SOLVABILITY RELATIONS FOR SOME NON FREDHOLM OPERATORS

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Abstract: Solvability conditions for nonhomogeneous elliptic partial differential equations involving Schrödinger type operators without Fredholm property were derived in our preceding works [10], [11], [12]. We reformulate these relations in terms of solutions of corresponding homogeneous problems belonging to the appropriate functional spaces.

AMS Subject Classification: 35J10, 35P10, 35P25
Key words: solvability conditions, non Fredholm operators, functional spaces

1. Introduction

The first problem considered in the note is

$$-\Delta u + V(x)u - au = f(x), \; x \in \mathbb{R}^3, \tag{1.1}$$

$a \geq 0$ is a constant and the corresponding homogeneous problem will be

$$-\Delta w + V(x)w - aw = 0, \tag{1.2}$$

where $V(x)$ is shallow and short-range and satisfies the conditions analogous to those used in works [10], [11], [12].

Assumption 1. *The potential function* $V(x) : \mathbb{R}^3 \to \mathbb{R}$ *satisfies the estimate*

$$|V(x)| \leq \frac{C}{1 + |x|^{3.5+\varepsilon}}$$

with some $\varepsilon > 0$ *and* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ *a.e. such that*

Received: Month, Date Year

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \quad and \quad \sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi.$$

Here C denotes a finite positive constant and c_{HLS} given on p.98 of [6] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\Big| \leq c_{HLS}\|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

Here and further down the norm of a function $f_1 \in L^p(\mathbb{R}^d),\ 1 \leq p \leq \infty,\ d \in \mathbb{N}$ is denoted as $\|f_1\|_{L^p(\mathbb{R}^d)}$. We will be using $(f_1(x), f_2(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f_1(x)\bar{f}_2(x)dx$, with a slight abuse of notations when the functions involved in the inner product are not square integrable, like for instance $w(x)$ involved in relation (1.5). The sphere of radius r in the space of d dimensions centered at the origin will be denoted by S_r^d. Due to the decay of the potential function at infinity the essential spectrum of the Schrödinger operator $-\Delta + V(x) - a$ on $L^2(\mathbb{R}^3)$ involved in the left side of equation (1.1) fills the semi-axis $[-a, \infty)$ (see e.g. [4]) such that there is no finite dimensional isolated kernel and the Fredholm alternative theorem fails to work for problem (1.1). Under our Assumption 1 the Schrödinger operator is self-adjoint and unitarily equivalent to $-\Delta - a$ on $L^2(\mathbb{R}^3)$ via the wave operators (see [1], [5], [8], [10]) and its functions of the continuous spectrum satisfying

$$(-\Delta + V(x))\varphi_k(x) = k^2\varphi_k(x),\ k \in \mathbb{R}^3, \tag{1.3}$$

the Lippmann-Schwinger equation for the perturbed plane waves (see e.g. [7] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy$$

and the orthogonality relations

$$(\varphi_k(x), \varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q),\ k, q \in \mathbb{R}^3$$

form the complete system in $L^2(\mathbb{R}^3)$. Similarly to [10] for the right side of (1.1) we have the following.

Assumption 2. *The function* $f(x) \in L^2(\mathbb{R}^3)$ *and* $|x|f(x) \in L^1(\mathbb{R}^3)$.

Let us introduce the functional space

$$\tilde{W}^{2,\infty}(\mathbb{R}^3) := \{w(x) : \mathbb{R}^3 \to \mathbb{C} \mid w, \nabla w, \Delta w \in L^\infty(\mathbb{R}^3)\} \tag{1.4}$$

used in establishing solvability conditions for the Laplacian problem with convection terms in [12]. Our first proposition will be as follows.

