PRESERVATION OF A.C. SPECTRUM FOR RANDOM DECAYING PERTURBATIONS OF SQUARE-SUMMABLE HIGH-ORDER VARIATION

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Abstract. We consider random selfadjoint Jacobi matrices of the form

$$(\mathbf{J}_\omega u)(n) = a_n(\omega)u(n+1) + b_n(\omega)u(n) + a_{n-1}(\omega)u(n-1)$$

on $\ell^2(\mathbb{N})$, where $\{a_n(\omega) > 0\}$ and $\{b_n(\omega) \in \mathbb{R}\}$ are sequences of random variables on a probability space $(\Omega, dP(\omega))$ such that there exists $q \in \mathbb{N}$, such that for any $l \in \mathbb{N}$,

$$\beta_{2l}(\omega) = a_l(\omega) - a_{l+q}(\omega) \text{ and } \beta_{2l+1}(\omega) = b_l(\omega) - b_{l+q}(\omega)$$

are independent random variables of zero mean satisfying

$$\sum_{n=1}^{\infty} \int_\Omega \beta_n^2(\omega)\, dP(\omega) < \infty.$$

Let $\mathbf{J}_p$ be the deterministic periodic (of period q) Jacobi matrix whose coefficients are the mean values of the corresponding entries in $\mathbf{J}_\omega$.

We prove that for a.e. ω, the a.c. spectrum of the operator $\mathbf{J}_\omega$ equals to and fills the spectrum of $\mathbf{J}_p$. If, moreover,

$$\sum_{n=1}^{\infty} \int_\Omega \beta_n^4(\omega)\, dP(\omega) < \infty,$$

then for a.e. ω, the spectrum of $\mathbf{J}_\omega$ is purely absolutely continuous on the interior of the bands that make up the spectrum of $\mathbf{J}_p$.

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1. Introduction.

In this paper we study sufficient conditions for preservation of a.c. spectrum of periodic Jacobi matrices under a natural class of random slowly decaying perturbations.

For any two bounded sequences of real numbers $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ and $\mathbf{b} = \{b_n\}_{n=1}^{\infty}$, where $a_n > 0$, we define a Jacobi matrix $\mathbf{J} = \mathbf{J}(\mathbf{a}, \mathbf{b})$ by

$$(\mathbf{J}u)(n) = a_n u(n+1) + b_n u(n) + a_{n-1} u(n-1). \tag{1}$$

We consider only such matrices whose elements are bounded, so they define bounded self-adjoint operators on $\ell^2(\mathbb{N})$. The special case of $\mathbf{a} = \mathbf{1}$ (namely, $a_n = 1$ for all n) is also called a *discrete Schrödinger operator* and its diagonal is then referred to as a *potential*. The discrete Schrödinger operator with zero potential $\Delta = \mathbf{J}(\mathbf{1}, \mathbf{0})$ is called the *free* discrete *Laplacian*.

We say that the absolutely continuous spectrum of an operator of the form (1) *fills* a set S if $\mu_{\mathrm{ac}}(Q) > 0$ for any $Q \subset S$ of positive Lebesgue measure. We say that its spectrum is purely absolutely continuous on S if, in addition, $\mu_{\mathrm{sing}}(S) = 0$. Here $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sing}}$ is the decomposition of the spectral measure of the operator into absolutely continuous and singular parts. The *essential support* of the absolutely continuous spectrum of such an operator, denoted $\Sigma_{ac}(\,\cdot\,)$, is the equivalence class (up to sets of zero Lebesgue measure) of the largest set filled by its absolutely continuous spectrum.

Preservation of the absolutely continuous spectrum of an operator under decaying perturbations has been the subject of extensive research over the last two decades. One can start, e.g, from the free Laplacian that has purely a.c. spectrum filling the interval $[-2, 2]$ and add to it a decaying potential. Two basic facts have been known for a long time:

Theorem 1. *If* $\sum_{n=1}^{\infty} |a_n - 1| + |b_n| < \infty$*, then the a.c. spectrum of* $\mathbf{J}(\mathbf{a}, \mathbf{b})$ *is the same as that of* Δ *in the sense that it is equal to and fills* $[-2, 2]$ *and that* $\mathbf{J}(\mathbf{a}, \mathbf{b})$ *has purely a.c. spectrum on the interior of* $[-2, 2]$.

