathbf{C}$), it is customary to use the "Borel" terminology.

Definition 13. A PO-measure on $\mathscr{B}_0(X)$ [resp. $\mathscr{B}(X)$, $\mathscr{B}_{\mathrm{w}}(X)$] is called a *Baire* [resp. *Borel*, *weakly Borel*] PO-measure on X.

We write $\mathscr{L}(X)$, or briefly $\mathscr{L}$, for the class of all continuous real functions on X with compact support. Every f in $\mathscr{L}$ is a Baire function [4, p. 220] and so is *a fortiori* a Borel function and a weakly Borel function; since f is moreover bounded, it is eligible to be integrated with respect to any of the PO-measures contemplated in Definition 13.

A Baire PO-measure is uniquely determined by the integral which it assigns fo functions in $\mathscr{L}$:

Theorem 17. *If F_1 and F_2 are Baire* PO-*measures on X such that*

$$\int f\,dF_1 = \int f\,dF_2$$

for all f in $\mathscr{L}$, then $F_1 = F_2$.

Proof. For each vector x, write

$$\begin{aligned}\nu_x^1(M) &= (F_1(M)x|x)\,,\\ \nu_x^2(M) &= (F_2(M)x|x)\,,\end{aligned}$$

for all Baire sets M. Thus ν_x^1 and ν_x^2 are Baire measures (Theorem 1), and citing (25) we have

$$\int f\,d\nu_x^1 = \Big((\textstyle\int f\,dF_1)x\big|x\Big) = \Big((\textstyle\int f\,dF_2)x\big|x\Big) = \displaystyle\int f\,d\nu_x^2$$

for all f in $\mathscr{L}$, and so $\nu_x^1 = \nu_x^2$ by the analogous result for numerical Baire measures [4, p. 248]. $\diamondsuit$

Every Baire PO-measure is "regular" in the following sense:

THEOREM 18. *If F is any Baire* PO-*measure on X, then*

$$F(M) = \mathrm{LUB}\,\{F(C) :\ C \subset M\,,\ C \text{ compact } G_\delta\}\,, \tag{40a}$$

$$F(M) = \mathrm{GLB}\,\{F(U) :\ U \supset M\,,\ U \text{ open Baire set}\}\,, \tag{40b}$$

for every Baire set M.

Proof. Fix a vector x, and consider the (finite) Baire measure ν_x,

$$\nu_x(M) = (F(M)x\big|x) \qquad (M \in \mathscr{B}_0)\,.$$

Fix a Baire set M. Since every Baire measure is regular (cf. [4, p. 228] and [2, p. 194]), we have

$$\nu_x(M) = \mathrm{LUB}\,\{\nu_x(C) :\ C \subset M\,,\ C \text{ compact } G_\delta\}\,.$$

On the other hand, since the class of all compact G_δ's $C \subset M$ is increasingly directed (and is bounded above by M), so is the corresponding family of operators $F(C)$ (and $F(C) \leqslant F(M)$ for all C); citing Proposition 1, we have

$$\begin{aligned}\Big(\big(\mathop{\mathrm{LUB}}_{C\subset M} F(C)\big)x\Big|x\Big) &= \mathop{\mathrm{LUB}}_{C\subset M}(F(C)x\big|x) = \mathop{\mathrm{LUB}}_{C\subset M} \nu_x(C) \\ &= \nu_x(M) = (F(M)x\big|x)\,,\end{aligned}$$

and (40a) results from the arbitrariness of x. The proof of (40b) is similar. ◇

The analogue of the Riesz representation theorem, in the context of PO-measures, is as follows (for the spectral theory, we need this result only for the case that $X = \mathbf{R}$):

THEOREM 19. *Suppose that for each f in $\mathscr{L}$ there is given a Hermitian operator T_f, and that the correspondence*

$$f \mapsto T_f \tag{41}$$

has the following properties:

$$T_{f+g} = T_f + T_g\,, \tag{42}$$

$$T_{af} = aT_f\,, \tag{43}$$

$$f \geqslant 0 \text{ implies } T_f \geqslant 0\,. \tag{44}$$

Assume, moreover, that the correspondence (41) *is bounded, that is, assume there exists a non negative real number K such that*

$$\|T_f\| \leqslant K\,\|f\|_\infty \tag{45}$$

for all f *in* $\mathscr{L}$. *Then there exists one and only one Baire* PO-*measure* F *on* X *such that*

$$\int f\,dF = T_f$$

for all f *in* $\mathscr{L}$. *If, moreover, the correspondence* (41) *is multiplicative, that is, if*

$$T_{fg} = T_f T_g \tag{46}$$

for all f *and* g, *then* F *is a spectral measure.*

Proof. Uniqueness is covered by Theorem 17. Let us show that such an F exists. For each vector x, define

$$L_x(f) = (T_f x \big| x) \qquad (f \in \mathscr{L})\,; \tag{47}$$

it follows from (42)–(44) that L_x is a positive linear form on $\mathscr{L}$, hence there exists a unique Baire measure ν_x on X such that

$$(T_f x\big|x) = \int f\,d\nu_x \tag{48}$$

for all f in $\mathscr{L}$ [4, Sec. 56].

It follows from (45) that the Baire measure ν_x is finite. Indeed, if C is any compact G_δ, choose a function f in $\mathscr{L}$ such that $0 \leqslant f \leqslant 1$ and $f \geqslant \chi_C$ [4, p. 216]. Since

$$\int f\,d\nu_x = (T_f x\big|x) \leqslant K\,\|f\|_\infty \|x\|^2 \leqslant K\,\|x\|^2\,,$$

and since

$$\nu_x(C) = \int \chi_C\,d\nu_x \leqslant \int f\,d\nu_x$$

we conclude that

$$\nu_x(C) \leqslant K\,\|x\|^2\,;$$

it then follows from regularity that

$$\nu_x(M) \leqslant K\,\|x\|^2 \tag{49}$$

for every Baire set M.

We propose to construct a Baire PO-measure by applying Theorem 2 to the family of finite Baire measures ν_x. Anyway, the condition (6) of

Theorem 2 is covered by (49); indeed, we may take $k_M = K$ for every Baire set M. To show, for example, that

$$(*) \qquad [\nu_{x+y}(M)]^{1/2} \leqslant [\nu_x(M)]^{1/2} + [\nu_y(M)]^{1/2}$$

for all Baire sets M, let us verify it first for a compact G_δ C. Choose a sequence f_n in $\mathscr{L}$ such that $f_n \downarrow \chi_C$ [4, p. 240]. Then by the monotone convergence theorem,

$$\int f_n \, d\nu_{x+y} \downarrow \nu_{x+y}(C)\,,$$
$$\int f_n \, d\nu_x \downarrow \nu_x(C)\,,$$
$$\int f_n \, d\nu_y \downarrow \nu_y(C)\,.$$

Now, for any positive operator A, we have

$$(A(x+y)\big|x+y)^{1/2} \leqslant (Ax\big|x)^{1/2} + (Ay\big|y)^{1/2}$$

by the generalized Schwarz inequality [8, Sec. 104]; applying this result to the positive operator T_{f_n}, we have

$$(T_{f_n}(x+y)\big|x+y)^{1/2} \leqslant (T_{f_n}x\big|x)^{1/2} + (T_{f_n}y\big|y)^{1/2}\,,$$

that is,

$$\Big(\int f_n \, d\nu_{x+y}\Big)^{1/2} \leqslant \Big(\int f_n \, d\nu_x\Big)^{1/2} + \Big(\int f_n \, d\nu_y\Big)^{1/2}\,,$$

and passage to the limit yields the desired inequality

$$[\nu_{x+y}(C)]^{1/2} \leqslant [\nu_x(C)]^{1/2} + [\nu_y(C)]^{1/2}.$$

The validity of (*) for an arbitrary Baire set M then follows at once from regularity.

The conditions (4) and (5) of Theorem 2 are proved similarly. For instance, one verifies (5) for a compact G_δ via a sequence of functions in $\mathscr{L}$. Then for an arbitrary Baire set M, one constructs, for each term of (5), a suitable increasing sequence of compact G_δ's contained in M; the term-by-term union of these four sequences yields (5) for M, on passing to the limit.

According to Theorem 2, there exists a Baire PO-measure F such that

$$\nu_x(M) = (F(M)x\big|x)$$

for all x and M. Incidentally, $\|F(M)\| \leqslant K$ by (49). Since

$$\left((\int f\, dF)x \middle| x \right) = \int f\, d\nu_x = (T_f x | x)$$

for every f in $\mathscr{L}$ (cf. (25) and (48)), we have proved the existence of the required F.

Assume, finally, that (46) holds. If C is a compact G_δ, let us show that $F(C)$ is a projection. Choose a sequence f_n in $\mathscr{L}$ such that $f_n \downarrow \chi_C$. Defining $g_n = f_n^2$, we have also $g_n \downarrow \chi_C$. By part (iii) of Theorem 11, we have

$$\int f_n\, dF \to \int \chi_C\, dF \qquad \text{strongly,}$$

$$\int g_n\, dF \to \int \chi_C\, dF \qquad \text{strongly,}$$

that is,

$$T_{f_n} \to F(C) \qquad \text{strongly,}$$

$$T_{g_n} \to F(C) \qquad \text{strongly.}$$

Since $T_{g_n} = (T_{f_n})^2$ by (46), we have, for each vector x,

$$(F(C)x|x) = \lim(T_{g_n}x|x) = \lim(T_{f_n}x|T_{f_n}x) = (F(C)x|F(C)x)$$

by the continuity of the inner product. Thus $F(C) = F(C)^2$. It then follows from regularity (cf. (40a)) and Proposition 2 that $F(M)$ is a projection, for every Baire set M. ◇

The appropriate concept of regularity for Borel PO-measures is as follows:

*DEFINITION 14. A Borel PO-measure E on X is said to be *regular* in case

$$E(N) = \text{LUB}\,\{E(C) : \ C \subset N\,,\ C \ \text{compact}\,\}$$

for each Borel set N.

Observe that for each Borel set N, the class of all compact sets $C \subset N$ is increasingly directed, hence the corresponding family of operators $E(C)$ is also increasingly directed, and $E(C) \leqslant E(N)$ for all C; it follows from Proposition 1 that the LUB indicated in (50) exists, whether or not E is regular. A convenient criterion for regularity is as follows:

*THEOREM 20. *A Borel* PO-*measure* E *on* X *is regular if and only if, for each vector* x, *the (finite) Borel measure*

$$\mu_x(N) = (E(N)x|x) \qquad (N \in \mathscr{B})$$

is regular.

Proof. In view of Proposition 1, the validity of (50) is equivalent to the validity of the relation

$$\mu_x(N) = \mathrm{LUB}\,\{\mu_x(C):\ C \subset N\,,\ \ C \text{ compact}\,\}$$

for each vector x. See [4, Sec. 52] for the concept of regularity for (numerical) Borel measures. ♢

The theory of extension of Baire measures carries over easily to Baire PO-measures:

*THEOREM 21. *If F is a Baire* PO-*measure on X, there exists one and only one regular Borel* PO-*measure E on X which extends F. If, moreover, F is a spectral measure, then so is E.*

Proof. We imitate the numerical case, as presented in [4]. For each compact set C, define

$$\lambda(C) = \mathrm{GLB}\,\{F(V):\ V \supset C\,,\ V \text{ open Baire set}\,\}\,, \tag{51}$$

and for each open Borel set U, define

$$\lambda_*(U) = \mathrm{LUB}\,\{\lambda(C):\ C \subset U\,,\ \ C \text{ compact}\,\}\,, \tag{52}$$

and, finally, for each Borel set N define

$$E(N) = \mathrm{GLB}\,\{\lambda_*(U):\ U \supset N\,,\ \ U \text{ open Borel set}\,\}\,. \tag{53}$$

Proposition 1 assures the existence of all of these operators (one checks that the set function λ is monotone, before defining λ_*, and then that λ_* is monotone, before defining E). It also follows from Proposition 1 that

$$(E(N)x\big|x) = \mathop{\mathrm{GLB}}_{U\supset N}\ \mathop{\mathrm{LUB}}_{C\subset U}\ \mathop{\mathrm{GLB}}_{V\supset C}\,(F(V)x\big|x) \tag{54}$$

for each vector x, and any Borel set N.

For each vector x, let ν_x be the Baire measure

$$\nu_x(M) = (F(M)x\big|x) \qquad (M \in \mathscr{B}_0)\,,$$

and let μ_x be the unique regular Borel measure which extends ν_x [4, p. 239]. Comparing (54) with the construction of μ_x in [4], we see that

$$(E(N)x\big|x) = \mu_x(N)\,,$$

thus E is a (Borel) PO-measure by Theorem 1, and E is regular by Theorem 20. Since μ_x extends ν_x, it follows that E extends F.

If E' is any regular Borel PO-measure extending F, and if, for each vector x, we define

$$\mu'_x(N) = (E'(N)x \big| x) \qquad (N \in \mathscr{B})$$

then μ'_x is a regular Borel measure (Theorem 20) such that

$$\mu'_x(M) = (F(M)x \big| x) = \mu_x(M)$$

for all Baire sets M, and so

$$\mu'_x(N) = \mu_x(N)$$

for all Borel sets N [4, p. 229]. Thus

$$(E'(N)x \big| x) = (E(N)x \big| x)\,,$$

and we conclude that $E' = E$.

Finally, if F is projection-valued, then it is clear from the formulas (51)–(53), and Proposition 2, that E is also projection-valued. ♢

We remark that Theorems 19 and 21 give the connection between operator valued positive linear mappings and regular Borel PO-measures.

Definition 15. A weakly Borel PO-measure E on X is said to be *regular* in case

$$E(A) = \mathrm{LUB}\,\{E(C):\ C \subset A\,,\ C \text{ compact}\,\} \tag{55}$$

for each weakly Borel set A.

Every Borel PO-measure has a canonical weakly Borel extension:

*Theorem 22. *If E is a Borel* PO-*measure on X, then the operator valued set function E_{w}, defined for each weakly Borel set A by the formula*

$$E_{\mathrm{w}}(A) = \mathrm{LUB}\,\{E(N):\ N \subset A\,,\ N\ \textit{Borel}\,\}\,, \tag{56}$$

is a weakly Borel PO-*measure which extends E. If E is regular, then so is E_{w}. If E is a spectral measure, then so is E_{w}.*

Proof. Write $\mathscr{A}$ for the class of all subsets A of X such that $A \cap N$ is a Borel set for every Borel set N, and let G be the PO-measure on the σ-algebra $\mathscr{A}$ defined as in Theorem 8. Thus

$$G(A) = \mathrm{LUB}\,\{E(N):\ N \subset A\,,\ N \text{ Borel}\,\}$$

for each A in $\mathscr{A}$ (recall that E is bounded, by Theorem 5). Since $\mathscr{B}_{\mathrm{w}} \subset \mathscr{A}$ [2, p. 181], and since G extends E, it follows that the restriction of G to $\mathscr{B}_{\mathrm{w}}$ is a PO-measure extending E. In other words, the formula (56) defines a weakly Borel extension of E. If E is projection-valued, then so is G (Theorem 8), hence also E_{w}.

