# Eigenvalues of Laplacian with constant magnetic field on noncompact hyperbolic surfaces with finite area

**Abderemane MORAME**[1] **and Françoise TRUC**[2]
[1] *Université de Nantes, Faculté des Sciences, Dpt. Mathématiques, UMR 6629 du CNRS, B.P. 99208, 44322 Nantes Cedex 3, (FRANCE), E.Mail: Abderemane.Morame@univ-nantes.fr*
[2] *Université de Grenoble I, Institut Fourier, UMR 5582 CNRS-UJF, B.P. 74, 38402 St Martin d'Hères Cedex, (France), E.Mail: Francoise.Truc@ujf-grenoble.fr*

**Abstract**

We consider a magnetic Laplacian $-\Delta_A = (id + A)^\star(id + A)$ on a noncompact hyperbolic surface $\mathbf{M}$ with finite area. $A$ is a real one-form and the magnetic field $dA$ is constant in each cusp. When the harmonic component of $A$ satifies some quantified condition, the spectrum of $-\Delta_A$ is discrete. In this case we prove that the counting function of the eigenvalues of $-\Delta_A$ satisfies the classical Weyl formula, even when $dA = 0$. [1]

## 1 Introduction

We consider a smooth, connected, complete and oriented Riemannian surface $(\mathbf{M}, g)$ and a smooth, real one-form $A$ on $\mathbf{M}$. We define the magnetic Laplacian, the Bochner Laplacian

$$-\Delta_A \ = \ (i\ d + A)^\star (i\ d + A)\ , \tag{1.1}$$

$$(\ (i\ d + A)u = i\ du + uA\ ,\ \forall\ u\ \in\ C_0^\infty(\mathbf{M};\mathbb{C})\ .$$

The magnetic field is the exact two-form $\rho_B \ = \ dA$ .
If $dm$ is the Riemannian measure on $\mathbf{M}$ , then

$$\rho_B \ = \ \widetilde{\mathbf{b}}\, dm\ , \quad \text{with} \quad \widetilde{\mathbf{b}}\ \in\ C^\infty(\mathbf{M};\mathbb{R})\ . \tag{1.2}$$

[1] *Keywords* : spectral asymptotics, magnetic field, Aharanov-Bohm, hyperbolic surface.

The magnetic intensity is $\mathbf{b} = |\widetilde{\mathbf{b}}|$ .

It is well known, (see [Shu] ), that $-\Delta_A$ has a unique self-adjoint extension on $L^2(\mathbf{M})$ , containing in its domain $C_0^\infty(\mathbf{M};\mathbb{C})$ , the space of smooth and compactly supported functions. The spectrum of $-\Delta_A$ is gauge invariant : for any $f \in C^1(\mathbf{M};\mathbb{R})$ , $-\Delta_A$ and $-\Delta_{A+df}$ are unitarily equivalent, hence they have the same spectrum.

We are interested in constant magnetic fields on $\mathbf{M}$ in the case when $(\mathbf{M},\ g)$ is a non-compact geometrically finite hyperbolic surface of finite area; (see [Per] or [Bor] for the definition and the related references). More precisely

$$\mathbf{M} = \bigcup_{j=0}^{J} M_j \tag{1.3}$$

where the $M_j$ are open sets of $\mathbf{M}$, such that the closure of $M_0$ is compact, and (when $J \geq 1$) the other $M_j$ are cuspidal ends of $\mathbf{M}$.

This means that, for any $j$, $1 \leq j \leq J$ , there exist strictly positive constants $a_j$ and $L_j$ such that $M_j$ is isometric to $\mathbb{S}\times]a_j^2, +\infty[$ , equipped with the metric

$$ds_j^2 = y^{-2}(\ L_j^2\ d\theta^2 + dy^2\ )\ ; \tag{1.4}$$

($\mathbb{S} = \mathbb{S}^1$ is the unit circle and $M_j \cap M_k = \emptyset$ if $j \neq k$ ) .
Let us choose some $z_0 \in M_0$ and let us define

$$d\ :\ \mathbf{M} \to \mathbb{R}_+\ ; \quad d(z) = d_g(z, z_0)\ ; \tag{1.5}$$

$d_g(\ .\ ,\ .\ )$ denotes the distance with respect to the metric $g$.