Theorem 2. *If $a_n \to 1$, $b_n \to 0$ as $n \to \infty$, and*

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| + |b_{n+1} - b_n| < \infty,$$

then the a.c. spectrum of $\mathbf{J}(\mathbf{a}, \mathbf{b})$ is the same as that of Δ in the same sense as in Theorem 1.

The first fact has been known at least from the 1950's and follows, in particular, from Kato-Birman theory of trace class perturbations (see [25, vol III]). The second has been essentially proven by Weidemann [34] in 1967. (Weidemann actually proved a variant of this for continuous Schrödinger operators. For a proof of the discrete case, see Dombrowski-Nevai [10] or Simon [28].)

On the other hand, the works of Delyon, Simon and Souillard [7, 8, 27] and Kotani-Ushiroya [20, 21] on decaying random potentials showed in the 1980's that perturbations that are not square-summable can result in purely singular spectrum. In the 1990's much work (see, e.g., [3, 5, 6, 17, 18, 26]) has been done towards showing that square-summable perturbations of the free Laplacian do not change its a.c. spectrum. Eventually, Killip and Simon [15] proved, in particular, the following.

Theorem 3. *If a perturbation is* square-summable*, that is*

$$\sum_{n=1}^{\infty} |a_n - 1|^2 + |b_n|^2 < \infty,$$

then $\Sigma_{ac}(\mathbf{J}(\mathbf{a}, \mathbf{b}))$, the essential support of the a.c. spectrum of $\mathbf{J}(\mathbf{a}, \mathbf{b})$, is equal to $[-2, 2]$.

In [19], Kiselev, Last and Simon conjectured the following.

Conjecture 1. *If a potential $\hat{\mathbf{b}}$ is* square-summable*, then, for* any *Jacobi matrix $\mathbf{J}(\mathbf{a}, \mathbf{b})$, $\Sigma_{ac}(\mathbf{J}(\mathbf{a}, \mathbf{b} + \hat{\mathbf{b}}))$ is equal to $\Sigma_{ac}(\mathbf{J}(\mathbf{a}, \mathbf{b}))$.*

Results towards the full Conjecture 1 seem to be scarce so far. Killip [14] has proven it for the case of discrete Schrödinger operators with *periodic* potentials. Breuer and Last [1] have recently shown that, for any Jacobi matrix, the a.c. spectrum which is associated with bounded generalized eigenfunctions (like the spectrum of a periodic Jacobi matrix) is stable under square-summable *random* perturbations. In [12] we have shown that if both a square-summable random perturbation and a decaying perturbation of bounded variation are added to the free Laplacian, the a.c. spectrum is still preserved. But, to the best of our knowledge, there has been no significant progress made so far with the general deterministic case.

Another natural direction of research in this area is to extend and generalize Theorem 2 in the direction in which Theorem 1 has been extended. A notable result in this direction has been obtained by Kupin [22] who showed that the essential support of the a.c. spectrum of Δ is still preserved if a decaying potential of a square-summable variation is added to it under an additional restriction that this perturbation lies in ℓ^m for some $m \in \mathbb{N}$. But for a perturbation of a bounded variation of a *general* Jacobi matrix, a guess even weaker than an analog of Conjecture 1 would be wrong. Indeed, one of us have recently constructed [24] an example of a Jacobi matrix $\mathbf{J}(\mathbf{a}, \tilde{\mathbf{b}} + \hat{\mathbf{b}})$ with $\mathbf{a} = \mathbf{1}$, $\lim_{n\to\infty} \tilde{b}_n = \lim_{n\to\infty} \hat{b}_n = 0$, so that $\{\tilde{b}_n\}_{n=1}^{\infty}$ is of bounded variation, both $\mathbf{J}(\mathbf{a}, \hat{\mathbf{b}})$ and $\mathbf{J}(\mathbf{a}, \tilde{\mathbf{b}})$ have purely a.c. spectrum on $(-2, 2)$ with essential support $(-2, 2)$, but $\mathbf{J}(\mathbf{a}, \tilde{\mathbf{b}} + \hat{\mathbf{b}})$ has empty absolutely continuous spectrum. In particular, adding a decaying perturbation of bounded variation to a Jacobi matrix can fully "destroy" its absolutely continuous spectrum.