Finally, suppose E is regular. To show that E_{w} is regular, it is obviously sufficient to show that

$$G(A) = \mathrm{LUB}\,\{E(C)\ C \subset A\,,\ \ C\ \text{ compact}\,\}$$

for each A in $\mathscr{A}$. Fix $A \in \mathscr{A}$, and observe that the class of compact subsets of A is precisely the class of all compect sets C such that $C \subset N \subset A$ for some Borel set N (recall that compact sets are themselves Borel sets). Then, by the associativity of LUB's, we have

$$\begin{aligned} G(A) &= \mathrm{LUB}\,\{E(N):\ N \subset A\,,\ \ N\ \text{ Borel}\,\} \\ &= \mathrm{LUB}\,\big\{\mathrm{LUB}\,\{E(C):\ C \subset N\,,\ C\ \text{ compact}\}:\ N \subset A\,,\ \ N\ \text{ Borel}\big\} \\ &= \mathrm{LUB}\,\{E(C):\ C \subset A\,,\ C\ \text{ compact}\,\}\,; \end{aligned}$$

the indicated association is justified by the fact that each LUB, being calculated for an increasingly directed family of operators, exists (cf. Proposition 1). $\diamondsuit$

The gist of Theorems 21 and 22 is that we may work with either Baire, or regular Borel, or regular weakly Borel PO-measures, at our pleasure. For simplicity in discussing spectrum, we choose to work with weakly Borel PO-measures:

Definition 16. If E is a weakly Borel PO-measure on X, we define $\bigvee(E)$ to be the union of all the open sets on which E vanishes, that is,

$$\bigvee(E) = \bigcup\{U:\ U\ \text{ open}\,,\ E(U) = 0\,\}\,. \tag{57}$$

We call $\bigvee(E)$ the *co-spectrum* of E.

Thus $\bigvee(E)$ is itself an open set, and is therefore a weakly Borel set.

Definition 17. With notation as in Definition 16, we define the *spectrum* of E to be the complement of $\bigvee(E)$, and denote it by $\bigwedge(E)$:

$$\bigwedge(E) = \complement \bigvee(E) = X - \bigvee(E)\,. \tag{58}$$

Thus $\bigwedge(E)$ is a closed set, and is therefore a weakly Borel set. The next theorem will show, in effect, that a regular weakly Borel PO-measure is concentrated on its spectrum.

LEMMA. *If* E *is a weakly Borel* PO-*measure on* X, *then* $E(C) = 0$ *for every compact subset* C *of* $\bigvee(E)$.

Proof. If C is any subset of $\bigvee(E)$, then each point of C belongs to some open set on which E vanishes; if, moreover, C is compact, then

$$C \subset U_1 \cup \cdots \cup U_n$$

for a suitable finite class of such open sets, hence

$$E(C) \leqslant E(U_1) + \cdots + E(U_n) = 0$$

by the (obvious) subadditivity of E. ♢

THEOREM 23. *If* E *is a regular weakly Borel* PO-*measure on* X, *then*

$$E\big(\bigvee(E)\big) = 0\,, \tag{59}$$

and so

$$E\big(\bigwedge(E)\big) = E(X)\,. \tag{60}$$

Proof. The relation (59) is immediate from the lemma, and the definition of regularity. Then

$$E(X) = E\big(\bigwedge(E) \cup \bigvee(E)\big) = E\big(\bigwedge(E)\big) + E\big(\bigvee(E)\big) = E\big(\bigwedge(E)\big).\ ♢$$

COROLLARY. *If* E *is a regular weakly Borel* PO-*measure on* X, *then* $\bigwedge(E)$ *is empty if and only if* E *is identically zero.*

Proof. If $\bigwedge(E) = \varnothing$, then by (60) we have $E(X) = E(\varnothing) = 0$; since E is monotone, we conclude that E is identically zero.

On the other hand, $E(X) = 0$ implies $X \subset \bigvee(E)$ and $\bigwedge(E) = \varnothing$, by the definitions. ♢

We remark that if X is σ-compact and metrizable (e.g., if $X = \mathbf{R}$ or $\mathbf{C}$), so that $\mathscr{B}_0 = \mathscr{B} = \mathscr{B}_{\mathrm{w}}$, then every Borel PO-measure E on X is regular (Theorem 18), and hence the formulas (59), (60) hold for E.

We conclude the section with a key definition:

DEFINITION 18. If E is a regular weakly Borel PO-measure on X, then E is said to be *compact* if $\bigwedge(E)$ is a compact subset of X.

It is a convenient abbreviation to reserve this term for regular weakly Borel PO-measures; thus, when we say "E is a compact PO-measure on X", it is implicit that

(a) X is a locally compact space,
(b) E is a PO-measure defined on the class of weakly Borel sets of X,
(c) E is regular in the sense of Definition 15, and
(d) $\bigwedge(E)$ is a compact subset of X.
The next section is devoted to compact PO-measures, and to a formulation of the "spectral mapping theorem".

*Exercise. If F is a Baire PO-measure on X, there exists one and only one regular weakly Borel PO-measure G which extends F. If, moreover, F is a spectral measure, then so is G.

7. Compact PO-measures

In this section we consider a locally compact space X, and a regular weakly Borel PO-measure E on X. We assume, moreover, that E is compact in the sense of Definition 18; the object of the section is to exploit this extra hypothesis. We shall further assume that E is not identically zero, in other words $E(X) \neq 0$; consequently the spectrum $\bigwedge(E)$ of E is a non empty compact subset of X (Corollary of Theorem 23). As usual, our sharpest results will be obtained for spectral measures, but it is interesting to see how far we can get with PO-measures. The answer is: not very.

The motivation for the next definition is that since $E\big(X - \bigwedge(E)\big) = 0$ (Theorem 23), we are concerned primarily with behaviour on the spectrum:

Definition 19. If f is any complex function on X, we define

$$\widehat{f} = \chi_{\wedge(E)} f \,. \tag{61}$$

Thus

$$\widehat{f}(\lambda) = \begin{cases} f(\lambda) & \text{when } \lambda \in \bigwedge(E) \\ 0 & \text{when } \lambda \notin \bigwedge(E) \,. \end{cases}$$

The correspondence $f \mapsto \widehat{f}$ evidently preserves sums, scalar multiples, products, and conjugation. If f is real-valued, then so is $\widehat{f}$; if $f \geqslant 0$, then also $\widehat{f} \geqslant 0$. If f is measurable (with respect to the σ-algebra $\mathscr{B}_{\mathrm{w}}$ of weakly Borel sets), then so is $\widehat{f}$. In particular, $\widehat{f}$ is measurable when f is continuous, and in this case $\widehat{f}$ is also bounded, by the compactness of $\bigwedge(E)$.

We recall that $\mathscr{M}$ denotes the class of all bounded measurable complex functions on X (Definition 8).

DEFINITION 20. We denote by $\mathscr{F}$ the class of all complex functions f on X such that $\widehat{f} \in \mathscr{M}$, that is, $\chi_{\wedge(E)} f$ is bounded and measurable.

Evidently $\mathscr{F}$ is closed under the formation of sums, scalar multiples, products, conjugates, and absolute values. Also $\mathscr{M} \subset \mathscr{F}$, and it is plausible that the correspondence

$$f \mapsto \int f\,dE \qquad (f \in \mathscr{M}) \tag{62}$$

described in Section 5 is trivially extensible to $\mathscr{F}$ (as we shall see below). The principal reason for bringing $\mathscr{F}$ into the picture is that $\mathscr{F}$ contains every continuous complex function on X. We remark that a function in $\mathscr{F}$ need not be measurable.

The following lemma is a necessary preliminary to extending the correspondence (62) to $\mathscr{F}$. Incidentally, we use the notations μ_x and $\int f\,d\mu_{x,y}$ in the sense defined in Section 5 (cf. formulas (11) and (14)).

LEMMA. *If $f \in \mathscr{M}$, then also $\widehat{f} \in \mathscr{M}$, and*

$$\int \widehat{f}\,dE = \int f\,dE\,.$$

Proof. The first assertion is obvious. Given any vector x, we wish to show that

$$\int \widehat{f}\,d\mu_x = \int f\,d\mu_x$$

(cf. (25)); indeed, since $E\big(X - \bigwedge(E)\big) = 0$, we have also $\mu_x\big(X - \bigwedge(E)\big) = 0$, and so $\widehat{f} = f$ almost everywhere with respect to μ_x. $\diamondsuit$

In view of the Lemma, the following definition is consistent with Definition 9:

DEFINITION 21. If $f \in \mathscr{F}$, we define $\int f\,dE$ to be the operator $\int \widehat{f}\,dE$ given by Definition 9. Thus $\int f\,dE$ is the unique operator such that

$$\Big(\big(\textstyle\int f\,dE\big)x\big|x\Big) = \displaystyle\int \widehat{f}\,d\mu_x \tag{63}$$

for all vectors x, equivalently (cf. (19), (26))

$$\Big(\big(\textstyle\int f\,dE\big)x\big|y\Big) = \displaystyle\int \widehat{f}\,d\mu_{x,y} \tag{64}$$

for all vectors x, y.

The following theorem is a slight but useful generalization of the above Lemma (we remark that the compactness of $\bigwedge(E)$ is immaterial here):

THEOREM 24. *If* M *is any weakly Borel set such that* $M \supset \bigwedge(E)$, *then*

$$\int \chi_M f \; dE = \int f \; dE \tag{65}$$

for every f *in* $\mathscr{F}$.

Proof. For any vector x,

$$\mu_x(X - M) \leqslant \mu_x\bigl(X - \bigwedge(E)\bigr) = 0\,,$$

thus $\chi_M f = f$ almost everywhere with respect to μ_x. Then also

$$\chi_{\wedge(E)}\chi_M f = \chi_{\wedge(E)} f$$

almost everywhere, and (65) results on integrating with respect to μ_x. $\diamondsuit$

Since the correspondence $f \mapsto \widehat{f}$ $(f \in \mathscr{F})$ preserves all operations in sight, it is immediate from Theorem 10 that the correspondence

$$f \mapsto \int f \, dE \qquad (f \in \mathscr{F}) \tag{66}$$

has the following properties:

$$\int (f+g)\, dE = \int f\, dE + \int g\, dE \tag{67}$$

$$\int (cf)\, dE = c \int f\, dE \tag{68}$$

$$\int \overline{f}\, dE = \Bigl(\int f\, dE\Bigr)^{*}. \tag{69}$$

If $f \in \mathscr{F}$ is real-valued, then $\int f\, dE$ is Hermitian; in fact, it clearly suffices to assume that f is real-valued on $\bigwedge(E)$. Similarly, if $f \geqslant 0$ on $\bigwedge(E)$, then $\int f\, dE \geqslant 0$. By Theorem 13, we have

$$\int f\, dE \in \{E\}'' \tag{70}$$

for all f in $\mathscr{F}$.

If, moreover, E is projection-valued (i.e. is a compact spectral measure), then by Theorems 14, 15 and 16, we have, for all f, g in $\mathscr{F}$,

$$\int f\,dE \qquad \text{is normal} \tag{71}$$

$$\int f\,dE \leftrightarrow \int g\,dE \tag{72}$$

$$\int fg\,dE = \Big(\int f\,dE\Big)\Big(\int g\,dE\Big) \tag{73}$$

$$\Big(\int f\,dE\Big)^*\Big(\int f\,dE\Big) = \int |f|^2\,dE \tag{74}$$

$$\Big\|\int f\,dE\Big\| \leqslant \|\widehat{f}\|_\infty\,. \tag{75}$$

Formula (75) reminds us again that we are concerned with the values of f on $\bigwedge(E)$. Although an arbitrary f in $\mathscr{F}$ may be unbounded, a satisfactory substitute "norm" is $\|\widehat{f}\|_\infty$.

Definition 22. If $f \in \mathscr{F}$, the *spectral norm* of f, denoted $N_E(f)$, is defined to be $\|\widehat{f}\|_\infty$. Thus,

$$N_E(f) = \text{LUB}\,\{|f(\lambda)| :\ \lambda \in \textstyle\bigwedge(E)\,\}\,. \tag{76}$$

The following properties of the spectral norm (which is really a seminorm) are obvious:

$$N_E(f) \geqslant 0 \tag{77}$$

$$N_E(f) = 0 \quad \text{if and only if} \quad f = 0 \ \text{ on } \textstyle\bigwedge(E)\,. \tag{78}$$

$$N_E(f+g) \leqslant N_E(f) + N_E(g) \tag{79}$$

$$N_E(cf) = |c|N_E(f) \tag{80}$$

$$N_E(\overline{f}) = N_E(f) = N_E(|f|) \tag{81}$$

$$N_E(fg) \leqslant N_E(f)N_E(g) \tag{82}$$

$$N_E(|f|^2) = \big(N_E(f)\big)^2 \tag{83}$$

The next two lemmas lead up to a proof that if E is projection-valued, then $\|\int f\,dE\| = N_E(f)$ for every continuous complex function f on X. We defer the extra hypotheses as long as possible:

Lemma 1. *If* $f \in \mathscr{F}$ *and* f *is real-valued on* $\bigwedge(E)$, *then*

$$\|\textstyle\int f\,dE\| \leqslant N_E(f)\,\|E(X)\|\,. \tag{84}$$

Proof. Let $K = \|E(X)\|$. Citing Theorem 10, we have

$$\|\textstyle\int f\,dE\| = \|\int \widehat{f}\,dE\| \leqslant K\,\|\widehat{f}\|_\infty = K\,N_E(f)$$

(cf. formula (12)). ♢

The inequality (84) can be sharpened to equality when f is continuous and non negative, and E has the following peculiar "zero-one" property:

LEMMA 2. *Assume that for each weakly Borel set* M, *either* $E(M) = 0$ *or* $\|E(M)\| = 1$ (*this is the case, for example, when* E *is a spectral measure*). *Then*

$$\left\| \int f\, dE \right\| = N_E(f) \tag{85}$$

for every continuous real function on X *such that* $f \geqslant 0$ *on* $\bigwedge(E)$.

Proof. Since $\left\| \int f\, dE \right\| \leqslant N_E(f)$ by Lemma 1, the relation (85) is trivial when $N_E(f) = 0$. Assuming $N_E(f) > 0$, and given any $\varepsilon > 0$, it will suffice to show that

$$\left\| \int f\, dE \right\| \geqslant N_E(f) - \varepsilon\,.$$

We may suppose that $\varepsilon < N_E(f)$. Define

$$M = \{\lambda \in X :\ f(\lambda) > N_E(f) - \varepsilon\}\,;$$

M is an open set, and it is clear from the definition of $N_E(f)$ that M must contain at least one point of $\bigwedge(E)$. It follows that $E(M) \neq 0$; for, $E(M) = 0$ would imply (since M is open) that $M \subset \bigvee(E) = X - \bigwedge(E)$, contrary to $M \cap \bigwedge(E) \neq \varnothing$.