For any $b \in \mathbb{R}^J$ , there exists a one-form $A$ , such that the corresponding magnetic field $dA$ satisfies

$$dA = \widetilde{\mathbf{b}}(z)dm \quad \text{with} \quad \widetilde{\mathbf{b}}(z) = b_j\ \forall\ z \in M_j\ . \tag{1.6}$$

The following statement on the essential spectrum is proven in [Mo-Tr1] :

**Theorem 1.1** *Assume (1.3) and (1.6). Then for any* $j$ , $1 \leq j \leq J$ *and for any* $z \in M_j$ *there exists a unique closed curve through* $z$ , $\mathcal{C}_{j,z}$ *in* $(M_j,\ g)$ , *not contractible and with zero* $g-$*curvature. (*$\mathcal{C}_{j,z}$ *is called an horocycle of* $M_j$ *). The following limit exists and is finite:*

$$[A]_{M_j} = \lim_{d(z)\to+\infty} \int_{\mathcal{C}_{j,z}} A\ . \tag{1.7}$$

*If* $J^A = \{ j \in \mathbb{N} \, , \; 1 \leq j \leq J \; s.t. \; [A]_{M_j} \in 2\pi\mathbb{Z} \; \} \neq \emptyset$ *, then*

$$\mathrm{sp}_{ess}(-\Delta_A) \; = \; [\frac{1}{4} + \min_{j \in J^A} b_j^2 \; , \; +\infty[ \; . \tag{1.8}$$

*If* $J^A = \emptyset$ *, then* $\mathrm{sp}_{ess}(-\Delta_A) = \emptyset$ *:*
$-\Delta_A$ *has purely discrete spectrum, (its resolvent is compact).*

When the magnetic Laplacian $-\Delta_A$ has purely discrete spectrum, it is called a magnetic bottle, (see [Col2]).

If $A = df + A^H + A^\delta$ is the Hodge decomposition of $A$ with $A^H$ harmonic, ($dA^H = 0$ and $d^\star A^H = 0$ ) , then $\forall \; j$ , $[A]_{M_j} = [A^H]_{M_j}$ , so the discreteness of the spectrum of $-\Delta_A$ depends only on the harmonic component of $A$ . So one can see the case $J^A = \emptyset$ as an Aharonov-Bohm phenomenon [Ah-Bo], a situation where the magnetic field $dA$ is not sufficicient to describe $-\Delta_A$ and the use of the magnetic potential $A$ is essential : we can have magnetic bottle with null intensity.

# 2 The Weyl formula in the case of finite area with a non-integer class one-form

Here we are interested in the pure point part of the spectrum. We assume that $J^A = \emptyset$ , then the spectrum of $-\Delta_A$ is discrete. In this case, we denote by $(\lambda_j)_j$ the increasing sequence of eigenvalues of $-\Delta_A$ , (each eigenvalue is repeated according to its multiplicity). Let

$$N(\lambda, -\Delta_A) \; = \; \sum_{\lambda_j < \lambda} 1 \; . \tag{2.1}$$

We will show that the asymptotic behavior of $N(\lambda)$ is given by the Weyl formula :

**Theorem 2.1** *Consider a geometrically finite hyperbolic surface* $(\mathbf{M}, \, g)$ *of finite area, and assume (1.6) with* $J^A = \emptyset$ *, (see (1.7 for the definition).*

*Then*

$$N(\lambda, -\Delta_A) \; = \; \lambda \frac{|\mathbf{M}|}{4\pi} \; + \; \mathbf{O}(\sqrt{\lambda} \ln \lambda) \; . \tag{2.2}$$

**Remark 2.2** *As $J^A$ depends only on the harmonic component of $A$ , $J^A$ is not empty when* $\mathbf{M}$ *is simply connected. In [Go-Mo] there are some results close to Theorem 2.1, but for simply connected manifolds.*

*The cases where the magnetic field prevails were studied in [Mo-Tr1] and in [Mo-Tr2].*

**Proof of Theorem 2.1.** Any constant depending only on the $b_j$ and on $\min_{1\leq j\leq J}\inf_{k\in\mathbb{Z}} |[A]_{M_j} - 2k\pi|$ will be denoted invariably $C$ .