Theorem 3. *Let Assumptions 1 and 2 hold. Then problem (1.1) admits a unique solution* $u(x) \in H^2(\mathbb{R}^3)$ *if and only if*

$$(f(x), w(x))_{L^2(\mathbb{R}^3)} = 0 \tag{1.5}$$

for any $w(x) \in \tilde{W}^{2,\,\infty}(\mathbb{R}^3)$ *satisfying the homogeneous equation (1.2), where the space* $\tilde{W}^{2,\,\infty}(\mathbb{R}^3)$ *is defined in (1.4).*

The second equation studied in the work is given by

$$-\Delta_x u + V(x)u - \Delta_y u + U(y)u - au = F(x,y),\ x, y \in \mathbb{R}^3, \tag{1.6}$$

where a is a positive constant and the corresponding homogeneous problem is

$$-\Delta_x \theta + V(x)\theta - \Delta_y \theta + U(y)\theta - a\theta = 0. \tag{1.7}$$

Note that we do not consider the case of $a = 0$ here since the orthogonality conditions are not required when a vanishes for problem (1.6) according to Theorem 3 of [11]. The Laplace operators Δ_x and Δ_y are in $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ respectively, the resulting operator $\Delta = \Delta_x + \Delta_y$ and potentials $V(x)$ and $U(y)$ are shallow and decaying at infinity with the same rate as before. Thus, problem (1.6) involves the operator without Fredholm property in its left side. Analogously to [11] for the right side of (1.6) we assume the following.

Assumption 4. *The function* $F(x,y) \in L^2(\mathbb{R}^6)$ *and* $|x|F(x,y),\ |y|F(x,y) \in L^1(\mathbb{R}^6)$.

For the studies of the problem above we will be using the functional space

$$\tilde{W}^{2,\,\infty}(\mathbb{R}^6) := \{\theta(x,y) : \mathbb{R}^6 \to \mathbb{C} \mid \theta,\ \nabla\theta,\ \Delta_x\theta,\ \Delta_y\theta \in L^\infty(\mathbb{R}^6)\}, \tag{1.8}$$

where $\nabla = \nabla_x + \nabla_y$ and the gradients ∇_x and ∇_y are acting in x and y in $\mathbb{R}^3$ variables respectively. Our second statement is as follows.

Theorem 5. *Let the potential functions* $V(x)$ *and* $U(y)$ *satisfy Assumption 1 and Assumption 4 holds. Then problem (1.6) has a unique solution* $u(x,y) \in H^2(\mathbb{R}^6)$ *if and only if*

$$(F(x,y), \theta(x,y))_{L^2(\mathbb{R}^6)} = 0 \tag{1.9}$$

for any $\theta(x,y) \in \tilde{W}^{2,\,\infty}(\mathbb{R}^6)$ *solving the homogeneous problem (1.7) with the space* $\tilde{W}^{2,\,\infty}(\mathbb{R}^6)$ *defined in (1.8).*

Finally, we aim to establish solvability conditions for the equation

$$-\Delta_x u - \Delta_y u + U(y)u - au = \phi(x,y),\ x \in \mathbb{R}^n,\ y \in \mathbb{R}^3 \tag{1.10}$$

with $a \geq 0$ and the related homogeneous problem is given by

$$-\Delta_x Q - \Delta_y Q + U(y)Q - aQ = 0. \tag{1.11}$$

When $a > 0$ the dimension $n \in \mathbb{N}$ is assumed to be arbitrary but when a vanishes we consider only $n = 1$ since according to Theorem 6 of [11] in higher dimensions in this case the orthogonality conditions are not necessary. The Laplace operators Δ_x and Δ_y are in $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, y_3)$ variables respectively, such that $\Delta = \Delta_x + \Delta_y$ and the spatial behavior of the potential function $U(y)$ is analogous to the one considered in the two previous models. Hence the essential spectrum of the operator $-\Delta_x - \Delta_y + U(y) - a$ on $L^2(\mathbb{R}^{n+3})$ consists of the unbounded interval $[-a, \infty)$ and the Fredholm alternative theorem fails to work for problem (1.10). We formulate the conditions on the right side of equation (1.10) analogically to the ones stated in [11].