Since $E\big(X - \bigwedge(E)\big) = 0$ (Theorem 23), we have

$$\begin{aligned} E(M) &= E\big(M \cap \textstyle\bigwedge(E)\big) + E\big(M - \textstyle\bigwedge(E)\big) \\ &= E\big(M \cap \textstyle\bigwedge(E)\big) + 0 \end{aligned}$$

by the additivity and monotonicity of E. Thus

$$E\big(M \cap \textstyle\bigwedge(E)\big) = E(M) \neq 0\,,$$

and therefore $\big\|E\big(M \cap \bigwedge(E)\big)\big\| = 1$ by our hypothesis. By the definition of M, we have

$$f \geqslant N_E(f) - \varepsilon \quad \text{on } M\,,$$

that is,

$$\chi_M f \geqslant \big(N_E(f) - \varepsilon\big)\chi_M\,;$$

multiplying through by the characteristic function of $\bigwedge(E)$, and noting that $\widehat{f} \geqslant 0$, we have

$$\widehat{f} \geqslant \chi_M \widehat{f} \geqslant \big(N_E(f) - \varepsilon\big)\chi_{M \cap \wedge(E)}\,,$$

and therefore
$$\int \widehat{f}\, dE \geqslant \big(N_E(f) - \varepsilon\big) \int \chi_{M \cap \wedge(E)}\, dE\,,$$

that is,

$$(*) \qquad \int f\, dE \geqslant \big(N_E(f) - \varepsilon\big) E\big(M \cap \bigwedge(E)\big)$$

(cf. the formulas following Theorem 24). Since

$$\big\|E\big(M \cap \bigwedge(E)\big)\big\| = 1 \quad \text{and} \quad N_E(f) - \varepsilon > 0\,,$$

it follows from $(*)$ that $\big\| \int f\, dE\big\| \geqslant N_E(f) - \varepsilon$. ♢

THEOREM 25. *If E is a compact spectral measure on X, then*

$$(86) \qquad \big\|\int f\, dE\big\| = N_E(f)$$

for every continuous complex function f on X.

Proof. Let $A = \int f\, dE$ and $g = \overline{f} f = |f|^2$. By Lemma 2,

$$\big\|\int g\, dE\big\| = N_E(g)\,.$$

But $\int g\, dE = A^*A$ and $N_E(g) = \big(N_E(f)\big)^2$ by (74) and (83); substituting in the above equation, we have

$$\|A^*A\| = \big(N_E(f)\big)^2,$$

thus
$$\|A\|^2 = \big(N_E(f)\big)^2 . \ \diamondsuit$$

The next theorem is a "spectral mapping theorem" for compact spectral measures; when combined with the spectral theorem for a normal operator, it will yield the usual spectral mapping theorem for a normal operator (see Section 12). We separate out part of the argument as a lemma:

LEMMA. *If E is a compact spectral measure on X, and if g is a continuous complex function on X which vanishes at some point of $\bigwedge(E)$, then the operator $\int g\, dE$ is singular (i.e., not invertible).*

Proof. Say $\lambda_0 \in \bigwedge(E)$, $g(\lambda_0) = 0$. Let $B = \int g\, dE$, and assume to the contrary that B is invertible. Then B is bounded below, thus there exists an $\varepsilon > 0$ such that $B^*B \geqslant \varepsilon I$ (cf. [5, p. 38]). Citing formula (74), we have
$$\int |g|^2\, dE \geqslant \varepsilon I\,.$$

Define

$$M = \{\lambda \in X : \ |g(\lambda)|^2 < \varepsilon/2\};$$

since g is continuous and $g(\lambda_0) = 0$, M is an open neighborhood of λ_0. Necessarily $E(M) \neq 0$, by the same argument as in the proof of Lemma 2 above. Choose a vector x such that $\|x\| = 1$ and $E(M)x = x$. Since

$$\chi_M |g|^2 \leqslant \frac{\varepsilon}{2}\,\chi_M\,,$$

and in particular $\chi_M |g|^2$ is bounded (and of course measurable), integration with respect to μ_x yields

$$\int \chi_M |g|^2 \, d\mu_x \leqslant \frac{\varepsilon}{2}\,\mu_x(M)\,;$$

since $\mu_x(M) = (E(M)x\big|x) = (x\big|x) = 1$, we therefore have

$$(*) \qquad \int \chi_M |g|^2 \, d\mu_x \leqslant \varepsilon/2\,.$$

Now, $\mu_x(X - M) = \mu_x(X) - \mu_x(M) = (E(X)x\big|x) - 1 \leqslant (x\big|x) - 1 = 0$, hence

$$(**) \qquad \chi_M |g|^2 = |g|^2 \qquad \text{a.e. } [\mu_x]\,.$$

Similarly, $\mu_x\big(X - \bigwedge(E)\big) = 0$ implies

$$\chi_{\wedge(E)} |g|^2 = |g|^2 \qquad \text{a.e. } [\mu_x]\,;$$

combining this with $(**)$, we have

$$(***) \qquad \chi_M |g|^2 = \chi_{\wedge(E)} |g|^2 \qquad \text{a.e. } [\mu_x]\,.$$

Substituting $(***)$ in $(*)$,

$$\int \chi_{\wedge(E)} |g|^2 \, d\mu_x \leqslant \varepsilon/2\,,$$

that is,

$$\int |\widehat{g}|^2 \, d\mu_x \leqslant \varepsilon/2\,;$$

this means, in view of (63), that

$$\big(\big(\textstyle\int |g|^2 \, dE\big)x\big|x\big) \leqslant \varepsilon/2\,.$$

Thus $(B^*Bx|x) \leqslant \varepsilon/2$; but $(B^*Bx|x) \geqslant \varepsilon(x|x) = \varepsilon$ by the choice of ε, a contradiction. ♢

DEFINITION 23. We say that a PO-measure E is *normalized* in case $E(X) = I$.

(For a spectral measure, normalization can always be achieved by throwing away an irrelevant part of the underlying Hilbert space.)

We have arrived at spectral paydirt:

THEOREM 26. (Spectral Mapping Theorem) *If E is a normalized compact spectral measure on the locally compact space X, then*

$$\bigwedge\left(\int f\,dE\right) = f\left(\bigwedge(E)\right) \tag{87}$$

for every continuous complex function f on X.

Proof. Recall that for any operator A, $\bigwedge(A)$ denotes the spectrum of A, and $\Pi(A)$ denotes the approximate point spectrum; when A is normal, $\bigwedge(A) = \Pi(A)$ [5, Sec. 31].

Let us show first that

$$f\left(\bigwedge(E)\right) \subset \bigwedge\left(\int f\,dE\right). \tag{87a}$$

Thus, assuming $\lambda_0 \in \bigwedge(E)$, it is to be shown that the operator $\int f\,dE - f(\lambda_0)I$ is singular. Indeed, defining $g = f - f(\lambda_0)1$, we have $g(\lambda_0) = 0$, hence by the Lemma, the operator $\int g\,dE$ is singular; since

$$\int g\,dE = \int f\,dE - f(\lambda_0)\int 1\,dE,$$

and since

$$\int 1\,dE = \int \chi_X\,dE = E(X) = I,$$

we have established the singularity of $\int f\,dE - f(\lambda_0)I$.

To show that

$$\bigwedge\left(\int f\,dE\right) \subset f\left(\bigwedge(E)\right), \tag{87b}$$

let us assume that c is a complex number such that $c \notin f\left(\bigwedge(E)\right)$, and let us prove that $c \notin \bigwedge\left(\int f\,dE\right)$. Define $g = f - c1$; we are assuming that g never vanishes on $\bigwedge(E)$, and we wish to prove that the operator $\int f\,dE - cI = \int g\,dE$ is invertible. By the continuity of g and the compactness of $\bigwedge(E)$, g must be bounded below on $\bigwedge(E)$, say

$$|g| \geqslant \varepsilon \quad \text{on } \bigwedge(E),$$

for a suitable $\varepsilon > 0$. Then

$$|g|^2 \geqslant \chi_{\wedge(E)}|g|^2 \geqslant \varepsilon^2 \chi_{\wedge(E)},$$

hence

$$\int |g|^2 \, dE \geqslant \varepsilon^2 E\big(\bigwedge(E)\big);$$

since $E\big(\bigwedge(E)\big) = E(X) = I$ (Theorem 23), we have

$$\int |g|^2 \, dE \geqslant \varepsilon^2 I,$$

that is,

$$\big(\textstyle\int f\, dE - cI\big)^* \big(\textstyle\int f\, dE - cI\big) \geqslant \varepsilon^2 I$$

(cf. (74)). Thus $\int f\, dE - cI$ is bounded below, that is, $c \notin \Pi\big(\int f\, dE\big)$. Since $\int f\, dE$ is normal (cf. (71)), we conclude that $c \notin \bigwedge\big(\int f\, dE\big)$. ♢

An interesting sidelight on Theorem 26 is that since $\bigwedge(E) \neq \varnothing$ (Corollary of Theorem 23), it follows that the normal operator $\int f\, dE$ must have non empty spectrum (cf. [3]). (This is not news, since it is known that every operator has non empty spectrum [9, p. 309].)

Another consequence of Theorem 26 is that the norm of the operator $\int f\, dE$ can be calculated from its spectrum (cf. [3]):

COROLLARY. *If E is a normalized compact spectral measure on the locally compact space X, then*

$$\big\|\textstyle\int f\, dE\big\| = \mathrm{LUB}\,\big\{|c| : \ c \in \bigwedge\big(\textstyle\int f\, dE\big)\big\} \tag{88}$$

for every continuous complex function f on X.

Proof. By Theorem 26, the LUB on the right is equal to

$$\mathrm{LUB}\,\big\{|f(\lambda)| : \ \lambda \in \bigwedge(E)\big\},$$

in other words, to $N_E(f)$; but $N_E(f) = \big\| \int f\, dE\big\|$ by Theorem 25. ♢

8. Real and complex compact spectral measures

The Spectral Theorem, as formulated in Section 12, asserts that each normal operator may be represented as an integral with respect to a certain uniquely determined spectral measure. In this section we are concerned only with the uniqueness of such representations.

More precisely, given a compact normalized spectral measure on a locally compact space X, the object of this section is to exploit the extra hypothesis that $X = \mathbf{R}$ or $\mathbf{C}$. Accordingly, for the rest of the section X will denote either $\mathbf{R}$ or $\mathbf{C}$. The key to this special situation is that the identity mapping of X becomes eligible for integration.

Definition 24. A *real* (resp. *complex*) PO-measure is a (necessarily regular) PO-measure defined on the class of all Borel sets of $\mathbf{R}$ (resp. $\mathbf{C}$).

In this section we shall be concerned only with real or complex spectral measures which are compact (Definition 18) and normalized (Definition 23). Every result in this section is valid for both the real and complex cases; nevertheless, we shall find it notationally convenient to split the first result into two separate theorems, according as $X = \mathbf{R}$ or $\mathbf{C}$. For instance, we shall write

$$u : \mathbf{C} \to \mathbf{C} \tag{89}$$

for the mapping $u(\lambda) = \lambda$ $(\lambda \in \mathbf{C})$, and

$$v : \mathbf{R} \to \mathbf{C} \tag{90}$$

for the mapping $v(\alpha) = \alpha$ $(\alpha \in \mathbf{R})$. Thus u is the identity mapping of $\mathbf{C}$, whereas v is the identity injection of $\mathbf{R}$ into $\mathbf{C}$. The first two theorems assert that every real (resp. complex) normalized compact spectral measure is uniquely determined by the integral which it assigns to the continuous function v (resp. u).

Theorem 27. (Uniqueness Theorem, real case) *If F_1 and F_2 are normalized compact real spectral measures such that*

$$\int \alpha \, dF_1 = \int \alpha \, dF_2 \,,$$

then $F_1 = F_2$.

Proof. We are using $\int \alpha \, dF_k$ as a suggestive notation for $\int v \, dF_k$, where v is the function (90); let us denote this operator by A.

Our first task is to show that

$$\int f \, dF_1 = \int f \, dF_2 \tag{91}$$

for every continuous $f : \mathbf{R} \to \mathbf{C}$. Consider first the case that f is a real polynomial, say

$$f = a_0 1 + a_1 v + \cdots + a_r v^r$$

for suitable real numbers $a_0, \ldots, a_r$. It is clear from the properties of spectral integrals that

$$\int f \, dF_k = a_0 \int 1 \, dF_k + a_1 A + a_2 A^2 + \cdots + a_r A^r$$

(cf. (67), (68), (73)); interpreting 1 as the characteristic function of $\mathbf{R}$, we have

$$\int 1 \, dF_k = F_k(\mathbf{R}) = I\,,$$

thus

$$\int f \, dF_k = a_0 I + a_1 A + a_2 A^2 + \cdots + a_r A^r$$

for $k = 1, 2$. Since the right side is independent of k, we see that (91) holds for real polynomials.

Suppose now that f is any continuous real-valued function on $\mathbf{R}$. Let $[a, b]$ be any closed interval containing both $\bigwedge(F_1)$ and $\bigwedge(F_2)$. (Granted that $\bigwedge(F_1) = \bigwedge(F_2)$ will follow from $F_1 = F_2$, for the moment we cannot, and need not, be so precise.) Choose, by the Weierstrass theorem, a sequence of real polynomials p_n such that $p_n \to f$ uniformly on $[a, b]$. Since $p_n \to f$ uniformly on $\bigwedge(F_k)$ for each k, we have

$$\left\| \int p_n \, dF_k - \int f \, dF_k \right\| \to 0$$

by Theorem 11 (or (75)); since (91) holds for each p_n, we conclude that (91) holds for f also. The case that f is complex-valued follows at once by linearity.

In particular, if $\mathscr{L}$ is the class of all continuous real functions f on $\mathbf{R}$ with compact support, we have

$$\int f \, dF_1 = \int f \, dF_2$$

for all f in $\mathscr{L}$, hence $F_1 = F_2$ by Theorem 17. $\diamondsuit$

Since $\mathbf{C}$ is homeomorphic with $\mathbf{R}\times\mathbf{R}$ via the mapping $\alpha+i\beta \to (\alpha,\beta)$, the Borel sets of the two spaces also correspond under this mapping. If M and N are Borel sets in $\mathbf{R}$, it will be convenient, in the proof of the complex analogue of Theorem 27, to write $M \times N$ for the Borel set

$$\{\alpha + i\beta : \ \alpha \in M\,,\ \beta \in N\}$$

of $\mathbf{C}$. First we prove a lemma which paves the way for deducing the complex analogue of Theorem 27 from Theorem 27 itself:

LEMMA. *Let E be a normalized compact complex spectral measure, define*

$$A = \int \lambda\, dE\,,$$

and suppose

$$A = S + iT$$

is the Cartesian decomposition of A. Define, for each Borel set M in $\mathbf{R}$,

$$F(M) = E(M \times \mathbf{R})\,.$$

Then F is a normalized compact real spectral measure, and

$$\int \alpha\, dF = S\,.$$

Proof. We are using $\int \lambda\, dE$ and $\int \alpha\, dF$ as suggestive notations for $\int u\, dE$ and $\int v\, dF$, respectively, where u and v are the functions (89), (90). It is easy to see that F is a (real) spectral measure; for instance, the proof of multiplicativity runs as follows:

$$\begin{aligned} F(M\cap N) &= E\big[(M\cap N)\times \mathbf{R}\big] = E\big[(M\times\mathbf{R})\cap(N\times\mathbf{R})\big] \\ &= E(M\times\mathbf{R})E(N\times\mathbf{R}) = F(M)F(N)\,. \end{aligned}$$

Since $F(\mathbf{R}) = E(\mathbf{R}\times\mathbf{R}) = E(\mathbf{C}) = I$, F is normalized. Moreover, F is compact; for, if $M\times N$ is a "rectangle" in $\mathbf{C}$ with compact sides, such that $\bigwedge(E) \subset M\times N$, then $\mathbf{R} - M$ is open in $\mathbf{R}$, and

$$\begin{aligned} F(\mathbf{R} - M) &= E\big((\mathbf{R} - M)\times\mathbf{R}\big) = E(\mathbf{C} - M\times\mathbf{R}) \\ &\leqslant E(\mathbf{C} - M\times N) \leqslant E\big(\mathbf{C} - \textstyle\bigwedge(E)\big) = 0\,, \end{aligned}$$

and so $\bigwedge(F) \subset M$.