Thus, it would be natural to confine the consideration first to the simple case of perturbations of a summable variation added to a periodic Jacobi matrix. In particular, we note the following result of Golinskii-Nevai [11] concerning variations of order q (before [11], some related results were obtained by Stolz [30, 31]):

Theorem 4. *Let $\mathbf{J}(\mathbf{a}, \mathbf{b})$ be a periodic Jacobi matrix of period q and let $\{\hat{a}_n\}_{n=1}^{\infty}$ and $\{\hat{b}_n\}_{n=1}^{\infty}$ be decaying sequences obeying,*

$$\sum_{n=1}^{\infty} |\hat{a}_{n+q} - \hat{a}_n| + |\hat{b}_{n+q} - \hat{b}_n| < \infty. \tag{2}$$

Then the essential support of the a.c. spectrum of $\mathbf{J}(\mathbf{a}+\hat{\mathbf{a}}, \mathbf{b}+\hat{\mathbf{b}})$ is equal to the spectrum of $\mathbf{J}(\mathbf{a}, \mathbf{b})$ and, moreover, the spectrum of $\mathbf{J}(\mathbf{a}+\hat{\mathbf{a}}, \mathbf{b}+\hat{\mathbf{b}})$

is purely absolutely continuous on the interior of the bands that make up the spectrum of $\mathbf{J}(\mathbf{a}, \mathbf{b})$.

Remark. Note that, in particular, this theorem extends Theorem 2 to the case where the free Laplacian is replaced by a periodic Jacobi matrix.

We believe that the following statement (which generalizes [24, Conjecture 1.6]) should be true:

Conjecture 2. *Let* $\mathbf{J}(\mathbf{a}, \mathbf{b})$ *be a periodic Jacobi matrix of period* q *and let* $\{\hat{a}_n\}_{n=1}^{\infty}$ *and* $\{\hat{b}_n\}_{n=1}^{\infty}$ *be decaying sequences obeying,*

$$\sum_{n=1}^{\infty} |\hat{a}_{n+q} - \hat{a}_n|^2 + |\hat{b}_{n+q} - \hat{b}_n|^2 < \infty. \tag{3}$$

Then the essential support of the a.c. spectrum of $\mathbf{J}(\mathbf{a} + \hat{\mathbf{a}}, \mathbf{b} + \hat{\mathbf{b}})$ *is equal to the spectrum of* $\mathbf{J}(\mathbf{a}, \mathbf{b})$.

We note that a variant of this conjecture for the special case $q = 1$ has also been made by Simon [29, Chapter 12]. Kim and Kiselev [16] made some progress towards Conjecture 2 by extending to the discrete case some of the results and techniques previously used by Christ and Kiselev [4] to treat continuous (namely, differential) Schrödinger operators. They studied the discrete Schrödinger case where $\mathbf{a} = \mathbf{1}$, $\mathbf{b} = \mathbf{0}$ (so $\mathbf{J}(\mathbf{a}, \mathbf{b})$ is just the discrete Laplacian), $\hat{\mathbf{a}} = \mathbf{0}$ and where $\hat{\mathbf{b}}$ is a bounded (but not necessarily decaying) sequence obeying $\sum_{n=1}^{\infty} |\hat{b}_{n+1} - \hat{b}_n|^p < \infty$ for some $p < 2$. They show that in this case the essential support of the a.c. spectrum coincides with $[-2 + \limsup \hat{b}_n, 2 + \limsup \hat{b}_n] \cap [-2 + \liminf \hat{b}_n, 2 + \liminf \hat{b}_n]$. We note, however, that the case $p = 2$ appears to be outside the scope of their techniques. Some very significant progress towards Conjecture 2 has been recently made by Denisov [9], who proved the full [24, Conjecture 1.6], namely, Conjecture 2 for the discrete Schrödinger case where $\mathbf{a} = \mathbf{1}$, $\mathbf{b} = \mathbf{0}$ and $\hat{\mathbf{a}} = \mathbf{0}$. We believe that his ideas are likely to be extensible to the more general setting of Conjecture 2 and we hope that it will thus be soon possible to prove it in full [13].

Our present work explores perturbations whose variations of order q are square-summable random variables. As mentioned above, the study of random perturbations have indicated the exact rate of the decay that still preserves the a.c. spectrum in the extension of Theorem 1. From this point of view, part of the interest in our present result is that it can be considered as "evidence" in support of Conjecture 2. We note, however, that for appropriate cases (namely, when the variations are also fourth-order-summable), our result also yields stable purity of the a.c. spectrum which is not expected to hold in the general deterministic setting of Conjecture 2. Indeed, we obtain for such cases the same kind of a.c. spectrum preservation as in Theorem 4. Thus, our result here goes beyond being a random version of Conjecture 2 and we believe that it cannot be deduced from any deterministic statement of its type.