Consider a cusp $M = M_j = \mathbb{S}\times]\alpha^2, +\infty[$ equipped with the metric $ds^2 = L^2 e^{-2t} d\theta^2 + dt^2$ for some $\alpha > 0$ and $L > 0$ .

Let us denote by $-\Delta_A^M$ the Dirichlet operator on $M$ , associated to $-\Delta_A$ . The first step will be to prove that

$$N(\lambda, -\Delta_A^M) \;=\; \lambda\frac{|M|}{4\pi} \;+\; \mathbf{O}(\sqrt{\lambda}\ln\lambda)\;. \tag{2.3}$$

Since $-\Delta_A^M$ and $-\Delta_{A+d\varphi+kd\theta}^M$ are gauge equivalent for any $\varphi \in C^\infty(\overline{\mathbf{M}};\mathbb{R})$ and any $k \in \mathbb{Z}$, we can assume that

$$-\Delta_A^M = L^{-2}e^{2t}(D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}\,, \quad \text{with} \quad A_1 = -\xi \pm bLe^{-t}\;,\; \xi \in ]0,1[\;,$$

$(b = b_j\;,\; 2\pi\xi - [A]_M \;\in\; 2\pi\mathbb{Z})$ . Then we get that

$$\mathrm{sp}(-\Delta_A^M) = \bigcup_{\ell\in\mathbb{Z}} \mathrm{sp}(P_\ell)\;;\; P_\ell = D_t^2 + \frac{1}{4} + \left(e^t\frac{(\ell+\xi)}{L} \pm b\right)^2\;,$$

for the Dirichlet condition on $L^2(I;dt)$ ; $I = ]\alpha^2, +\infty[$ . This implies that

$$N(\lambda, -\Delta_A^M) \;=\; \sum_{\ell\in\mathbb{Z}} N(\lambda, P_\ell) = \sum_{\ell\in X_\lambda} N(\lambda, P_\ell) \tag{2.4}$$

with $X_\lambda = \{\ell \;/\; e^{\alpha^2}\frac{|\ell+\xi|}{L} < \sqrt{\lambda - 1/4} - b\;\}$ .

Denoting by $Q_\ell$ the Dirichlet operator on $I$ associated to

$$Q_\ell = D_t^2 + \frac{1}{4} + \frac{(\ell+\xi)^2}{L^2}e^{2t}\;,$$

we easily get that

$$Q_\ell - C\sqrt{Q_\ell} \;\leq\; P_\ell \;\leq\; Q_\ell + C\sqrt{Q_\ell}\;. \tag{2.5}$$

Therefore one can find a constant $C(b)$ , depending only on $b$ , such that, for any $\lambda >> 1 + C(b)$ ,

$$N(\lambda - \sqrt{\lambda}C(b), Q_\ell) \ \leq \ N(\lambda, P_\ell) \ \leq \ N(\lambda + \sqrt{\lambda}C(b), Q_\ell) \ . \qquad (2.6)$$

Following Titchmarsh's method ( [Tit], Theorem 7.4) we establish the following bounds

**Lemma 2.3** *There exists $C > 1$ so that for any $\mu >> 1$ and any $\ell \in X_\mu$ ,*

$$w_\ell(\mu) - \pi \ \leq \ \pi N(\mu - \frac{1}{4}, Q_\ell) \ \leq \ w_\ell(\mu) + \frac{1}{12}\ln\mu + C \ , \qquad (2.7)$$

*with*

$$w_\ell(\mu) \ = \ \int_{\alpha^2}^{+\infty} \left[\mu - \frac{(\ell+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dt \qquad (2.8)$$

$$= \ \int_{\alpha^2}^{T_{\mu,L}} \left[\mu - \frac{(\ell+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dt \ ;$$

$(e^{T_{\mu,L}} = L\sqrt{\mu}/(\inf_{k\in\mathbb{Z}} |\xi - k|) \ )$ .