By definition, $A = \int u\,dE$; we are trying to show that $S = \int v\,dF$. In any case, if $f : \mathbf{C} \to \mathbf{C}$ is the function defined by the formula

$$f(\lambda) = \frac{\lambda + \overline{\lambda}}{2}\,,$$

we have

$$\int f\,dE = \frac{A + A^*}{2} = S$$

by the elementary formal properties of operator valued integrals (cf. formulas (67)–(69)). The basic idea of the rest of the proof is that f is constant on each vertical line of the Gaussian plane, and so in approximating f it is sufficient to approximate its slice along the real axis.

If $g : \mathbf{R} \to \mathbf{C}$ is any function, let us write g' for the function on $\mathbf{C}$ defined by the formula

$$g'(\alpha + i\beta) = g(\alpha)\,.$$

Observe that $v' = f$.

If g is any simple Borel function on $\mathbf{R}$, say

$$g = \sum_1^r c_j\,\chi_{M_j}\,,$$

then g' is the simple Borel function

$$g' = \sum_1^r c_j\,\chi_{M_j \times \mathbf{R}}$$

on $\mathbf{C}$, and it is clear from the definition of F that

$$\sum_1^r c_j F(M_j) = \sum_1^r c_j E(M_j \times \mathbf{R})\,,$$

that is,

$$\text{(92)} \qquad \int g\,dF = \int g'\,dE\,.$$

Let $[a, b]$ be a closed interval such that

$$\bigwedge(E) \subset [a, b] \times [a, b]\,;$$

as shown earlier in the proof,

$$\bigwedge(F) \subset [a,b] \,.$$

Since $\chi_{[a,b]}v$ is a bounded Borel function on $\mathbf{R}$, there exists a sequence g_n of simple Borel functions on $\mathbf{R}$ such that $g_n \to v$ uniformly on $[a,b]$. Then also

$$g_n \to v \quad \text{uniformly on } \bigwedge(F)\,,$$

and

$$g_n' \to v' \quad \text{uniformly on } \bigwedge(E)$$

(indeed on $[a,b] \times \mathbf{R}$), consequently

$$\| \int g_n \, dF - \int v \, dF \| \to 0$$

and

$$\| \int g_n' \, dE - \int f \, dE \| \to 0$$

(cf. Theorem 11, or (75)). Since (92) holds for each g_n, we conclude that

$$\int v \, dF = \int f \, dE = S \,. \ \diamondsuit$$

(We remark that the above Lemma, and its proof, are valid if we replace "spectral measure" by "PO-measure" throughout, and delete the argument that F is multiplicative.)

THEOREM 28. (Uniqueness Theorem, complex case) *If E_1 and E_2 are normalized compact complex spectral measures such that*

$$\int \lambda \, dE_1 = \int \lambda \, dE_2$$

then $E_1 = E_2$.

Proof. Let $A = \int \lambda \, dE_1 = \int \lambda \, dE_2$, where λ stands for the function u in (89). Suppose

$$A = S + iT$$

is the Cartesian decomposition of A. For each $k = 1, 2$, define

$$F_k(M) = E_k(M \times \mathbf{R})$$

as in the Lemma; thus F_k is a normalized compact real spectral measure, and

$$\int \alpha\, dF_1 = S = \int \alpha\, dF_2$$

by the Lemma. Then $F_1 = F_2$ by Theorem 27, thus

$$E_1(M \times \mathbf{R}) = E_2(M \times \mathbf{R})$$

for all Borel sets M in $\mathbf{R}$. Similarly

$$E_1(\mathbf{R} \times N) = E_2(\mathbf{R} \times N)$$

for all Borel sets N in $\mathbf{R}$. Then

$$\begin{aligned} E_1(M \times N) &= E_1\big[(M \times \mathbf{R}) \cap (\mathbf{R} \times N)\big] \\ &= E_1(M \times \mathbf{R})E_1(\mathbf{R} \times N) \\ &= E_2(M \times \mathbf{R})E_2(\mathbf{R} \times N) \\ &= E_2(M \times N) \end{aligned}$$

for all Borel sets M and N in $\mathbf{R}$. It follows from additivity that $E_1 = E_2$ on the ring $\mathscr{R}$ generated by all such rectangles $M \times N$ (cf. [4, p. 139]), consequently $E_1 = E_2$ on $\mathfrak{S}(\mathscr{R})$ by Theorem 6. Since $\mathfrak{S}(\mathscr{R})$ is precisely the class of all Borel sets in $\mathbf{C}$ (cf. formula (1)), we conclude, indeed, that $E_1 = E_2$. $\diamondsuit$

For simplicity we shall write $\int \lambda\, dE$ for either $\int u\, dE$ or $\int v\, dE$, according as the spectral measure E is complex or real, respectively. In the remaining theorems of the section we need not sort out the real and complex cases, for they can be treated simultaneously.

The following theorem is evidently a generalization of Theorems 27 and 28:

Theorem 29. *Assume $X = \mathbf{R}$ or $\mathbf{C}$. Let E_1 and E_2 be normalized compact spectral measures on X, and define*

$$A_1 = \int \lambda\, dE_1\,,$$
$$A_2 = \int \lambda\, dE_2\,.$$

If A_1 and A_2 are unitarily equivalent, then so are E_1 and E_2, and conversely. More precisely, if U is a unitary operator, then

(93) $$A_1 = U^{-1}A_2U$$

if and only if

$$E_1(M) = U^{-1} E_2(M) U \tag{94}$$

for all Borel sets M in X.

Proof. Let U be any unitary operator, and define

$$E(M) = U^{-1} E_2(M) U$$

for all Borel sets M. It is to be shown that (93) holds if and only if $E_1 = E$. In any case it is clear that E is a normalized Borel spectral measure; since $E(M) = 0$ if and only if $E_2(M) = 0$, we have $\bigwedge(E) = \bigwedge(E_2)$, and in particular E is also compact. Let $A = \int \lambda \, dE$.

Choose a compact set Ω which contains both $\bigwedge(E_1)$ and $\bigwedge(E_2) = \bigwedge(E)$. By Theorem 24. we have

$$A_1 = \int \chi_\Omega \lambda \, dE_1 , \tag{95}$$

$$\mathrm{A}_2 = \int \chi_\Omega \lambda \, dE_2 , \tag{96}$$

$$A = \int \chi_\Omega \lambda \, dE . \tag{97}$$

Consider the Borel measures

$$\begin{aligned} \mu_x^1(M) &= (E_1(M)x \big| x) , \\ \mu_y^2(M) &= (E_2(M)y \big| y) , \\ \mu_x(M) &= (E(M)x \big| x) , \end{aligned}$$

where x and y are any vectors. Citing (25), the formulas (95)–(97) yield

$$(A_1 x \big| x) = \int \chi_\Omega \, \lambda \, d\mu_x^1 , \tag{98}$$

$$(\mathrm{A}_2 y \big| y) = \int \chi_\Omega \, \lambda \, d\mu_y^2 , \tag{99}$$

$$(A x \big| x) = \int \chi_\Omega \, \lambda \, d\mu_x . \tag{100}$$

In particular, for $y = Ux$ we have

$$\mu_{Ux}^2(M) = (E_2(M)Ux \big| Ux) = \big(U^{-1} E_2(M) U x \big| x\big) = (E(M)x \big| x) = \mu_x(M) ,$$

thus (99) yields

$$(99') \qquad (U^{-1}A_2Ux|x) = \int \chi_\Omega \lambda \, d\mu_x \, .$$

From (100) and (99′), we have

$$(101) \qquad A = U^{-1}A_2U \, .$$

Suppose now that U satisfies (93), that is, $A_1 = U^{-1}A_2U$; citing (101) we have $A_1 = A$, hence $E_1 = E$ by either Theorem 27 or 28, as the case may be.

If, conversely, $E_1 = E$, then $\mu_x^1 = \mu_x$ for all x, hence

$$(A_1x|x) = \int \chi_\Omega \lambda \, d\mu_x^1 = \int \chi_\Omega \lambda \, d\mu_x = (Ax|x) \, ;$$

thus $A_1 = A = U^{-1}A_2U$. ♢

For the next theorem we need an elementary (and well known) result on the commutant of a self-adjoint set of operators:

Lemma. *If $\mathscr{D}$ is any set of operators such that $T \in \mathscr{D}$ implies $T^* \in \mathscr{D}$, and if $\mathscr{D}'$ is the commutant of $\mathscr{D}$, then every operator in $\mathscr{D}'$ is a linear combination of unitary operators in $\mathscr{D}'$.*

Proof. Given $S \in \mathscr{D}'$, let us show that S is a linear combination of unitaries in $\mathscr{D}'$. Since $\mathscr{D}'$ is a self-adjoint algebra, we may suppose that S is Hermitian, and that $\|S\| \leqslant 1$. Then $S^2 \leqslant I$, and so $I - S^2 \geqslant 0$. Let $R = (I - S^2)^{1/2}$; thus $R \geqslant 0$, $R^2 = I - S^2$, and

$$R \in \{I - S^2\}'' = \{S^2\}'' \subset (\mathscr{D}')'' = \mathscr{D}' \, .$$

Define $U = S + iR$; then $U \in \mathscr{D}'$,

$$U^*U = UU^* = S^2 + R^2 = I \, ,$$

and

$$S = \frac{1}{2}U + \frac{1}{2}U^* \, . \ ♢$$

Theorem 30. *Assume $X = \mathbf{R}$ or $\mathbf{C}$. Let E be a normalized compact spectral measure on X, and define*

$$A = \int \lambda \, dE \, .$$

Then

$$E(M) \in \{A, A^*\}''$$

for all Borel sets M *.*

Proof. Given $T \in \{A, A^*\}'$, we are to show that

$$E(M) \leftrightarrow T$$

for all Borel sets M . By the Lemma, applied to the set $\mathscr{D} = \{A, A^*\}$, we may assume that $T = U$, U a unitary operator. Since $U \in \{A, A^*\}'$, we have $U \leftrightarrow A$, that is,

$$A = U^{-1}AU\,;$$

by Theorem 29 (with $E_1 = E_2 = E$), we conclude that

$$E(M) = U^{-1}E(M)U$$

for all Borel sets M , thus $E(M) \leftrightarrow U$ for all Borel sets M , as was to be shown. ♢

We remark that when $X = \mathbf{R}$ in Theorem 30, the operator A is Hermitian (Theorem 10); thus the conclusion is that

$$E(M) \in \{A\}'' \tag{102}$$

for all Borel sets M . When $X = \mathbf{C}$, all we know is that A is normal (cf. (71)); nevertheless, (102) still holds in this case, because, by a theorem of B. Fuglede,

$$\{A\}' = \{A^*\}'\,, \tag{103}$$

and consequently $\{A, A^*\}' = \{A\}'$. The reader is referred to [5, Sec. 41] for a proof of (103) for any normal operator A , based on the Spectral Theorem (Section 12), and an ingenious geometrical characterization of the projections $E(M)$.

We close the section with a result which could have been proved immediately following Theorem 26:

THEOREM 31. *Assume* $X = \mathbf{R}$ *or* $\mathbf{C}$ *. Let* E *be a normalized compact spectral measure on* X *, and define*

$$A = \int \lambda \, dE\,.$$

Then

$$\bigwedge(A) = \bigwedge(E)\,, \tag{104}$$

and in particular $\bigwedge(A)$ *is non empty, and*

$$\|A\| = \mathrm{LUB}\,\{|\lambda| :\ \lambda \in \bigwedge(A)\,\}\,. \tag{105}$$

Moreover,

$$\bigwedge\big(\textstyle\int f\,dE\big) = f\big(\bigwedge(A)\big) \tag{106}$$

for every continuous complex function f *on* X .

Proof. Since $E(X) = I$, we know from the Corollary of Theorem 23 that $\bigwedge(E) \neq \varnothing$. Taking $f(\lambda) = \lambda$ in Theorem 26, we have

$$\bigwedge(A) = \bigwedge\big(\textstyle\int f\,dE\big) = f\big(\bigwedge(E))\big) = \bigwedge(E)$$

and

$$\|A\| = \mathrm{LUB}\,\{|\lambda| :\ \lambda \in \bigwedge(A)\,\}$$

by the Corollary of Theorem 26. Then for an arbitrary continuous complex function f, Theorem 26 yields

$$\bigwedge\big(\textstyle\int f\,dE\big) = f\big(\bigwedge(A)\big).\ \diamondsuit$$

9. The Spectral Theorem for a Hermitian operator

As far as the Spectral Theorem for a Hermitian operator is concerned, we may replace "locally compact space" by "the space $\mathbf{R}$" in all of the foregoing material.

THEOREM 32. (The Spectral Theorem for a Hermitian operator) *If* A *is a Hermitian operator, there exists one and only one normalized compact real spectral measure* E *such that*

$$A = \int \lambda\,dE\,.$$

Proof. In view of Theorem 27, we have only to prove existence. The first step, preparatory to invoking Theorem 19, is to construct an elementary "functional calculus" for A. We start with real polynomials p; if

$$p(\lambda) = a_0 + a_1\lambda + \cdots + a_r\lambda^r \,,$$

we define

$$p(A) = a_0 I + a_1 A + \cdots + a_r A^r \,.$$

The following properties of the Hermitian operator valued correspondence $p \mapsto p(A)$ are immediate:

$$(p+q)(A) = p(A) + q(A) \tag{107}$$

$$(a\,p)(A) = a\,p(A) \tag{108}$$

$$(pq)(A) = p(A)q(A) \tag{109}$$

$$p(A) = A \quad \text{when} \quad p(\lambda) = \lambda \,. \tag{110}$$

To state the key property of the correspondence, define

$$a = \text{GLB}\,\{(Ax\big|x) : \ \|x\| = 1\,\}\,,$$
$$b = \text{LUB}\,\{(Ax\big|x) : \ \|x\| = 1\,\}\,;$$

then

$$p \geqslant 0 \text{ on } [a,b] \quad \text{implies} \quad p(A) \geqslant 0\,. \tag{111}$$

A proof of (111), using the fundamental theorem of algebra, is given in [8, Sec. 106]. A slightly different proof, still using the fundamental theorem, is given in [1], with a slightly sharper conclusion; but all we need is (111), and an altogether elementary proof, avoiding the fundamental theorem, is given in [7]. In any case, we owe to F. Riesz the recognition of the fundamental role played by (111).