Our main result in this paper is the following:

Theorem 5. *Let J_ω be a selfadjoint random Jacobi matrix*

$$(J_\omega u)(n) = a_n(\omega)u(n+1) + b_n(\omega)u(n) + a_{n-1}(\omega)u(n-1)$$

on $\ell^2(\mathbb{N})$, where $\{a_n(\omega) > 0\}$ and $\{b_n(\omega) \in \mathbb{R}\}$ are sequences of random variables on a probability space $(\Omega, dP(\omega))$ such that there exists $q \in \mathbb{N}$, so that for any $l \in \mathbb{N}$,

$$\beta_{2l}(\omega) = a_l(\omega) - a_{l+q}(\omega) \text{ and } \beta_{2l+1}(\omega) = b_l(\omega) - b_{l+q}(\omega) \tag{4}$$

are independent random variables of zero mean satisfying

$$\sum_{n=1}^{\infty} \int_\Omega \beta_n^2(\omega)\, dP(\omega) < \infty. \tag{5}$$

Let $\mathbf{J}_p = \mathbf{J}(\tilde{\mathbf{a}}, \tilde{\mathbf{b}})$ be the deterministic periodic (of the period q) Jacobi matrix whose coefficients are

$$\tilde{a}_l = \int_\Omega a_l(\omega)\, dP(\omega) = \tilde{a}_{l+q},$$

$$\tilde{b}_l = \int_\Omega b_l(\omega)\, dP(\omega) = \tilde{b}_{l+q}.$$

Then, for a.e. ω, the a.c. spectrum of the operator $\mathbf{J}_\omega$ equals to and fills the spectrum of $\mathbf{J}_p$.

If, moreover,

$$\sum_{n=1}^{\infty} \int_\Omega \beta_n^4(\omega)\, dP(\omega) < \infty,$$

then for a.e. ω, the spectrum of $\mathbf{J}_\omega$ is purely absolutely continuous on the interior of the bands that make up the spectrum of $\mathbf{J}_p$.

Remark. For cases where Theorem 5 yields purity of the a.c. spectrum, our proof actually shows something a bit stronger, namely, that for a.e. fixed ω, the purity of the absolutely continuous spectrum will be stable under changing any finite number of entries in the Jacobi matrix.

Theorem 5 is proven in Section 2.

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2. Proof of Theorem 5

Let us start from building explicitly a random Jacobi operator $\mathbf{J}_\omega$ on $\ell^2(\mathbb{N})$, satisfying the condition (4) and introducing some notations.

Let $\mathbf{J}_p$ be a periodic (of period q) Jacobi matrix

$$(\mathbf{J}_p u)(n) = \tilde{a}_n u(n+1) + \tilde{b}_n u(n) + \tilde{a}_{n-1} u(n-1),$$

such that $\tilde{a}_{n+q} = \tilde{a}_n$, $\tilde{b}_{n+q} = \tilde{b}_n$ and $\min \tilde{a}_n = \varepsilon_0 > 0$.

Consider a sequence $\{\beta_n\}_{n=1}^\infty$ of independent random variables on a probability space $(\Omega, dP(\omega))$ that satisfies

$$\forall n \; \mathbb{E}(\beta_n) \stackrel{\text{Def}}{=} \int_\Omega \beta_n(\omega)\, dP(\omega) = 0 \tag{6}$$

and

$$\sum_{n=1}^\infty \mathbb{E}\big(\beta_n^2\big) < \infty.$$

In such a case, for $n, m \in \mathbb{N}$ and $0 \le m < q$,

$$\alpha_n(\omega) = \sum_{i=0}^\infty \beta_{2qi+n}(\omega), \tag{7}$$

$$a_{nq+m}(\omega) = \tilde{a}_m + \alpha_{2(nq+m)}(\omega), \quad b_{nq+m}(\omega) = \tilde{b}_m + \alpha_{2(nq+m)+1}(\omega)$$

are well defined for a.e. ω (see, e.g., [33]). To get $a_n(\omega) > 0$ for all $n \in \mathbb{N}$, we can take, e.g., β_i such that $|\beta_i(\omega)| < C/i$, for a.e. ω, for an appropriate constant C. The operator $\mathbf{J}_\omega = \mathbf{J}(\{a_n(\omega)\}_{n=1}^\infty, \{b_n(\omega)\}_{n=m}^\infty)$ will be then a well-defined self-adjoint random Jacobi matrix.