**Proof of Lemma 2.3**

The lower bound is easily obtained (see [Tit], Formula 7.1.2 p 143) so we focus on the upper bound.

Let us define $V_\ell = \frac{(\ell+\xi)^2}{L^2}e^{2t}$ and denote by $\phi_\mu^\ell$ a solution of $Q_\ell\phi = (\mu - \frac{1}{4})\phi$. Consider $x_\ell$ and $y_\ell$ so that $V_\ell(x_\ell) = \mu$ and $V_\ell(y_\ell) = \nu$, for a given $0 < \nu < \mu$ to be determined later. We denote by $m$ the number of zeros of $\phi_\mu^\ell$ on $]\alpha^2, y_\ell[$. Recall that the number $n$ of zeros of $\phi_\mu^\ell$ on $]\alpha^2, x_\ell[$ is equal to $N(\mu - \frac{1}{4}, Q_\ell)$. Applying Lemma 7.3 p 146 in [Tit] we deduce that

$$m\pi = \int_{\alpha^2}^{y_\ell} [\mu - V_\ell]^{1/2}\, dt + R_\ell$$

with $R_\ell = \frac{1}{4}\ln(\mu - V_\ell(\alpha^2)) - \frac{1}{4}\ln(\mu - V_\ell(y_\ell)) + \pi$, hence

$$|n\pi - \int_{\alpha^2}^{x_\ell} [\mu - V_\ell]^{1/2}\, dt| \leq (x_\ell - y_\ell)(\mu - \nu)^{1/2} + R_\ell + (n-m)\pi$$

According to the Sturm comparison theorem ([Tit], p 107-108), we have

$$(n-m)\pi \leq (x_\ell - y_\ell)(\mu - \nu)^{1/2}$$

and

$$|n\pi - \int_{\alpha^2}^{x_\ell} [\mu - V_\ell]^{1/2}\, dt| \leq \ln(\frac{\mu}{\nu})(\mu-\nu)^{1/2} + \frac{1}{4}\ln\mu - \frac{1}{4}\ln(\mu-\nu) + 2\pi$$

Now taking $\nu = \mu - \mu^{2/3}$ we get the desired estimate.

In view of (2.4) we now compute $\sum_{\ell\in\mathbb{Z}} w_\ell(\mu)$ . We first get the following

**Lemma 2.4** *There exists* $C > 1$ *such that, for any* $\mu >> 1$ *and any* $t \in [\alpha^2, T_{\mu,L}]$ ,

$$\left| \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dx \;-\; \sum_{\ell\in\mathbb{Z}} \left[\mu - \frac{(\ell+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} \right| \;\leq\; C(\sqrt{\mu} + \frac{e^t}{L}) \;.$$

This leads to

**Lemma 2.5** *There exists* $C > 1$ *such that, for any* $\mu >> 1$ ,

$$\left| \int_{\alpha^2}^{T_{\mu,L}} \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dxdt \;-\; \sum_{\ell\in\mathbb{Z}} w_\ell(\mu) \right| \;\leq\; C\sqrt{\mu}\ln\mu \;.$$

We now compute the integral in the left-hand side.
Making the change of variables $y^2 = \frac{(x+\xi)^2}{L^2\mu}e^{2t}$ we obtain that it is equal to $\mu L \int_{\alpha^2}^{T_{\mu,L}} e^{-t}dt \int_{\mathbb{R}} [1-x^2]_+^{1/2}\, dx$, so we get