Assumption 6. *We have $\phi(x, y) \in L^2(\mathbb{R}^{n+3})$ and $|x|\phi(x, y),\ |y|\phi(x, y) \in L^1(\mathbb{R}^{n+3})$.*

Let us introduce the functional space which we will use for deriving the solvability relations for our problem.

$$\tilde{W}^{2,\,\infty}(\mathbb{R}^{n+3}) := \{Q(x, y) : \mathbb{R}^{n+3} \to \mathbb{C} \mid Q,\ \nabla Q,\ \Delta_x Q,\ \Delta_y Q \in L^\infty(\mathbb{R}^{n+3})\} \tag{1.12}$$

with $\nabla = \nabla_x + \nabla_y$ and the operators ∇_x and ∇_y act in variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^3$ respectively. Our final proposition is as follows.

Theorem 7. *Let the potential functions $U(y)$ satisfy Assumption 1 and Assumption 6 holds. Then problem (1.10) possesses a unique solution $u(x, y) \in H^2(\mathbb{R}^{n+3})$ if and only if*

$$(\phi(x, y), Q(x, y))_{L^2(\mathbb{R}^{n+3})} = 0 \tag{1.13}$$

for an arbitrary $Q(x, y) \in \tilde{W}^{2,\,\infty}(\mathbb{R}^{n+3})$ solving the homogeneous problem (1.11), where the space $\tilde{W}^{2,\,\infty}(\mathbb{R}^{n+3})$ is defined in (1.12).

Note that the solvability conditions for problem (1.1) were obtained in [10] and for equations (1.6) and (1.10) in [11], such that in both works they were stated as orthogonality conditions to the appropriate functions of the continuous spectra of the self-adjoint operators. In the present article the solvability relations for these equations are derived as orthogonality conditions to the solutions of the corresponding homogeneous problems belonging to the appropriate functional spaces. The similarity with the usual Fredholm solvability conditions here is only formal since the operators involved here do not satisfy the Fredholm property and their ranges are not closed.

The studies of operators without Fredholm property are crucial, for instance for proving the existence in the appropriate functional spaces of stationary and travelling wave solutions of reaction-diffusion equations (see e.g. [2], [3], [9], [12]).

2. Proof of the solvability conditions

Let us introduce the sequence of infinitely smooth cut-off functions in the space of an arbitrary dimension $d \in \mathbb{N}$, $\{\xi_n\}_{n=1}^{\infty}$, which are dependent only upon the radial variable such that $\xi_n \equiv 1$ inside the ball $|x| \le r_n$, it vanishes identically when $|x| \ge R_n$ and is monotonically decreasing inside the spherical layer $r_n \le |x| \le R_n$. The sequences of radii r_n, R_n tend to infinity as $n \to \infty$ and are chosen such that R_n increases at a higher rate. This enables us to have $\|\nabla \xi_n\|_{L^2(\mathbb{R}^d)},\ \|\Delta \xi_n\|_{L^2(\mathbb{R}^d)} \to 0$ as $n \to \infty$. The cut-off functions will be needed to perform the limiting arguments below since the solutions of the homogeneous Schrödinger equations discussed are bounded but may not be decaying at infinity, like for instance the perturbed plane waves $\varphi_k(x)$. The quadratic forms studied below will be finite due to the fact that $w(x) \in \tilde{W}^{2,\,\infty}$ and the integration takes place over the compact support of ξ_n. Let us procede with proving the solvability relations for the three dimensional problem.

Proof of Theorem 3. First we assume that equation (1.1) admits a unique solution $u(x) \in H^2(\mathbb{R}^3)$. Let $w(x) \in \tilde{W}^{2,\,\infty}(\mathbb{R}^3)$ be a solution of the homogeneous problem (1.2) with the space $\tilde{W}^{2,\,\infty}(\mathbb{R}^3)$ defined in (1.4). Then we easily obtain

$$(-\Delta u + V(x)u - au, w\xi_n)_{L^2(\mathbb{R}^3)} = (f, w\xi_n)_{L^2(\mathbb{R}^3)}. \tag{2.14}$$