If, for any continuous real function f on $\mathbf{R}$, we define

$$N(f) = \text{LUB}\,\{|f(\lambda)| : \ a \leqslant \lambda \leqslant b\,\}\,,$$

it follows at once from (111) that for every real polynomial p,

$$-N(p)I \leqslant p(A) \leqslant N(p)I\,,$$

and consequently

$$\|p(A)\| \leqslant N(p)\,. \tag{112}$$

We now construct a Hermitian operator valued correspondence

$$f \mapsto T_f \qquad (f \in \mathscr{L})$$

where $\mathscr{L}$ is the class of all continuous real functions on $\mathbf{R}$ with compact support. Given any f in $\mathscr{L}$, choose, by the Weierstrass theorem, a sequence of real polynomials p_n such that $p_n \to f$ uniformly on $[a,b]$, thus

$$N(p_n - f) \to 0\,;$$

then also

$$N(p_m - p_n) \to 0\,,$$

and so

$$\|p_m(A) - p_n(A)\| \to 0$$

by (112). It follows from the completeness of the normed algebra of all operators that there exists a (Hermitian) operator S such that $\|p_n(A) - S\| \to 0$. If q_n is another sequence of real polynomials such that $q_n \to f$ uniformly on $[a,b]$, it follows from the calculation

$$\begin{aligned}\|p_n(A) - q_n(A) &= \|(p_n - q_n)(A)\| \leqslant N(p_n - q_n)\\ &\leqslant N(p_n - f) + N(f - q_n) \to 0\end{aligned}$$

that the operator S depends only on f, and not on the particular sequence of polynomials used to approximate f. We may therefore unambiguously define $T_f = S$. Observe that

$$\|T_f\| \leqslant N(f)\,; \tag{113}$$

indeed,

$$\begin{gathered}|N(p_n) - N(f)| \leqslant N(p_n - f) \to 0\,,\\ \big|\|p_n(A)\| - \|T_f\|\big| \leqslant \|p_n(A) - T_f\| \to 0\,,\end{gathered}$$

and the inequality (113) follows on passing to the limit in the inequalities

$$\|p_n(A)\| \leqslant N(p_n)\,.$$

Writing

$$\|f\|_\infty = \mathrm{LUB}\,\{|f(\lambda)| : \ \lambda \in \mathbf{R}\,\}\,,$$

it follows at once from (113) that

$$\|T_f\| \leqslant \|f\|_\infty \tag{114}$$

for all f in $\mathscr{L}$. We leave to the reader the verification of the following properties of the correspondence $f \mapsto T_f$ $(f \in \mathscr{L})$:

$$T_{f+g} = T_f + T_g \tag{115}$$

$$T_{af} = a\,T_f \tag{116}$$

$$T_{fg} = T_f T_g \tag{117}$$

$$f \geqslant 0 \quad \text{implies} \quad T_f \geqslant 0\,; \tag{118}$$

properties (115)–(117) may be deduced from (107)–(109) by obvious continuity arguments, and (118) follows immediately from (117) (cf. [1, p. 1050]).

We have amassed the hypotheses of Theorem 19; accordingly, there is a real spectral measure E such that

$$\int f\,dE = T_f \tag{119}$$

for all f in $\mathscr{L}$.

Let us show next that E is compact, indeed,

$$\bigwedge(E) \subset [a,b]\,. \tag{120}$$

Since E is necessarily regular (Theorem 18), it will suffice to show that $E(C) = 0$ for every compact subset C of the open set $\mathbf{R} - [a,b]$. For such a set C, we may choose a function f in $\mathscr{L}$ such that $f \geqslant \chi_C$; since C and $[a,b]$ are disjoint, we may assume, moreover, that $f = 0$ on $[a,b]$. Evidently $T_f = 0$ (e.g. by (113)), thus

$$\int f\,dE = 0$$

by (119); since, for each vector x,

$$\big((\textstyle\int f\,dE)x \big| x\big) = \textstyle\int f\,d\mu_x \geqslant \mu_x(C)$$

(where, as usual, $\mu_x(M) = (E(M)x\big|x)$ for all Borel sets M), we conclude that $\mu_x(C) = 0$. Thus

$$(E(C)x\big|x) = \mu_x(C) = 0$$

for each vector x, and so $E(C) = 0$.

Now that E is known to be compact, arbitrary continuous real functions f on $\mathbf{R}$ become eligible for integration; moreover, since $\bigwedge(E) \subset [a,b]$, it follows from Theorem 24 that

$$\int f\,dE = \int \chi_{[a,b]} f\,dE\,. \tag{121}$$

Finally, to show that E is normalized, and that $\int \lambda \, dE = A$, it is sufficient to show that

$$\int p \, dE = p(A) \tag{122}$$

for every real polynomial p. Indeed, given a real polynomial p, let f be any function in $\mathscr{L}$ such that $f = p$ on $[a, b]$. In the definition of Tf, we are free to take $p_n = p$ for all n; then

$$T_f = \lim p_n(A) = p(A).$$

Since $\int f \, dE = T_f$ by (119), we have $\int f \, dE = p(A)$. Since

$$\chi_{[a,b]} f = \chi_{[a,b]} p$$

by the choice of f, we see from (121) that

$$\int p \, dE = \int \chi_{[a,b]} p \, dE = \int \chi_{[a,b]} f \, dE = \int f \, dE = p(A). \ \diamondsuit$$

There remains the task of constructing a complex spectral measure for each normal operator. If A is a normal operator, say with Cartesian decomposition $A = A_1 + iA_2$, we may construct for each A_k a real spectral measure E_k as in Theorem 32. Since $A_1 \leftrightarrow A_2$ by normality, it is easy to show, on the basis of Theorem 30, that $E_1 \leftrightarrow E_2$ (that is, $E_1(M_1) \leftrightarrow E_2(M_2)$ for all Borel sets M_1 and M_2 in $\mathbf{R}$). Our plan, copied from [5], is to "blend" the real spectral measures E_1 and E_2 into a complex spectral measure E; this requires a considerable amount of elementary but delicate technique, which will occupy us for the next two sections.

10. Measure on semirings

We summarize in this section some elementary measure-theoretic definitions and propositions which will be technically useful in the next section. The reason for setting them down explicitly is that they are slightly (but only slightly) out of the path beaten in [4]. All of the proofs are easy, and most of them are suppressed.

The term "semiring" has been defined in various ways by different writers, depending on the requirements of the context; for our purposes, the following concept will suffice:

Definition 25. A non empty class $\mathscr{P}$ of sets will be called a *semiring* in case: if M and N are sets in $\mathscr{P}$, then

(i) $M \cap N$ is in $\mathscr{P}$, and

(ii) $M - N$ is the union of finitely many mutually disjoint sets in $\mathscr{P}$.

Every semiring contains the empty set $\varnothing$, and in (ii) one can suppose $M \supset N$ without loss of generality. A semiring is a ring if and only if it is closed under finite unions. The important example for our purposes is covered by the following proposition, to the effect that the "Cartesian product" of two semirings (and in particular, of two rings) is itself a semiring:

Proposition 3. *If $\mathscr{P}_k$ is a semiring of subsets of X_k $(k = 1, 2)$, and if*

$$\mathscr{P} = \{M_1 \times M_2 :\ M_1 \in \mathscr{P}_1\,,\ M_2 \in \mathscr{P}_2\,\}\,,$$

then $\mathscr{P}$ is also a semiring. In particular, if the $\mathscr{P}_k$ are rings, then $\mathscr{P}$ is a semiring.

A reasonably close model for the proof of Proposition 3 can be read out of [4, p. 139]. Since our basic measure-theoretic techniques are formulated in terms of rings, it is important to know the structure of the ring generated by a semiring (cf. [4, p. 139]):

Proposition 4. *If $\mathscr{P}$ is a semiring, then the ring generated by $\mathscr{P}$ coincides with the class of all finite unions*

$$\bigcup_1^n M_i$$

where $M_1, \ldots, M_n$ are mutually disjoint sets in $\mathscr{P}$.

Propositions 3 and 4 yield at once:

Proposition 5. *If $\mathscr{P}_k$ is a semiring of subsets of X_k $(k = 1, 2)$, and if*

$$\mathscr{P} = \{M_1 \times M_2 :\ M_1 \in \mathscr{P}_1\,,\ M_2 \in \mathscr{P}_2\,\}\,,$$

then the ring generated by $\mathscr{P}$ coincides with the class of all finite unions

$$\bigcup_{i=1}^n M_1^i \times M_2^i\,,$$

where $M_k^i \in \mathscr{P}_k$, and the rectangles $M_1^i \times M_2^i$ are mutually disjoint.

Our application of Proposition 5 in the next section will be to the case that $\mathscr{P}_1$ and $\mathscr{P}_2$ are rings. Moreover, we shall be given measures on each of these rings, and wish to define in some way (actually a rather peculiar way) a measure on the ring generated by $\mathscr{P}$. The first step is to define an appropriate set function on $\mathscr{P}$. Accordingly, it is convenient to broaden the notion of "measure" to semirings:

DEFINITION 26. If $\mathscr{P}$ is a semiring, a *measure* on $\mathscr{P}$ is a non negative, extended real valued set function τ on $\mathscr{P}$ such that (i) $\tau(\varnothing) = 0$, and (ii) if M_n is a sequence of mutually disjoint sets in $\mathscr{P}$ whose union M is also in $\mathscr{P}$, then $\tau(M) = \sum_1^\infty \tau(M_n)$.

When $\mathscr{P}$ is a ring, this agrees with the usual concept of measure [4, p. 30]. A measure on a semiring $\mathscr{P}$ is uniquely extendable to a measure on the ring generated by $\mathscr{P}$ (cf. [4, p. 35] or [11, p. 94]):

PROPOSITION 6. *If τ is a measure on a semiring $\mathscr{P}$, and $\mathscr{R}$ is the ring generated by $\mathscr{P}$, there exists one and only one measure on $\mathscr{R}$ which extends τ.*

In the presence of finiteness, we may proceed uniquely to $\mathfrak{S}(\mathscr{R}) = \mathfrak{S}(\mathscr{P})$ (cf. [4, p. 54]):

PROPOSITION 7. *If τ is a measure on a semiring $\mathscr{P}$, and if $\tau(M) < \infty$ for every M in $\mathscr{P}$, then there exists one and only one measure on $\mathfrak{S}(\mathscr{P})$ which extends τ.*

When there is a topology on the underlying set, we may make the following definition [11, p. 98]:

DEFINITION 27. Let $\mathscr{P}$ be a semiring of subsets of a topological space. A real-valued set function τ on $\mathscr{P}$ is said to be *regular* in case: given any set M in $\mathscr{P}$, and given any $\varepsilon > 0$, there exist sets C and U in $\mathscr{P}$ such that

$$C \subset M \subset U,$$

C is compact, U is open, and

$$\tau(U) - \varepsilon \leqslant \tau(M) \leqslant \tau(C) + \varepsilon.$$

The following proposition [11, p. 98] is decisive for the next section (and therefore for the proof of the spectral theorem for a normal operator given in Section 12):

PROPOSITION 8. *Let $\mathscr{P}$ be a semiring of subsets of a topological space, and suppose that τ is a non negative real valued set function on $\mathscr{P}$. Assume that τ is finitely additive, and that τ is regular in the sense of the preceding definition. Then τ is a measure on the semiring $\mathscr{P}$.*

Proof. Finite additivity means that if $M_1, \ldots, M_n$ is a finite class of mutually disjoint sets in $\mathscr{P}$ whose union M is also in $\mathscr{P}$, then

$$\tau(M) = \sum_{i=1}^{n} \tau(M_i)\,.$$

The problem is to show that τ is countably additive in the sense of Definition 26.*

We assert that τ is *monotone* and *subadditive*. That is, if M, N, and $M_1, \ldots, M_n$ are sets in $\mathscr{P}$, then $M \subset N$ implies $\tau(M) \leqslant \tau(N)$, and

$$M \subset \bigcup_{i=1}^{n} M_i \quad \text{implies} \quad \tau(M) \leqslant \sum_{i=1}^{n} \tau(M_i)\,.$$

Rather than grind out the easy details, we indicate the following alternative. Let $\mathscr{R}$ be the ring generated by $\mathscr{P}$, and let μ be the finitely additive extension of τ to $\mathscr{R}$ (constructed just as in the usual proof of Proposition 6). Then μ is also non negative, and the usual ring theoretic arguments for monotonicity and subadditivity are available.

Suppose now that M_n is a sequence of mutually disjoint sets in $\mathscr{P}$ whose union M is also in $\mathscr{P}$. For each n, we have

$$\bigcup_{i=1}^{n} M_i \subset M\,,$$

hence by the finite additivity and monotonicity of μ,

$$\sum_{i=1}^{n} \tau(M_i) = \sum_{i=1}^{n} \mu(M_i) = \mu\big(\bigcup_{i=1}^{n} M_i\big) \leqslant \mu(M) = \tau(M)\,;$$

since n is arbitrary,

$$\sum_{i=1}^{\infty} \tau(M_i) \leqslant \tau(M)\,.$$

It remains to show that

$$\tau(M) \leqslant \sum_{i=1}^{\infty} \tau(M_i)\,,$$

* For the special case that $\mathscr{P}$ is a ring, the result (finite additivity plus regularity implies countable additivity) is attributed to A.D. Alexandroff in a paper by Z. Riečanová [19] devoted to generalizing it. For generalizations to set functions taking values in certain topological groups, see the paper of J.E. Hunneycutt, Jr. [17].

and the argument for this is a familiar chestnut from the classical Borel-Lebesgue theory. Given any $\varepsilon > 0$, it is sufficient to prove that

$$\tau(M) \leqslant \sum_{i=1}^{\infty} \tau(M_i) + 2\varepsilon .$$

By regularity, we may choose a compact set C in $\mathscr{P}$ such that $C \subset M$ and

$$\tau(M) \leqslant \tau(C) + \varepsilon .$$

We may also choose, for each i, an open set U_i in $\mathscr{P}$ such that $M_i \subset U_i$ and

$$\tau(U_i) \leqslant \tau(M_i) + \varepsilon/2^i .$$

Then

$$C \subset M = \bigcup_{i=1}^{\infty} M_i \subset \bigcup_{i=1}^{\infty} U_i ,$$

and so by compactness we have

$$C \subset \bigcup_{i=1}^{n} U_i$$

for a suitable integer n. Since τ is subadditive, we have

$$\tau(C) \leqslant \sum_{i=1}^{n} \tau(U_i) \leqslant \sum_{i=1}^{n} [\tau(M_i) + \varepsilon/2^i] < \sum_{i=1}^{\infty} \tau(M_i) + \varepsilon$$

and so

$$\tau(M) \leqslant \tau(C) + \varepsilon < \sum_{i=1}^{\infty} \tau(M_i) + \varepsilon + \varepsilon . \ \diamondsuit$$

11. Amalgamation

Our main objective, attained in the next section, is to associate with each normal operator A a suitable complex spectral measure. This is accomplished by decomposing A into its Hermitian components, $A = A_1 + iA_2$, constructing a real spectral measure for each A_k via Theorem 32, and then blending these two spectral measures into a spectral measure for A. In this section we describe the appropriate blending technique.