**Lemma 2.6** *There exists* $C > 1$ *such that, for any* $\mu >> 1$ ,

$$\left| \int_{\alpha^2}^{T_{\mu,L}} \int_{\mathbb{R}} \left[\mu - \frac{(x+\xi)^2}{L^2}e^{2t}\right]_+^{1/2} dxdt \;-\mu L e^{-\alpha^2} \int_{\mathbb{R}} \left[1-x^2\right]_+^{1/2} dx \right| \;\leq\; C\sqrt{\mu} \;.$$

Noticing that $|M| = 2\pi L e^{-\alpha^2}$ and using Lemmas 2.5 and 2.6 we have

**Lemma 2.7**

$$\frac{1}{\pi}\sum_{\ell} \in w_\ell(\mu) \;=\; \frac{|M|}{4\pi}\mu \;+\; \mathbf{O}(\sqrt{\mu}\ln\mu) \;, \quad \text{as} \quad \mu \to +\infty \;.$$

In view of (2.4),(2.6) and (2.7) Lemma 2.7 ends the proof of formula (2.3). Now it remains to consider the whole surface $\mathbf{M}$.

We have : $\mathbf{M} \; = \; \left( \bigcup_{j=0}^{J} M_j \right)$

where the $M_j$ are open sets of $\mathbf{M}$, such that the closure of $M_0$ is compact, and the other $M_j$ are cuspidal ends of $\mathbf{M}$ and

$M_j \cap M_k = \emptyset$ , if $j \neq k$ . We denote $M_0^0 = \mathbf{M} \setminus (\bigcup_{j=1}^{J} \overline{M_j})$ , then

$$\mathbf{M} \; = \; \overline{M_0^0} \bigcup \left( \bigcup_{j=1}^{J} \overline{M_j} \right) \; . \tag{2.9}$$

Let us denote respectively by $-\Delta_{A,D}^{\Omega}$ and by $-\Delta_{A,N}^{\Omega}$ the Dirichlet operator and the Neumann-like operator on an open set $\Omega$ of $\mathbf{M}$ associated to $-\Delta_A$ . The minimax principle and (2.9) imply that

$$N(\lambda, -\Delta_{A,D}^{M_0^0}) + \sum_{1 \leq j \leq J} N(\lambda, -\Delta_{A,D}^{M_j}) \leq N(\lambda, -\Delta_A) \tag{2.10}$$

$$\leq N(\lambda, -\Delta_{A,N}^{M_0^0}) + \sum_{1 \leq j \leq J} N(\lambda, -\Delta_{A,N}^{M_j})$$

The Weyl formula with remainder, (see [Hor] for Dirichlet boundary condition and [Sa-Va] p. 9 for Neumann-like boundary condition), gives that

$$\left\{ \begin{array}{l} N(\lambda, -\Delta_{A,D}^{M_0^0}) = (4\pi)^{-1} |M_0^0| \lambda + \mathbf{O}(\sqrt{\lambda}) \\ N(\lambda, -\Delta_{A,N}^{M_0^0}) = (4\pi)^{-1} |M_0^0| \lambda + \mathbf{O}(\sqrt{\lambda}) \end{array} \right\} . \tag{2.11}$$

The asymptotic formula for $N(\lambda, -\Delta_{A,N}^{M_j})$ ,

$$N(\lambda, -\Delta_{A,N}^{M_j}) \; = \; \lambda \frac{|M_j|}{4\pi} \; + \; \mathbf{O}(\sqrt{\lambda} \ln \lambda) \; , \tag{2.12}$$

is obtained as for the Dirichlet case (2.3) (with $M = M_j$ ), by noticing that $N(\lambda, P_{\ell,D}) \leq N(\lambda, P_{\ell,N}) \leq N(\lambda, P_{\ell,D}) + 1$ , where $P_{\ell,D}$ and $P_{\ell,N}$ are Dirichlet and Neumann operators on a half-line $I = ]\alpha^2, +\infty[$ , associated to the same differential Schödinger operator $P_\ell = D_t^2 + \frac{1}{4} + (e^t \frac{(\ell + \xi)}{L} \pm b)^2$ , more precisely $P_{\ell,N}$ is of Robin type condition $\partial_t u(\alpha^2) + u(\alpha^2)/2 = 0$ cause of change of density.