So, suppose an operator $\mathbf{J}_\omega$ satisfying the conditions of Theorem 5 is given. Denote $\tilde{a}_n = \int_\Omega a_n(\omega)\, dP(\omega)$, $\tilde{b}_n = \int_\Omega b_n(\omega)\, dP(\omega)$ and $\alpha_n(\omega)$ as in (7).

Our analysis of the a.c. spectrum of the operator $\mathbf{J}_\omega$ will be built upon establishing the near-boundedness of its generalized eigenfunctions, which are the solutions of the difference equation (here and in what follows we denote by the same letter an operator on $\ell^2(\mathbb{N})$ and the corresponding difference operator on the vector space of all real valued sequences)

$$\mathbf{J}_\omega u = Eu.$$

Definition. The sequence $\{\mathcal{T}_n(E,\omega)\}$ of *transfer matrices* for the operator $\mathbf{J}_\omega$ at the energy E is defined by

$$\mathcal{T}_n(E,\omega) = \begin{pmatrix} \frac{E-b_n(\omega)}{a_n(\omega)} & \frac{-a_{n-1}(\omega)}{a_n(\omega)} \\ 1 & 0 \end{pmatrix} : \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} \longmapsto \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix},$$

where u is a solution of the difference equation $\mathbf{J}_\omega u = Eu$. For $n \geq m$, we define

$$\mathcal{T}_{n,m}(E,\omega) \stackrel{\text{Def}}{=} \mathcal{T}_{n-1}(E,\omega)\mathcal{T}_{n-2}(E,\omega)\ldots\mathcal{T}_m(E,\omega); \quad \mathcal{T}_{n,n} \stackrel{\text{Def}}{=} \mathcal{I}.$$

The transfer matrices $\tilde{\mathcal{T}}_n(E)$ and $\tilde{\mathcal{T}}_{n,m}(E)$ for the operator $\mathbf{J}_p$ are defined similarly. Of course, $\tilde{\mathcal{T}}_n(E) = \tilde{\mathcal{T}}_{n+q}(E)$, so we will be writing, e.g., $\tilde{\mathcal{T}}_0(E)$ understanding it as $\tilde{\mathcal{T}}_q(E)$.

The growth of generalized eigenfunctions for $\mathbf{J}_\omega$ will be controlled by the growth of $\|\mathcal{T}_{n,m}(E,\omega)\|$ as $n \to \infty$. In particular, we shall use the following theorems from [23]:

Theorem 6. *Let $\mathbf{J}$ be a Jacobi matrix and $\mathcal{T}_{n,m}(E)$ its transfer matrices. Suppose S is a set such that for a.e. $E \in S$,*

$$\varliminf_{n\to\infty} \|\mathcal{T}_{n,m}(E)\| < \infty$$

(a fact that does not depend on m). Then the absolutely continuous spectrum of the operator $\mathbf{J}$ fills S.

Theorem 7. *Suppose there is some $m \in \mathbb{N}$ so that*

$$\underline{\lim}_{n\to\infty} \int_c^d \|\mathcal{T}_{n,m}(E,\omega)\|^p \, dE < \infty$$

for some $p > 2$. Then (c,d) is in the essential support of the absolutely continuous spectrum of $\mathbf{J}_\omega$ and the spectrum of $\mathbf{J}_\omega$ is purely absolutely continuous on (c,d).

Remarks. 1. Theorem 7 is essentially Theorem 1.3 of [23]. While [23] only discusses Jacobi matrices with $a_n = 1$, the result easily extends to our more general context.

2. As noted in [23], Theorem 7 is an extension of an idea of Carmona [2].

3. While the fact that (c,d) is in the essential support of the absolutely continuous spectrum isn't explicitly stated in [23, Theorem 1.3], this easily follows from spectral averaging and the fact that the

$$\underline{\lim}_{n\to\infty} \int_c^d \|\mathcal{T}_{n,m}(E,\omega)\|^p \, dE < \infty$$

condition is invariant to changing any finite number of entries in the Jacobi matrix.