The following notations are fixed for the rest of the section. We assume given a pair of PO-measures $(X_1, \mathscr{S}_1, E_1)$ and $(X_2, \mathscr{S}_2, E_2)$, where $\mathscr{S}_k$ is a σ-ring of subsets of X_k $(k = 1, 2)$. We wish to construct a PO-measure E on the product σ-ring $\mathscr{S}_1 \times \mathscr{S}_2$ such that

$$E(M_1 \times M_2) = E_1(M_1)E_2(M_2)$$

for every measurable rectangle $M_1 \times M_2$ (cf. [4, p. 140] for the terminology). Since the product of two Hermitian operators is Hermitian if and only if the two operators commute, it is clearly necessary to assume, and we do henceforth assume, that E_1 *commutes* with E_2 in the sense that

$$E_1(M_1) \leftrightarrow E_2(M_2)$$

for all M_1 and M_2; a concise notation for this is

$$E_1 \leftrightarrow E_2\,. \tag{C}$$

It will not be necessary to assume in this section that the operators $E_1(M_1)$ commute among themselves, although this will be the case when E_1 is a spectral measure (Theorem 3). Similarly, it is not assumed that the operators $E_2(M_2)$ commute among themselves.

I do not know whether such an E can be constructed without further hypotheses.* Before stating some extra conditions which will be sufficient for our purposes, a definition is in order. If F is a PO-measure defined on a σ-ring $\mathscr{S}$ of subsets of a topological space X, we shall say that F is *biregular* in case:

$$F(M) = \text{LUB}\,\{F(C) : \ C \subset M\,,\ C \text{ compact},\ C \in \mathscr{S}\}\,, \tag{123a}$$

$$F(M) = \text{GLB}\,\{F(U) : \ U \supset M\,,\ U \text{ open},\ U \in \mathscr{S}\}\,, \tag{123b}$$

for each M in $\mathscr{S}$ (observe that the indicated LUB and GLB exist by Proposition 1).

We assume, for the rest of the section, that X_1 and X_2 are topological spaces, and that E_1 and E_2 are biregular in the sense of the foregoing definition.

Examples

1. If X is a locally compact space, every Baire PO-measure F on X is automatically biregular (Theorem 18).

* It can't; for counterexamples see the papers of R. M. Dudley [16] and A.G. Miamee and H. Salehi [18].

2. For the Spectral Theorem, all we need is the case that $X_1 = X_2 = \mathbf{R}$, and E_1, E_2 are normalized compact spectral measures (cf. Definition 18).

*3. If X is a locally compact space, and F is a Borel PO-measure on X (Definition 13), then the conditions (123a) and (123b) imply one another; this can be deduced from the numerical case via Proposition 1 and Theorem 1 (cf. [4, p. 228]). In particular, every regular Borel PO-measure (Definition 14) is biregular.

*4. Every regular weakly Borel PO-measure (Definition 15) is biregular.[1]

We write $\mathscr{P}$ for the class of all measurable rectangles $M_1 \times M_2$, and we write $\mathscr{R}$ for the ring generated by $\mathscr{P}$. Thus $\mathscr{P}$ is a semiring, and $\mathscr{R}$ is the class of all finite disjoint unions of sets in $\mathscr{P}$ (cf. Proposition 5 of the preceding section).

To simplify notation in the next three lemmas, it is convenient fo fix a vector z of the underlying Hilbert space; we define a set function τ on $\mathscr{P}$ by the formula

$$\tau(M_1 \times M_2) = \big(E_1(M_1)E_2(M_2)z \big| z\big).$$

Observe that τ is unambiguously defined by this formula; this is clear if $M_1 \times M_2 \neq \varnothing$, whereas if $M_1 \times M_2 = \varnothing$, then either $M_1 = \varnothing$ of $M_2 = \varnothing$, and in either case $E_1(M_1)E_2(M_2) = 0$. In particular, we note that $\tau(\varnothing) = 0$. It follows from the commutativity relation (C) that

$$E_1(M_1)E_2(M_2) \geqslant 0$$

for all M_1 and M_2 (cf. [8, Sec. 104]), consequently τ is a non negative real valued set function on $\mathscr{P}$. The object of the first three lemmas is to prove that τ is in fact a measure on the semiring $\mathscr{P}$.

Lemma 1. *If $M_1 \times M_2 \in \mathscr{P}$, and if $M_1^1, M_1^2, \dots, M_1^n$ are mutually disjoint sets in $\mathscr{S}_1$ whose union is M_1, then*

$$\tau(M_1 \times M_2) = \sum_{i=1}^{n} \tau(M_1^i \times M_2)\,.$$

Proof. So to speak, we are asserting that for fixed M_2, $\tau(M_1 \times M_2)$ is a finitely additive function of M_1. Writing $y = \big(E_2(M_2)\big)^{1/2} z$, and citing

[1] In the first edition, the contrary was asserted (and, fortunately, never invoked); the error is corrected in [14, p. 989, Remark 5].

the commutativity relation (C), we have

$$\begin{aligned}\tau(M_1 \times M_2) &= \big(E_1(M_1)E_2(M_2)z\big|z\big) = (E_1(M_1)y\big|y) \\ &= \sum_{i=1}^{n}(E_1(M_1^i)y\big|y) \\ &= \sum_{i=1}^{n}(E_1(M_1^i)E_2(M_2)z\big|z) \\ &= \sum_{i=1}^{n}\tau(M_1^i \times M_2)\,.\ \diamondsuit\end{aligned}$$

The obvious "dual" of Lemma 1 also holds, and is proved similarly.

LEMMA 2. *τ is finitely additive on $\mathscr{P}$.*

Proof. (The reader is urged to draw a picture to follow what is really a very elementary maneuver.) Suppose $M_1^1 \times M_2^1, M_1^2 \times M_2^2, \ldots, M_1^n \times M_2^n$ is a finite class of mutually disjoint sets in $\mathscr{P}$, whose union is also a set $M_1 \times M_2$ in $\mathscr{P}$:

$$\text{(a)} \qquad M_1 \times M_2 = \bigcup_{i=1}^{n} M_1^i \times M_2^i\,.$$

We are to show that

$$\tau(M_1 \times M_2) = \sum_{i=1}^{n}\tau(M_1^i \times M_2^i)\,.$$

Since $\tau(\varnothing) = 0$, we may assume without loss of generality that these rectangles are all non empty. It then follows that

$$\text{(b)} \qquad M_1 = \bigcup_{i=1}^{n} M_1^i\,,$$

though the terms on the right of (b) need not be mutually disjoint.

Let $S^1, S^2, \ldots, S^m$ be a finite sequence of mutually disjoint nonempty sets in $\mathscr{S}_1$, whose union is M_1, and such that each M_1^i is the union of certain of the S^j (cf. [2, p. 71]). Then

$$\text{(c)} \qquad M_1 \times M_2 = \bigcup_{j=1}^{m} S^j \times M_2\,,$$

and the terms of the union are mutually disjoint.

For each $i = 1, \dots, n$, define

$$J_i = \{j : \ S^j \subset M_1^i\}.$$

Thus for each i, we have

$$M_1^i = \bigcup_{j \in J_i} S^j ,$$

and the terms of the union are of course mutually disjoint; it follows that

$$\text{(d)} \qquad M_1^i \times M_2^i = \bigcup_{j \in J_i} S^j \times M_2^i ,$$

and the terms on the right side of (d) are also mutually disjoint.

On the other hand, for each $j = 1, \dots, m$, define

$$I_j = \{i : \ S^j \subset M_1^i\};$$

since $S^j \subset M_1^i$ if and only if $S^j \cap M_1^i \neq \varnothing$, it follows from (a), on taking intersection with $S^j \times M_2$, that

$$\text{(e)} \qquad S^j \times M_2 = \bigcup_{i \in I_j} S^j \times M_2^i ,$$

and the terms on the right of (e) are also mutually disjoint.

Applying Lemma 1 to (c), we have

$$\text{(c}'\text{)} \qquad \tau(M_1 \times M_2) = \sum_{j=1}^{m} \tau(S^j \times M_2).$$

Applying Lemma 1 to (d), we have, for each i,

$$\text{(d}'\text{)} \qquad \tau(M_1^i \times M_2^i) = \sum_{j \in J_i} \tau(S^j \times M_2^i).$$

Applying the "dual" of Lemma 1 to (e), we have, for each j,

$$\text{(e}'\text{)} \qquad \tau(S^j \times M_2) = \sum_{i \in I_j} \tau(S^j \times M_2^i).$$

Substituting (e$'$) in (c$'$), we have

$$\text{(f)} \qquad \tau(M_1 \times M_2) = \sum_{j=1}^{m} \sum_{i \in I_j} \tau(S^j \times M_2^i).$$

Observe that not all combinations of i and j need occur among the terms $\tau(S^j \times M_2^i)$ in (f). Anyway $i \in I_j$ if and only if $j \in J_i$, and so reassociation of (f) yields

$$\begin{aligned}\tau(M_1 \times M_2) &= \sum_{i=1}^{n} \sum_{j \in J_i} \tau(S^j \times M_2^i) \\ &= \sum_{i=1}^{n} \tau(M_1^i \times M_2^i)\,,\end{aligned}$$

the last equality resulting from (d′). ♢

So far, E_1 and E_2 could have been any two PO-measures (defined on any two rings) satisfying the commutativity relation (C). It is in proving the countable additivity of τ that we shall appeal to topology. In view of Proposition 1, we have, by the assumed biregularity of E_1 and E_2:

$$\begin{aligned}&(124\text{a}) && (E_1(M_1)z|z) = \mathrm{LUB}\,\{(E_1(C_1)z|z):\ C_1 \subset M_1\,,\\ &&& \qquad C_1 \text{ compact},\ C_1 \in \mathscr{S}_1\}\,,\\ &(124\text{b}) && (E_1(M_1)z|z) = \mathrm{GLB}\,\{(E_1(U_1)z|z):\ U_1 \supset M_1\,,\\ &&& \qquad U_1 \text{ open},\ U_1 \in \mathscr{S}_1\}\,,\\ &(125\text{a}) && (E_2(M_2)z|z) = \mathrm{LUB}\,\{(E_2(C_2)z|z):\ C_2 \subset M_2\,,\\ &&& \qquad C_2 \text{ compact},\ C_2 \in \mathscr{S}_2\}\,,\\ &(125\text{b}) && (E_2(M_2)z|z) = \mathrm{GLB}\,\{(E_2(U_2)z|z):\ U_2 \supset M_2\,,\\ &&& \qquad U_2 \text{ open},\ U_2 \in \mathscr{S}_2\}\,,\end{aligned}$$

for all $M_1 \in \mathscr{S}_1$ and $M_2 \in \mathscr{S}_2$.

LEMMA 3. *Let $M_1 \times M_2 \in \mathscr{P}$. Given any $\varepsilon > 0$, there exist compact sets C_1, C_2 and open sets U_1, U_2 (in $\mathscr{S}_1, \mathscr{S}_2$, respectively) such that*

$$C_1 \times C_2 \subset M_1 \times M_2 \subset U_1 \times U_2\,,$$

and

$$\tau(U_1 \times U_2) - \varepsilon \leqslant \tau(M_1 \times M_2) \leqslant \tau(C_1 \times C_2) + \varepsilon\,.$$

Proof. For use later in the proof, we note that if R, S, T are positive operators such that $R \leqslant S$, $T \leftrightarrow R$, and $T \leftrightarrow S$, then $TR \leqslant TS$; indeed, since $T^{1/2} \in \{T\}''$, we have, for any vector y,

$$(TRy|y) = (RT^{1/2}y|T^{1/2}y) \leqslant (ST^{1/2}y|T^{1/2}y) = (TSy|y)\,.$$

We know from Theorem 5 that the E_k are bounded, thus there exists a real number $K > 0$ such that

$$\|E_1(N_1)\| \leqslant K\,, \qquad \|E_2(N_2)\| \leqslant K\,,$$

for all $N_1 \in \mathscr{S}_1$ and $N_2 \in \mathscr{S}_2$.

Let $M_1 \times M_2 \in \mathscr{P}$ and $\varepsilon > 0$ be given. According to (124a) and (125a), there exist compact sets $C_1 \in \mathscr{S}_1$ and $C_2 \in \mathscr{S}_2$ such that $C_k \subset M_k$ and

$$\big(E_k(M_k - C_k)z\big|z\big) \leqslant \varepsilon/2K\,.$$

Since the E_k are additive, we have

$$\begin{aligned} &E_1(M_1)E_2(M_2) = \\ &\qquad E_1(M_1)E_2(M_2 - C_2) + E_1(M_1 - C_1)E_2(C_2) + E_1(C_1)E_2(C_2)\,; \end{aligned}$$

then

$$\begin{aligned} \tau(M_1 \times M_2) &= \big(E_1(M_1)E_2(M_2)z\big|z\big) \\ &= \tau(C_1 \times C_2) + \big(E_1(M_1)E_2(M_2 - C_2)z\big|z\big) + \big(E_1(M_1 - C_1)E_2(C_2)z\big|z\big) \\ &\leqslant \tau(C_1 \times C_2) + K\big(E_2(M_2 - C_2)z\big|z\big) + K\big(E_1(M_1 - C_1)z\big|z\big) \\ &\leqslant \tau(C_1 \times C_2) + K(\varepsilon/2K) + K(\varepsilon/2K) \\ &= \tau(C_1 \times C_2) + \varepsilon\,. \end{aligned}$$

On the other hand, by (124b) and (125b), there exist open sets $U_1 \in \mathscr{S}_1$ and $U_2 \in \mathscr{S}_2$ such that $M_k \subset U_k$ and

$$\big(E_k(U_k - M_k)z\big|z\big) \leqslant \varepsilon/2K\,.$$

Then

$$\begin{aligned} &E_1(U_1)E_2(U_2) = \\ &\qquad E_1(U_1)E_2(U_2 - M_2) + E_1(U_1 - M_1)E_2(M_2) + E_1(M_1)E_2(M_2)\,, \end{aligned}$$

and a calculation similar to the earlier one yields

$$\tau(U_1 \times U_2) \leqslant \tau(M_1 \times M_2) + \varepsilon\,. \ \diamondsuit$$

The sets $C_1 \times C_2$ and $U_1 \times U_2$ in Lemma 3 are compact and open, respectively; citing Proposition 8 of the preceding section, we may summarize as follows:

LEMMA 4. *For each vector* x, *the formula*

$$\tau_x(M_1 \times M_2) = \big(E_1(M_1)E_2(M_2)x\big|x\big)$$

defines a measure on the semiring $\mathscr{P}$.

Recall that $\mathscr{R}$ is the ring generated by $\mathscr{P}$; citing Propositions 5 and 6 of the preceding section, we have at once:

LEMMA 5. *For each vector* x, *there exists one and only one* (*finite*) *measure* μ_x *on the ring* $\mathscr{R}$ *which extends* τ_x; *we may describe* μ_x *as the unique measure on* $\mathscr{R}$ *such that*

$$\mu_x(M_1 \times M_2) = \big(E_1(M_1)E_2(M_2)x\big|x\big)$$

for all measurable rectangles $M_1 \times M_2$.

We may now construct a positive operator valued set function E on the ring $\mathscr{R}$:

LEMMA 6. *For each* M *in* $\mathscr{R}$, *there exists one and only one* (*positive*) *operator* $E(M)$ *such that*

$$(E(M)x\big|x) = \mu_x(M) \tag{126}$$

for all vectors x (*where the measures* μ_x *are defined as in Lemma* 5).