It can be trivially proven that the right side of (2.14) converges to $(f, w)_{L^2(\mathbb{R}^3)}$, which is finite (see the Proof of Theorem 3 in [12]). Integration by parts using that $w(x)$ solves the corresponding homogeneous problem yields that the left side of (2.14) equals to

$$-(u, w\Delta\xi_n)_{L^2(\mathbb{R}^3)} - 2(u, \nabla w . \nabla \xi_n)_{L^2(\mathbb{R}^3)}.$$

Here and further down the dot stands for the scalar product of two vectors in finite dimensions. We easily estimate by means of the Schwarz inequality

$$|(u, w\Delta\xi_n)_{L^2(\mathbb{R}^3)}| \le \|w\|_{L^\infty(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)} \|\Delta\xi_n\|_{L^2(\mathbb{R}^3)} \to 0,$$

$$|(u, \nabla w . \nabla \xi_n)_{L^2(\mathbb{R}^3)}| \le \|\nabla w\|_{L^\infty(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)} \|\nabla\xi_n\|_{L^2(\mathbb{R}^3)} \to 0$$

as $n \to \infty$, which yields relation (1.5).

On the other hand, let orthogonality condition (1.5) of the theorem hold and $a > 0$ since when a vanishes the argument will be analogous. Let us consider the functions of the continuous spectrum $\varphi_k(x),\ k \in S^3_{\sqrt{a}}$ a.e. By means of (1.3) these functions satisfy the homogeneous Schrödinger equation (1.2) and they belong to the (1.4) space, which was proven in Lemma A3 of [12]. Thus we have the orthogonality condition

$$(f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0,\ k \in S^3_{\sqrt{a}}\ a.e.,$$

which implies that equation (1.1) admits a unique solution $u(x) \in H^2(\mathbb{R}^3)$ by means of Lemma 4 of [12] (see also Theorem 1 of [10]). ■

Then we turn our attention to the non Fredholm situation in six dimensions.

Proof of Theorem 5. Let us first suppose that equation (1.6) possesses a unique solution $u(x,y) \in H^2(\mathbb{R}^6)$ and $\theta(x,y) \in \tilde{W}^{2,\infty}(\mathbb{R}^6)$ solves the homogeneous problem (1.7), where the space $\tilde{W}^{2,\infty}(\mathbb{R}^6)$ is defined in (1.8). Clearly we have

$$(-\Delta_x u + V(x)u - \Delta_y u + U(y)u - au, \theta\xi_n)_{L^2(\mathbb{R}^6)} = (F(x,y), \theta\xi_n)_{L^2(\mathbb{R}^6)}. \quad (2.15)$$

Let us integrate by parts in the left side of (2.15) using that $\theta(x,y)$ satisfies the corresponding homogeneous equation. This yields

$$-(u, \theta\Delta\xi_n)_{L^2(\mathbb{R}^6)} - 2(u, \nabla\theta.\nabla\xi_n)_{L^2(\mathbb{R}^6)}.$$

Both of these terms tend to zero as $n \to \infty$, which can be easily shown analogously to the argument in three dimensions performed in the proof of the previous theorem. By means of Assumption 4 and Fact 2 of [11] we have $F(x,y) \in L^1(\mathbb{R}^6)$. Then for the right side of (2.15) we estimate

$$|(F(x,y), \theta\xi_n)_{L^2(\mathbb{R}^6)} - (F(x,y), \theta)_{L^2(\mathbb{R}^6)}| \le \|\theta\|_{L^\infty(\mathbb{R}^6)} \int_{|x|\ge r_n} |F(x,y)| dx dy \to 0$$

as $n \to \infty$, which yields orthogonality relation (1.9).