We get (2.2) from (2.3) with $M = M_j$ , (2.12), (for any $j = 1, \ldots, J$) , (2.10) and (2.11).□

**Remark 2.8** *Theorem 2.1 still holds if the metric of* $\mathbf{M}$ *is modified in a compact set.*

*When* $A = 0$ , $-\Delta = -\Delta_0$ *has embedded eigenvalues in its essential spectrum,* $(sp_{ess}(-\Delta) = [\frac{1}{4}, +\infty[)$ . *If* $N_{ess}(\lambda, -\Delta)$ *denotes the number of these eigenvalues in* $[\frac{1}{4}, \lambda[$ , *then it is well known that one has an upper bound* $N_{ess}(\lambda, -\Delta) \leq \lambda \frac{|\mathbf{M}|}{4\pi}$ ; *see [Col1] and [Hej] for the history and related improvement of the upper bound.*
*Recently [Mul] established a sharp asymptotic formula, similar to our case,*

$$N_{ess}(\lambda, -\Delta) = \lambda \frac{|\mathbf{M}|}{4\pi} + \mathbf{O}(\sqrt{\lambda} \ln \lambda) ,$$

*for some particular* $\mathbf{M}$ .

**Acknowledgement.**

*We are grateful to Yves Colin de Verdière for his useful comments and for pointing out the results for embedded eigenvalues of* $-\Delta$ .

# References

[Ah-Bo] Y. Aharonov, D. Bohm : Significance of electromagnetic potentials in quantum theory, Phys. Rev., 115, (1959), p. 485-491.

[Bor] D. Borthwick : Spectral Theory of Infinite-Area Hyperbolic Surfaces, Birkhäuser, Boston, 2007.

[Col1] Y. Colin de Verdière : Pseudo-laplaciens II, Ann. Institut Fourier, 33 (2), (1983), p. 87-113.

[Col2] Y. Colin de Verdière : L'asymptotique de Weyl pour les bouteilles magnétiques, Comm. Math. Phys., 105, (1986), p. 327-335.

[Go-Mo] S. Golénia, S. Moroianu : Spectral Analysis of Magnetic Laplacians on Conformally Cusp Manifolds, Ann. Henri Poincaré, 9, (2008), p. 131-179.

[Hej] D. Hejhal : The Selberg trace formula for $PSL(2, \mathbb{R})$, II, Lecture Notes in Math. 1001, Springer-verlag, Berlin, 1983.

[Hor] Lars Hörmander : The Analysis of Linear P.D.O. IV , Springer-verlag, Berlin, 1985.

[Ivr] V. J. Ivrii : Microlocal Analysis and Precise Spectral Asymptotics, Springer-verlag, Berlin, 1998.

[Mo-Tr1] A. Morame, F. Truc : Magnetic bottles on geometrically finite hypersurfaces, J. of Geometry and Physics, 59-7, (2009), p 1095-1105.

[Mo-Tr2] A. Morame, F. Truc : Magnetic bottles on the Poincaré half-plane: spectral asymptotics, J. Math. Kyoto Univ., 48-3, (2008), p 597-616.

[Mul] W. Müller : Weyl's law in the theory of automorphic forms, London Math. Soc. Lecture Notes, 354,(2008), p. 133-163.

[Per] P. Perry : The Spectral Geometry of Geometrically Finite Hyperbolic Manifolds, Proc. Sympos. Pure Math., 76, Part1, AMS, (2007), p. 289-327.

[Sa-Va] Y. Safarov, D. Vassiliev : The Asymptotic distribution of Eigenvalues of Partial Differential Operators, AMS Trans.155, 1996.

[Shu] M. Shubin : The essential Self-adjointness for Semi-bounded Magnetic Schrödinger operators on Non-compact Manifolds, J. Func. Anal., 186, (2001), p. 92-116.

[Tit] E. C. Titchmarsh : Eigenfunction Expansions associated with Second-order Differential equations I, second edition, Oxford at the Clarendon Press, 1962.