Proof. Suppose $M \in \mathscr{R}$. By Proposition 5 of the preceding section, we may write

$$M = \bigcup_{i=1}^{n} M_1^i \times M_2^i\,, \tag{127}$$

where $M_k^i \in \mathscr{R}_k$ and the rectangles $M_1^i \times M_2^i$ are mutually disjoint. Define an operator T by the formula

$$T = \sum_{i=1}^{n} E_1(M_1^i)E_2(M_2^i)\,. \tag{128}$$

For each vector x, we have, by the additivity of the measure μ_x,

$$(Tx\big|x) = \sum_{i=1}^{n} \big(E_1(M_1^i)E_2(M_2^i)x\big|x\big) = \sum_{i=1}^{n} \mu_x(M_1^i \times M_2^i) = \mu_x(M)\,,$$

thus

$$(Tx|x) = \mu_x(M)\,. \tag{129}$$

In particular, it is clear from the formula (129) that the operator T defined in (128) is independent of the particular decomposition (127) of M. We define $E(M) = T$. $\diamondsuit$

With notation as in Lemma 6, we have

$$\big(E(M_1 \times M_2)x\big|x\big) = \mu_x(M_1 \times M_2) = \tau_x(M_1 \times M_2) = \big(E_1(M_1)E_2(M_2)x\big|x\big)$$

for all measurable rectangles $M_1 \times M_2$ and all vectors x, thus

$$E(M_1 \times M_2) = E_1(M_1)E_2(M_2)\,. \tag{130}$$

LEMMA 7. *E is a bounded* PO-*measure on $\mathscr{R}$. If the E_k are spectral measures, then so is E.*

Proof. Of course E is the positive operator valued set function defined in Lemma 6. Since the μ_x are finite measures on $\mathscr{R}$ (Lemma 5), it follows from Theorem 1 that E is a PO-measure.

As noted in the proof of Lemma 3, the E_k are bounded, say

$$\|E_1(M_1)\| \leqslant K_1 < \infty\,,$$
$$\|E_2(M_2)\| \leqslant K_2 < \infty\,,$$

for all M_1 and M_2. It follows from (130) that

$$\|E(M_1 \times M_2)\| \leqslant K_1K_2$$

for all measurable rectangles $M_1 \times M_2$.

Suppose, now, that M is any set in $\mathscr{R}$. Obviously $M \subset M_1 \times M_2$ for a suitable measurable rectangle $M_1 \times M_2$, and so by the monotonicity of E we have

$$0 \leqslant E(M) \leqslant E(M_1 \times M_2)\,;$$

then

$$\|E(M)\| \leqslant \|E(M_1 \times M_2)\| \leqslant K_1K_2\,,$$

and this proves that E is bounded.

Assume, finally, that the E_k are projection-valued. If $M_1 \times M_2$ and $N_1 \times N_2$ are disjoint measurable rectangles, then either $M_1 \cap N_1 = \varnothing$ or

$M_2 \cap N_2 = \varnothing$, and in either case (cf. Theorem 3, and the commutativity relation (C)),

$$\begin{aligned} 0 &= E_1(M_1 \cap N_1)E_2(M_2 \cap N_2) \\ &= E_1(M_1)E_1(N_1)E_2(M_2)E_2(N_2) \\ &= E_1(M_1)E_2(M_2)E_1(N_1)E_2(N_2) \\ &= E(M_1 \times M_2)E(N_1 \times N_2)\,; \end{aligned}$$

thus the projections [5, p. 47] $E(M_1 \times M_2)$ and $E(N_1 \times N_2)$ are orthogonal, and hence their sum is a projection. It follows at once from formula (127), and the additivity of E, that E is projection-valued. $\diamondsuit$

Since the σ-ring generated by $\mathscr{R}$ is $\mathscr{S}_1 \times \mathscr{S}_2$, Lemma 7 and Theorem 7 yield the following key result:

THEOREM 33. *If, for $k = 1, 2$, X_k is a topological space, $\mathscr{S}_k$ is a σ-ring of subsets of X_k, and E_k is a biregular* PO-*measure on $\mathscr{S}_k$, and if $E_1 \leftrightarrow E_2$, then there exists one and only one* PO-*measure E on $\mathscr{S}_1 \times \mathscr{S}_2$ such that*

$$E(M_1 \times M_2) = E_1(M_1)E_2(M_2) \tag{130}$$

for all measurable rectangles $M_1 \times M_2$. If, moreover, the E_k are spectral measures, then so is E.

DEFINITION 28. With notation as in Theorem 33, we call E the *amalgam* of E_1 and E_2.

In view of formula (1), we have at once:

COROLLARY. *If E_k is a Baire* PO-*measure on a locally compact space X_k $(k = 1, 2)$, and if $E_1 \leftrightarrow E_2$, there exists one and only one Baire* PO-*measure E on $X_1 \times X_2$ such that*

$$E(M_1 \times M_2) = E_1(M_1)E_2(M_2)$$

for all Baire sets M_1 and M_2 (in X_1 and X_2, respectively). If, moreover, the E_k are spectral measures, then so is E.

Incidentally, if the underlying Hilbert space happens to be one-dimensional, the Corollary yields the usual Cartesian product of two finite Baire measures (cf. 4, p. 144]).

Continuing with the notations of Theorem 33, let us assume further that the $\mathscr{S}_k$ are σ-algebras, and that $E_1(X_1) = E_2(X_2) = I$; we consider now the integral with respect to E of certain special functions on $X_1 \times X_2$. We first introduce a convenient notation for the functions in question:

Definition 29. If f is a complex function on X_1, we define a complex function f' on $X_1 \times X_2$ by the formula

$$f'(\alpha, \beta) = f(\alpha).$$

Similarly, if g is a complex function on X_2, we define a complex function g'' on $X_1 \times X_2$ by the formula

$$g''(\alpha, \beta) = g(\beta).$$

Remarks

1. The functions f' and g'' are constant on the vertical and horizontal slices of $X_1 \times X_2$, respectively. A simple example, and the one we are most interested in, is the case that $X_1 = X_2 = \mathbf{R}$, and $f(\alpha) = \alpha$; on identifying $\mathbf{R} \times \mathbf{R}$ with $\mathbf{C}$, f' becomes the mapping $\alpha + i\beta \mapsto \alpha$, that is, $f'(\lambda)$ is the real part of λ, for all $\lambda \in \mathbf{C}$.

2. The correspondences $f \mapsto f'$ and $g \mapsto g''$ evidently preserve sums, scalar multiples, products and conjugates. Moreover, if $f_n \to f$ pointwise [resp. uniformly] then $f'_n \to f'$ pointwise [resp. uniformly]; similarly for the correspondence $g \mapsto g''$. If f and g are bounded, then so are f' and g'', and $\|f'\|_\infty = \|f\|_\infty$, $\|g''\|_\infty = \|g\|_\infty$.

3. If f and g are continuous, then so are f' and g''.

4. If $M_1 \in \mathscr{S}_1$ and $M_2 \in \mathscr{S}_2$, then

$$(\chi_{M_1})' = \chi_{M_1 \times X_2}, \quad (\chi_{M_2})'' = \chi_{X_1 \times M_2}.$$

It then follows easily from Remark 2 that if f and g are measurable relative to $\mathscr{S}_1$ and $\mathscr{S}_2$, respectively, then f' and g'' are measurable relative to $\mathscr{S}_1 \times \mathscr{S}_2$.

Theorem 34. *With notations as in Theorem* 33, *assume, moreover, that* $\mathscr{S}_1, \mathscr{S}_2$ *are* σ*-algebras, and that* E_1, E_2 *are normalized, that is,* $E_1(X_1) = E_2(X_2) = I$.

If f *and* g *are bounded measurable complex functions on* X_1 *and* X_2, *respectively, then*

$$\int f' \, dE = \int f \, dE_1,$$
$$\int g'' \, dE = \int g \, dE_2.$$

Proof. Since f' and g'' are also bounded measurable functions, all integrals in sight exist (see Section 5). Fix a vector x, and consider the measures

$$\mu_1(M_1) = (E_1(M_1)x \big| x),$$
$$\mu_2(M_2) = (E_2(M_2)x \big| x),$$
$$\mu(M) = (E(M)x \big| x),$$

on $\mathscr{S}_1$, $\mathscr{S}_2$, and $\mathscr{S}_1 \times \mathscr{S}_2$, respectively. It is to be shown that

$$(*) \qquad \int f'\, d\mu = \int f\, d\mu_1\,,$$

$$(**) \qquad \int g''\, d\mu = \int g\, d\mu_2\,,$$

(cf. (25)). To verify $(*)$, consider first the case that f is the characteristic function of a measurable set M_1, $f = \chi_{M_1}$; then $f' = \chi_{M_1 \times X_2}$, and so

$$\begin{aligned} \int f'\, d\mu &= \mu(M_1 \times X_2) = \big(E(M_1 \times X_2)x\big|x\big) \\ &= \big(E_1(M_1)E_2(X_2)x\big|x\big) = (E_1(M_1)x\big|x) \\ &= \mu_1(M_1) = \int f\, d\mu_1\,. \end{aligned}$$

The case that f is simple follows by linearity. If f is an arbitrary bounded measurable function, choose a sequence of simple functions f_n such that $\|f_n - f\|_\infty \to 0$; then also $\|f_n' - f'\|_\infty \to 0$. Since

$$\int f_n'\, d\mu = \int f_n\, d\mu_1$$

for all n, we obtain $(*)$ on passing to the limit, by the simplest of all "convergence theorems". $\diamondsuit$

For the rest of the section (except for the remarks at the end) we shall assume that the spaces X_1 and X_2 are locally compact, σ-compact, and metrizable. It follows that the space $X_1 \times X_2$ also has these properties. Thus $\mathscr{B}_0 = \mathscr{B} = \mathscr{B}_{\mathrm{w}}$ for each of the spaces $X_1, X_2, X_1 \times X_2$ (see Section 6). As usual in such situations, we switch to the "Borel" terminology. We assume further that $\mathscr{S}_k$ is the class of all Borel sets in X_k $(k = 1, 2)$, and so $\mathscr{S}_1 \times \mathscr{S}_2$ is (in this highly special case) the class of all Borel sets in $X_1 \times X_2$. Thus E_1, E_2, and E are regular Borel (= Baire = weakly Borel) PO-measures (notations as in Theorem 33). Concerning their spectra $\bigwedge(E_1), \bigwedge(E_2), \bigwedge(E)$, as defined in Section 6, we have the following result:

THEOREM 35. *Let X_1 and X_2 be σ-compact metrizable locally compact spaces, and let E_k be a (necessarily regular) Borel* PO-*measure on X_k $(k = 1, 2)$. Assume that $E_1 \leftrightarrow E_2$, and let E be the amalgam of E_1 and E_2. Then*

$$\bigwedge(E) \subset \bigwedge(E_1) \times \bigwedge(E_2)\,;$$

in particular, if the E_k are compact, then so is E. If $E_1(X_1) = E_2(X_2) \neq 0$, then $E \neq 0$, and therefore $\bigwedge(E) \neq \varnothing$.

Proof. Let us abbreviate $\bigwedge_k = \bigwedge(E_k)$ and $\bigwedge = \bigwedge(E)$. To prove the indicated inclusion, equivalently

$$\complement(\bigwedge_1 \times \bigwedge_2) \subset \complement \bigwedge,$$

it is sufficient to show that E vanishes on the open set $\complement(\bigwedge_1 \times \bigwedge_2)$. Indeed, since

$$\complement(\bigwedge_1 \times \bigwedge_2) = [(\complement \bigwedge_1) \times X_2] \cup [\bigwedge_1 \times (\complement \bigwedge_2)],$$

we have

$$\begin{aligned} E[\complement(\bigwedge_1 \times \bigwedge_2)] &= E_1(\complement \bigwedge_1)E_2(X_2) + E_1(\bigwedge_1)E_2(\complement \bigwedge_2) \\ &= 0 \cdot E_2(X_2) + E_1(\bigwedge_1) \cdot 0 \end{aligned}$$

by formulas (130) and (59). Since $\bigwedge(E)$ is a closed set, the assertion concerning compactness is clear.

Finally, if $E_1(X_1) = E_2(X_2) = T \neq 0$, then

$$E(X_1 \times X_2) = E_1(X_1) \times E_2(X_2) = T^2 = T^*T \neq 0,$$

hence $\bigwedge(E) \neq \varnothing$ by the Corollary of Theorem 23. ◇

Finally, we adapt Theorem 34 to the context of compact PO-measures and continuous functions:

THEOREM 36. *Let X_1 and X_2 be σ-compact metrizable locally compact spaces, and let E_k be a normalized compact* PO-*measure on X_k $(k = 1, 2)$. Assume that $E_1 \leftrightarrow E_2$, and let E be the amalgam of E_1 and E_2.*

If f and g are continuous complex functions on X_1 and X_2, respectively, then

$$\int f'\, dE = \int f\, dE_1, \tag{131a}$$

$$\int g''\, dE = \int g\, dE_2. \tag{131b}$$

Proof. We know from Theorem 35 that E is a compact PO-measure on $X_1 \times X_2$, and that

$$\bigwedge(E) \subset \bigwedge(E_1) \times \bigwedge(E_2).$$

In particular, since f' and g'' are also continuous, all of the integrals in question exist (see Section 7). We use the abbreviations $\bigwedge_1$, $\bigwedge_2$, and $\bigwedge$ as in the proof of Theorem 35.

Let us show, for example, that (131a) holds. Define $h = \chi_{\wedge_1} f$; then h is a bounded Borel function, and so

$$(*) \qquad \int h'\,dE = \int h\,dE_1$$

by Theorem 34. Since

$$h' = \left(\chi_{\wedge_1}\right)' f' = \chi_{\wedge_1 \times X_2} f' ,$$

and since

$$\bigwedge \subset \bigwedge\nolimits_1 \times \bigwedge\nolimits_2 \subset \bigwedge\nolimits_1 \times X_2 ,$$

we have, by two applications of Definition 21,

$$\begin{aligned} \int h'\,dE &= \int \chi_{\wedge} h'\,dE = \int \chi_{\wedge}\chi_{\wedge_1 \times X_2} f'\,dE \\ &= \int \chi_{\wedge} f'\,dE = \int f'\,dE . \end{aligned}$$

Similarly

$$\int h\,dE_1 = \int \chi_{\wedge_1} f\,dE_1 = \int f\,dE_1 .$$

Substituting in $(*)$, we obtain (131a). $\diamondsuit$

*REMARKS

Suppose E_k is a regular Borel PO-measure on a locally compact space X_k $(k = 1, 2)$, such that $E_1 \leftrightarrow E_2$, and let E be the amalgam of E_1 and E_2. The domain of definition of E is the σ-ring $\mathscr{B}(X_1) \times \mathscr{B}(X_2)$, which may fall short of $\mathscr{B}(X_1 \times X_2)$ (recall that $\mathscr{B}(X)$ denotes the class of Borel sets in a space X). However, if F_k is the restriction of E_k to the class of Baire sets of X_k, then the amalgam F of F_1 and F_2 is a Baire PO-measure on $X_1 \times X_2$ and so may be uniquely extended to a regular Borel PO-measure E' on $X_1 \times X_2$ (Theorem 21). It is easy to see that E' extends E (cf. [2, p. 203]).

As the details are delicate, it may be of interest (even in the numerical case) to sketch the proof that E' extends E. I am indebted to Roy A. Johnson for the proof of the numerical case on which the following proof is modeled.