On the other hand, assume that orthogonality condition (1.9) holds, where θ is an arbitrary solution of equation (1.7), $\theta \in \tilde{W}^{2,\infty}(\mathbb{R}^6)$. As discussed in the Proof of Theorem 3 of [11], under the given conditions the Schrödinger operator involved in the left side of (1.6) is unitarily equivalent to $-\Delta_x - \Delta_y - a$ on $L^2(\mathbb{R}^6)$ via the wave operators. The functions of the continuous spectrum $\varphi_k(x)\eta_q(y),\ k, q \in \mathbb{R}^3$ form the complete system in $L^2(\mathbb{R}^6)$, where $\eta_q(y)$ are the functions of the continuous spectrum of the operator $-\Delta_y + U(y)$, such that

$$(-\Delta_y + U(y))\eta_q(y) = q^2\eta_q(y),\ q \in \mathbb{R}^3. \quad (2.16)$$

They satisfy the orthogonality relations and the Lippmann-Schwinger equation similarly to $\varphi_k(x)$. It was shown in Lemma A3 of [12] that $\varphi_k(x),\ \eta_q(y) \in \tilde{W}^{2,\infty}(\mathbb{R}^3)$. Then functions given by

$$\varphi_k(x)\eta_q(y),\ (k,q) \in S^6_{\sqrt{a}}\ a.e. \quad (2.17)$$

will belong to $L^\infty(\mathbb{R}^6)$. When differentiating we easily arrive at

$$\nabla\varphi_k(x)\eta_q(y) = \eta_q(y)\nabla_x\varphi_k(x) + \varphi_k(x)\nabla_y\eta_q(y) \in L^\infty(\mathbb{R}^6),$$

$$\eta_q(y)\Delta_x\varphi_k(x) \in L^\infty(\mathbb{R}^6),\ \varphi_k(x)\Delta_y\eta_q(y) \in L^\infty(\mathbb{R}^6),$$

such that functions (2.17) belong to $\tilde{W}^{2,\infty}(\mathbb{R}^6)$. By means of (1.3) and (2.16) they satisfy equation (1.7). Therefore, we have the orthogonality relation

$$(F(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0, \ (k,q) \in S^6_{\sqrt{a}} \ a.e.$$

and via Theorem 3 of [11] equation (1.6) admits a unique solution $u(x,y) \in L^2(\mathbb{R}^6)$. From problem (1.6) under our assumptions on the scalar potentials involved in it and its right side we deduce that $\Delta u(x,y) \in L^2(\mathbb{R}^6)$ such that $u(x,y) \in H^2(\mathbb{R}^6)$ for this unique solution. ■

We finish the article with the proof of the result when the free Laplacian is added to the Schrödinger operator with a shallow, short-range potential.

Proof of Theorem 7. Let us first assume that problem (1.10) possesses a unique solution $u(x,y) \in H^2(\mathbb{R}^{n+3})$ and $Q(x,y) \in \tilde{W}^{2,\infty}(\mathbb{R}^{n+3})$ satisfies the homogeneous equation (1.11) with $\tilde{W}^{2,\infty}(\mathbb{R}^{n+3})$ defined in (1.12). Hence

$$(-\Delta_x u - \Delta_y u + U(y)u - au, Q\xi_n)_{L^2(\mathbb{R}^{n+3})} = (\phi, Q\xi_n)_{L^2(\mathbb{R}^{n+3})}.$$

We integrate by parts in the inner product on the left side of the identity above and perform the limiting argument with ideas similar to those used in the proofs of the previous two theorems to obtain the orthogonality relation (1.13).

In the second part of the proof we assume that condition (1.13) holds. According to the Proof of Theorem 6 of [11], under the conditions stated above the Schrödinger type operator in the left side of (1.10) is unitarily equivalent to $-\Delta_x - \Delta_y - a$ on $L^2(\mathbb{R}^{n+3})$ by means of the wave operators. First we consider the case of $a > 0$. Since $\eta_q(y) \in \tilde{W}^{2,\infty}(\mathbb{R}^3)$ as discussed above, the functions of the continuous spectrum of the original Schrödinger operator on $L^2(\mathbb{R}^{n+3})$

$$\frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\eta_q(y), \ (k,q) \in S^{n+3}_{\sqrt{a}} \ a.e. \tag{2.18}$$

belong to $L^\infty(\mathbb{R}^{n+3})$. Obviously,

$$\nabla \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\eta_q(y) = ik\frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\eta_q(y) + \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\nabla_y\eta_q(y) \in L^\infty(\mathbb{R}^{n+3}),$$