To single out the independent random variables we will rather consider for $n \geq m$ the matrices

$$\mathcal{P}_{n,m}(E,\omega) = \mathcal{A}_m(\omega)\mathcal{T}_{n,m}^{-1}(E,\omega)\mathcal{A}_n^{-1}(\omega), \quad \mathcal{A}_n(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & a_{n-1}(\omega) \end{pmatrix}.$$

In particular, $\mathcal{P}_{n,m}(E,\omega) = \mathcal{P}_m(E,\omega) \dots \mathcal{P}_{n-1}(E,\omega)$, where

$$\mathcal{P}_n(E,\omega) = \mathcal{P}_{n+1,n}(E,\omega) = \begin{pmatrix} 0 & 1/a_n(\omega) \\ -a_n(\omega) & \frac{E-b_n(\omega)}{a_n(\omega)} \end{pmatrix}.$$

Note that, as a random variable, $\mathcal{P}_{n,m}(E,\omega)$ depends only on $\{\beta_j\}_{j=2m}^{\infty}$.

Let $\varepsilon_1 = \varepsilon_0/2 = \min_{1\leq n\leq q} \tilde{a}_n/2$. Under the assumption that

$$\sup_{n\geq m} |\alpha_n(\omega)| \leq \varepsilon_1, \tag{8}$$

the norms of the matrices $\mathcal{T}_l(E,\omega)$ are uniformly bounded and to get a uniform bound on the norms of the matrices $\mathcal{T}_{l,mq}(E,\omega)$ as $l \to \infty$ it is sufficient to prove that $\overline{\lim}_{n\to\infty} \left\|\mathcal{T}_{(m+n)q,mq}(E,\omega)\right\| < \infty$, since between

$(m+n)q$ and $(m+n+1)q$ the norm of a transfer matrix cannot be more than $\sup_l \|\mathcal{T}_l(E,\omega)\|^{q-1}$ times larger.

But (8) also implies that

$$\det \mathcal{T}_{n,m}(E,\omega) = \frac{-a_{m-1}(\omega)}{a_{n-1}(\omega)} \sim 1,$$

$\|\mathcal{A}_m(\omega)\| \sim 1$ and $\|\mathcal{A}_n^{-1}(\omega)\| \sim 1$, so

$$\left\|\mathcal{T}_{(m+n)q,mq}(E,\omega)\right\| \sim \left\|\mathcal{T}^{-1}_{(m+n)q,mq}(E,\omega)\right\| \sim \left\|\mathcal{P}_{(m+n)q,mq}(E,\omega)\right\|.$$

Define

$$\tilde{\mathcal{M}}(E) = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{a}_0 \end{pmatrix} \tilde{\mathcal{T}}^{-1}_{q,0}(E) \begin{pmatrix} 1 & 0 \\ 0 & \tilde{a}_q \end{pmatrix}^{-1}.$$

Note that because of periodicity of $\tilde{a}_n$ and $\tilde{b}_n$, we have, for every E

$$\det \tilde{\mathcal{M}}(E) = \det \tilde{\mathcal{T}}_{q,0}(E) = 1. \tag{9}$$

The $\operatorname{tr} \tilde{\mathcal{M}}(E)$ is a polynomial function of E and, because of (9),

$$\operatorname{tr} \tilde{\mathcal{M}}(E) = \operatorname{tr} \tilde{\mathcal{T}}^{-1}_{q,0}(E) = \operatorname{tr} \tilde{\mathcal{T}}_{q,0}(E). \tag{10}$$

So, from the general Floquet theory (see, e.g., [32, chap. 7]) we know that the spectrum of the operator $\mathbf{J}_p$ is the union of the intervals of $\mathbb{R}$ that are defined by $|\operatorname{tr} \tilde{\mathcal{M}}(E)| \le 2$. In other words, the matrix $\tilde{\mathcal{M}}(E)$ is *quasi-unitary*[1] inside the spectrum of $\mathbf{J}_p$.

Let a compact set $I \subset \left\{ E \,\middle|\, \left|\operatorname{tr} \tilde{\mathcal{M}}(E)\right| < 2 \right\}$ and some $m \in \mathbb{N}$ be given.

Following Golinskii and Nevai [11], we will use a theorem due to Kooman to prove our result. For a proof of Theorem 8, see [29, chap. 12].