Fix a vector x, and define

$$\begin{aligned} \mu(M) &= (E(M)x \big| x) , \\ \nu(N) &= (E(N)x \big| x) , \\ \rho(A) &= (E'(A)x \big| x) , \end{aligned}$$

for all $M \in \mathscr{B}(X_1) \times \mathscr{B}(X_2)$, $N \in \mathscr{B}_0(X_1 \times X_2)$, and $A \in \mathscr{B}(X_1 \times X_2)$. Thus ρ is the unique regular Borel extension of the Baire measure ν, and the problem is to show that ρ is an extension of μ. We show first that

$$(*) \qquad \rho(C_1 \times C_2) = \mu(C_1 \times C_2),$$

for compact sets C_1 and C_2. Since the E_k are regular, we may choose, for each k, a compact G_δ D_k such that $C_k \subset D_k$ and

$$(E_k(C_k)x|x) = (E_k(D_k)x|x)$$

(cf. [2, p. 188]). Thus,

$$\big(E_k(D_k - C_k)x|x\big) = 0;$$

since $E_k(D_k - C_k) \geqslant 0$, we conclude (e.g. by the generalized Schwarz inequality [8, Sec. 104]) that

$$E_k(D_k - C_k)x = 0,$$

and so $E_k(C_k)x = E_k(D_k)x$. Then

$$\begin{aligned}\mu(C_1 \times C_2) &= \big(E(C_1 \times C_2)x|x\big) = \big(E_1(C_1)x|E_2(C_2)x\big)\\ &= \big(E_1(D_1)x|E_2(D_2)x\big) = \big(E(D_1 \times D_2)x|x\big)\\ &= \mu(D_1 \times D_2).\end{aligned}$$

Choose a compact G_δ D in $X_1 \times X_2$ such that $C_1 \times C_2 \subset D$ and

$$\rho(C_1 \times C_2) = \rho(D).$$

Let $D' = D \cap (D_1 \times D_2)$; then D' is a Baire set (in fact, a compact G_δ), and $C_1 \times C_2 \subset D'$. Since μ and ρ agree on Baire sets, we have

$$\mu(C_1 \times C_2) \leqslant \mu(D') \leqslant \mu(D) = \rho(D) = \rho(C_1 \times C_2),$$

and

$$\rho(C_1 \times C_2) \leqslant \rho(D') \leqslant \rho(D_1 \times D_2) = \mu(D_1 \times D_2) = \mu(C_1 \times C_2),$$

thus $(*)$ is verified.

Let $\mathscr{A}$ be the class of all finite unions of such rectangles $C_1 \times C_2$. Clearly $\mathscr{A}$ is closed under finite unions and intersections. It follows from $(*)$ that $\rho = \mu$ on $\mathscr{A}$ (Hint: $S \cup T - S = T - S \cap T$). Then $\rho = \mu$ on $\mathfrak{S}(\mathscr{A})$ [2, p. 185], and the proof is concluded by observing that

$$\mathfrak{S}(\mathscr{A}) = \mathscr{B}(X_1) \times \mathscr{B}(X_2)$$

(cf. [2, p. 118]).

*Exercise

With notation as in Theorem 33, E is also biregular (cf. [2, p. 199, Exer. 7, (iv)]).

12. The Spectral Theorem for a normal operator

With all the machinery we have clanking in the background, the proof of the Spectral Theorem is, and ought to be, easy:

THEOREM 37. (Spectral Theorem for a normal operator) *If* A *is a normal operator*, *there exists one and only one normalized compact complex spectral measure* E *such that*

$$A = \int \lambda \, dE \, .$$

Proof. Uniqueness is assured by Theorem 28, thus we are concerned here with existence. It will be convenient to use the notations u and v as in formulas (89, (90). Let

$$A = A_1 + iA_2$$

be the Cartesian decomposition of A. For each $k = 1, 2$, there exists, by Theorem 32, a normalized compact real spectral measure E_k such that

$$A_k = \int v \, dE_k \, .$$

It will be shown that $E_1 \leftrightarrow E_2$, and that the amalgam of E_1 and E_2 is the desired spectral measure E.

Given Borel sets M_1 and M_2 in $\mathbf{R}$, let us show that

$$E_1(M_1) \leftrightarrow E_2(M_2) \, .$$

Since $A_1 \leftrightarrow A_2$ by the normality of A, and since $E_2(M_2) \in \{A_2\}''$ by Theorem 30, we have

$$A_1 \leftrightarrow E_2(M_2) \, ;$$

it then follows from the relation $E_1(M_1) \in \{A_1\}''$ that $E_1(M_1) \leftrightarrow E_2(M_2)$.

Summarizing, $E_1 \leftrightarrow E_2$, and we may form the amalgam E of E_1 and E_2 (Corollary of Theorem 33). Granted that E is the desired spectral measure, it is interesting to note that our proof of the *existence* of E in the complex case has already made use of *uniquenss* for the real case via Theorem 30.

At any rate, we know that E is a spectral measure on the class of Borel sets of $\mathbf{R} \times \mathbf{R}$ (Corollary of Theorem 33), and is moreover regular (Theorem 18) and compact (Theorem 35). Since $E(\mathbf{R} \times \mathbf{R}) = E_1(\mathbf{R})E_2(\mathbf{R}) = I$, E is normalized; in particular, we note that $\bigwedge(E) \neq 0$ (Corollary of Theorem 23).

Consider the function $v(\alpha) = \alpha$ $(\alpha \in \mathbf{R})$, and the functions v' and v'' defined on $\mathbf{R} \times \mathbf{R}$ by the formulas

$$v'(\alpha, \beta) = v(\alpha) = \alpha\,,$$
$$v''(\alpha, \beta) = v(\beta) = \beta\,,$$

(cf. Definition 29). By Theorem 36 we have

$$\int v'\, dE = \int v\, dE_1\,,$$
$$\int v''\, dE = \int v\, dE_2\,,$$

thus

$$\int v'\, dE = A_1\,,$$
$$\int v''\, dE = A_2\,,$$

and consequently

$$\int (v' + iv'')\, dE = A_1 + iA_2 = A\,. \tag{132}$$

Thus if u is the function (89), we have, for any complex number $\lambda = \alpha + i\beta$,

$$u(\lambda) = \lambda = \alpha + i\beta = v'(\lambda) + i\,v''(\lambda)\,;$$

thus $u = v' + i\,v''$, and formula (132) yields the desired relation

$$\int u\, dE = A\,. \ \diamondsuit$$

Combining the Spectral Theorem with Theorem 31, we have at once:

THEOREM 38. *Let A be a normal operator, and let E be the unique normalized compact complex spectral measure such that*

$$A = \int \lambda \, dE .$$

Then:

$$\bigwedge(A) = \bigwedge(E) , \tag{133}$$

and

$$\|A\| = \mathrm{LUB}\,\{|\lambda| : \ \lambda \in \bigwedge(A)\,\} ; \tag{134}$$

in particular, $\bigwedge(A)$ *is non empty. Moreover,*

$$\bigwedge\big(\textstyle\int f\,dE\big) = f\big(\bigwedge(A)\big) \tag{135}$$

for every continuous $f : \mathbf{C} \to \mathbf{C}$.

Incidentally, an utterly elementary proof of (134) (and the non emptiness of $\bigwedge(A)$) has recently been found [3] (which works for hyponormal operators too). In a sense, the proof of non emptiness via Theorem 38 goes back to the automatic regularity of Baire measures (cf. the Corollary of Theorem 23).

Another dividend of the Spectral Theorem is that we are in a position to define "functions" of a normal operator in an unambiguous way. Thus, let A be a normal operator, and E the unique spectral measure constructed in Theorem 37. As in Definition 20, let us write $\mathscr{F}$ for the class of all functions $f : \mathbf{C} \to \mathbf{C}$ such that $\chi_{\wedge(E)} f$ is a bounded Borel function. Inasmuch as $\bigwedge(E) = \bigwedge(A)$, the class $\mathscr{F}$ is describable directly in terms of the spectrum of A ; it includes, in particular, every continuous complex function on $\mathbf{C}$.

DEFINITION 30. With notation as in the preceding paragraph, we define, for each f in $\mathscr{F}$,

$$f(A) = \int f \, dE . \tag{136}$$

(The only possible conflict with a previously used notation is cleared up by (122).)

Using the notation in Definition 30, we may reformulate Theorem 38 as a "spectral mapping theorem":

THEOREM 39. (Spectral Mapping Theorem for a normal operator) *If* A *is a normal operator, then*

$$\bigwedge\big(f(A)\big) = f\big(\bigwedge(A)\big) \tag{137}$$

for every continuous $f : \mathbf{C} \to \mathbf{C}$.

Granted that this definition of $f(A)$ is unambiguous, how natural is it? For a polynomial function in λ and $\overline{\lambda}$, say

$$p(\lambda) = \sum c_{rs}\lambda^r \overline{\lambda}^s\,, \tag{138}$$

we have

$$p(A) = \sum c_{rs}A^r A^{*s} \tag{139}$$

by the formal properties of a spectral integral, thus $p(A)$ is natural enough in this case. More generally, if $f : \mathbf{C} \to \mathbf{C}$ is continuous, we may choose, by the Weierstrass theorem in *two dimensions*, a sequence of polynomial functions p_n of the form (138), such that $p_n \to f$ uniformly on $\bigwedge(A) = \bigwedge(E)$; then

$$\|p_n(A) - f(A)\| \to 0$$

by a property of operator valued integrals (cf. Theorem 11, or (75)), thus

$$f(A) = \lim p_n(A)\,. \tag{140}$$

The formula (140) may be regarded either as a plausibility argument for Definition 30, or as a somewhat more explicit formula for calulating $f(A)$ (for continuous f). In any case, the Weierstrass theorem in two dimensions makes its appearance only after all the shooting is over, and the Riesz theorem in two dimensions is nowhere in view.

13. Multiplications

A currently popular formulation of spectral theory, of proven fertility, is in terms of multiplication operators in Hilbert function space. Though it lacks the crystalline purity of uniqueness, it leaves little to be desired in transparency [6].

The situation is as follows (cf. [5, p. 95] and [6]). Let A be a normal operator, and z a vector of the underlying Hilbert space. If we push z

around with the powers of A and A^*, and take the closed linear span, we arrive at a closed linear subspace which reduces A (i.e. is invariant under both A and A^*). As the underlying Hilbert space is clearly the orthogonal direct sum of a family of such reducing subspaces, we may assume, without loss of transparency, that the only reducing subspace which contains z is the entire Hilbert space. Briefly, we may assume z is a "cyclic vector". This means that if p stands for polynomial functions of the form

$$p(\lambda) = \sum c_{rs} \lambda^r \overline{\lambda}^s , \tag{141}$$

the correspondence

$$p \mapsto p(A)z \tag{142}$$

is a linear mapping of the complex linear space $\mathscr{P}$ of all such polynomial functions, onto a dense linear subspace $\mathscr{D}$ of the underlying Hilbert space. Let E be the spectral measure given by Theorem 37, and let μ be the Borel measure

$$\mu(M) = (E(M)z \big| z)$$

(M ranging over the Borel sets of $\mathbf{C}$). If p is a function of the form (141), we have

$$\|p(A)z\|^2 = \big(p(A)^* p(A)z \big| z\big) = \big(\big(\textstyle\int |p|^2 \, dE\big)z \big| z\big) = \displaystyle\int |p|^2 \, d\mu$$

(since $\mu\big(\mathbf{C} - \bigwedge(E)\big) = 0$, it is superfluous to insert $\chi_{\wedge(E)}$ in the integrand of the last integral); viewing $\mathscr{P}$ as a linear subspace of $\mathscr{L}^2(\mu)$ (the unidentified space of absolutely square-integrable Borel functions on $\mathbf{C}$), we see that the correspondence (142) is a linear "isometry" of $\mathscr{P}$ onto $\mathscr{D}$. Evidently this correspondence induces, by density, an isometric mapping U of $L^2(\mu)$ (the space $\mathscr{L}^2(\mu)$ with a.e. equal functions identified) onto the underlying Hilbert space. If $u(\lambda) = \lambda$ $(\lambda \in \mathbf{C})$, so that $\int u \, dE = A$, we have

$$\big[(up)(A)\big] z = u(A)p(A)z = Ap(A)z$$

for all polynomials p of the form (141); it follows that U transforms the corespondence

$$f \mapsto uf \qquad (f \in \mathscr{L}^2(\mu))$$

into the operator A. Switching our point of view, we may regard A as the operator in the Hilbert space $L^2(\mu)$ which sends the equivalence class of the function

$$\lambda \mapsto f(\lambda)$$

(where $f \in \mathscr{L}^2(\mu)$) to the equivalence class of the function

$$\lambda \mapsto \lambda f(\lambda)\,.$$

Informally, but suggestively,

$$(Af)(\lambda) = \lambda f(\lambda)\,.$$

14. Postlude

What about a non normal operator $A = A_1 + iA_2$? The catch is that A_1 does not commute with A_2; consequently if E_k is the unique real spectral measure associated with A_k, the relation $E_1 \leftrightarrow E_2$ does not hold (cf. (33)). If we wish to imitate the foregoing theory, we are confronted with the problem of amalgamating two non commuting real spectral measures.

Let us face the problem cheerfully, and consider two arbitrary PO-measures E_1 and E_2 defined on rings $\mathscr{R}_1$ and $\mathscr{R}_2$, respectively. Let $\mathscr{P}$ be the class of all rectangles $M_1 \times M_2$ $(M_1 \in \mathscr{R}_1,\ M_2 \in \mathscr{R}_2)$. With an eye on Section 11, the first task is to assign a positive operator to each such rectangle. There are two natural choices:

$$E_2(M_2)^{1/2}E_1(M_1)E_2(M_2)^{1/2}$$

and

$$E_1(M_1)^{1/2}E_2(M_2)E_1(M_1)^{1/2}\,.$$

Inspecting the proofs of the lemmas in Section 11, we find ourselves in desperate need of the relation

$$(*) \qquad E_2(M_2)^{1/2}E_1(M_1)E_2(M_2)^{1/2} = E_1(M_1)^{1/2}E_2(M_2)E_1(M_1)^{1/2}\,.$$

Alas, the relation $(*)$ is too stringent; when E_1 and E_2 are spectral measures, it holds only when $E_1 \leftrightarrow E_2$. More precisely, if P and Q are projections, then $QPQ = PQP$ if and only if $PQ = QP$, as one sees on multiplying out the expression

$$(PQ - QP)^*(PQ - QP) = -(PQ - QP)^2\,.$$

In the absence of a notion of amalgam for noncommuting spectral measures, non normal operators are impermeable to the foregoing techniques.

The trouble appears to be that the given Hilbert space is too small for maneuver. Granted the Naimark-Nagy dilation theory [10], there does exist a technique for representing an arbitrary operator A in the form

$$A = \int \lambda dF \,,$$

where F is a scalar multiple (by the scalar $\|A\|$) of a normalized compact PO-measure defined on the σ-algebra of Borel sets of the unit circle $|\lambda| = 1$.* For an initiation into these mysteries, the reader is referred to the work of Sz.-Nagy [10], the article of M. Schreiber ("A functional calculus for general operators in Hilbert space", *Trans. Amer. Soc.*, vol. 87 (1958), pp. 108–118), and to the literature related thereto.

* cf. [13, p. 181].

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