$$\Delta_x \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\eta_q(y) = -k^2\frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\eta_q(y) \in L^\infty(\mathbb{R}^{n+3}), \ \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\Delta_y\eta_q(y) \in L^\infty(\mathbb{R}^{n+3}),$$

such that the functions given by (2.18) belong to $\tilde{W}^{2,\infty}(\mathbb{R}^{n+3})$. A trivial calculation using (2.16) yields that they satisfy (1.11). Thus we arrive at

$$(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\eta_q(y))_{L^2(\mathbb{R}^{n+3})} = 0$$

with $(k, q) \in S_{\sqrt{a}}^{n+3}$ a.e. Due to Theorem 6 of [11] problem (1.10) has a unique solution $u(x, y) \in L^2(\mathbb{R}^{n+3})$. From equation (1.10) under our assumptions on its right side and the potential function $U(y)$ it can be easily observed that $\Delta u(x, y) \in L^2(\mathbb{R}^{n+3})$. Hence for this unique solution $u(x, y) \in H^2(\mathbb{R}^{n+3})$.

We finish the proof of the theorem with the studies of the case when the constant $a = 0$ and the dimension $n = 1$. For the function of the continuous spectrum with the vanishing wave vector $\eta_0(y) \in L^\infty(\mathbb{R}^4)$ we have $\nabla \eta_0(y) = \nabla_y \eta_0(y) \in L^\infty(\mathbb{R}^4)$. Since $\Delta_y \eta_0(y) \in L^\infty(\mathbb{R}^4)$ and $\Delta_x \eta_0(y)$ vanishes, we arrive at $\eta_0(y) \in \tilde{W}^{2,\infty}(\mathbb{R}^4)$. By means of (2.16) it solves equation (1.11). Hence

$$(\phi(x, y), \eta_0(y))_{L^2(\mathbb{R}^4)} = 0$$

and via Theorem 6 of [11] equation (1.10) with $a = 0$ and $n = 1$ admits a unique solution $u(x, y) \in L^2(\mathbb{R}^4)$. Analogously to the case of $a > 0$ discussed above this unique solution will belong to $u(x, y) \in H^2(\mathbb{R}^4)$. ■

References

[1] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry*, Springer-Verlag, Berlin (1987).

[2] A. Ducrot, M. Marion, V. Volpert, Systemes de réaction-diffusion sans propriété de Fredholm, CRAS, **340** (2005), 659–664.

[3] A. Ducrot, M. Marion, V. Volpert, Reaction-diffusion problems with non Fredholm operators, Advances Diff. Equations, **13**, No. 11-12 (2008), 1151–1192.

[4] B.L.G. Jonsson, M. Merkli, I.M. Sigal, F. Ting, Applied Analysis, In preparation.

[5] T. Kato, Wave operators and similarity for some non-selfadjoint operators, Math. Ann., **162** (1965/1966), 258–279.

[6] E. Lieb, M. Loss, *Analysis. Graduate Studies in Mathematics*, **14**, American Mathematical Society, Providence (1997).

[7] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, III: Scattering Theory*, Academic Press (1979).

[8] I. Rodnianski, W. Schlag, Time decay for solutions of Schrödinger equations with rough and time-dependent potentials, Invent. Math., **155**, No. 3 (2004), 451–513.

[9] V. Volpert, B. Kazmierczak, M. Massot, Z.Peradzynski, Solvability conditions for elliptic problems with non-Fredholm operators, Appl.Math., **29**, No. 2 (2002), 219–238.

[10] V. Vougalter, V. Volpert, Solvability conditions for some non Fredholm operators, To appear in: Proc. Edinb. Math. Soc., http://hal.archives-ouvertes.fr/hal-00362446/fr/

[11] V. Vougalter, V. Volpert. *On the solvability conditions for some non Fredholm operators*, Int. J. Pure Appl. Math., **60**, No. 2 (2010), 169–191.

[12] V. Vougalter, V. Volpert. *On the solvability conditions for the diffusion equation with convection terms*, To appear in: Commun. Pure Appl. Anal.