Theorem 8. *Let $\mathcal{A}(E)$ be a quasi-unitary matrix which depends continuously on some parameter E varying in a compact Hausdorff space. Then there exists $\gamma > 0$ and functions $\mathcal{C}_E(\mathcal{Q})$ and $\mathcal{B}_E(\mathcal{Q})$, jointly continuous in E and $\mathcal{Q}$, such that*

[1]A (2×2 in our case) matrix $\mathcal{A}$ is called quasi-unitary if it has two different eigenvalues both of absolute value 1. Obviously, a real matrix is quasi-unitary iff it is similar to a unitary matrix and iff $\det \mathcal{A} = 1$ and $|\operatorname{tr} \mathcal{A}| < 2$. A quasi-unitary matrix is indeed unitary in a norm associated with its eigenvectors as will be defined in what follows.

(i) *for each fixed E, $\mathcal{C}_E(\mathcal{Q})$ and $\mathcal{B}_E(\mathcal{Q})$ are analytic on*

$$N_\gamma(\mathcal{A}(E)) = \{\mathcal{Q} \mid \|\mathcal{A}(E) - \mathcal{Q}\| < \gamma\},$$

(ii) $\mathcal{C}_E(\mathcal{A}(E)) = \mathcal{I}$, $\mathcal{B}_E(\mathcal{A}(E)) = \mathcal{A}(E)$,

(iii) $\forall \mathcal{Q} \in N_\gamma(\mathcal{A}(E))$, $\mathcal{C}_E(\mathcal{Q})$ *is invertible,*

(iv) $\forall \mathcal{Q} \in N_\gamma(\mathcal{A}(E))$, $\mathcal{Q} = \mathcal{C}_E(\mathcal{Q})^{-1}\mathcal{B}_E(\mathcal{Q})\mathcal{C}_E(\mathcal{Q})$,

(v) $\forall \mathcal{Q} \in N_\gamma(\mathcal{A}(E))$, $\mathcal{B}_E(\mathcal{Q})$ *commutes with* $\mathcal{A}(E)$.

We shall apply Kooman's theorem to the quasi-unitary matrix $\tilde{\mathcal{M}}(E)$ for $E \in I$.

Let γ be the provided for $\tilde{\mathcal{M}}(E)$ by Theorem 8. Since, because of (ii),

$$\operatorname{tr} \mathcal{B}_E(\tilde{\mathcal{M}}(E)) = \operatorname{tr} \tilde{\mathcal{M}}(E) \in (-2, 2),$$

we can choose γ small enough to assure $|\operatorname{tr} \tilde{\mathcal{B}}_E(\mathcal{Q})| < 2$ for all $\mathcal{Q} \in \bar{N}_{\gamma/2}(\tilde{\mathcal{M}}(E))$, where $\bar{N}_{\gamma/2}(\mathcal{M}) = \{\mathcal{Q} \mid \|\mathcal{M} - \mathcal{Q}\| \leq \gamma/2\}$.

Note that for $n \in \mathbb{N}$, the matrix

$$\mathcal{M}_{n+m}(E, \omega) = \mathcal{P}_{(m+n+1)q,(m+n)q}(E, \omega),$$

corresponding to the $(m+n)$-th period for $\mathbf{J}_\omega$ is like the matrix $\tilde{\mathcal{M}}(E)$ corresponding to a period of $\mathbf{J}_p$, with $\tilde{a}$'s and $\tilde{b}$'s perturbed by

$$\alpha_{2(m+n)q}(\omega), \ldots, \alpha_{2(m+n+1)q-1}(\omega).$$

We think of $\tilde{a}_1, \ldots, \tilde{a}_q$ and $\tilde{b}_1, \ldots, \tilde{b}_q$ as of fixed values and the dependence of $\mathcal{M}_n(E, \omega)$ on α's and E is continuous and even analytic. Hence, there exists $\varepsilon_2 > 0$ (we take $\varepsilon_2 \leq \varepsilon_1$ to satisfy also the assumption (8)), such that

$$\sup_{l > 2qm} |\alpha_l(\omega)| \leq \varepsilon_2 \leq \varepsilon_1 \tag{11}$$

implies that for all $E \in I$ and for all $n \in \mathbb{N}$,

$$\mathcal{M}_n(E, \omega) \in \bar{N}_{\gamma/2}(\tilde{\mathcal{M}}(E)).$$

Then, as mentioned above, $|\operatorname{tr} \mathcal{B}_E(\mathcal{M}_n(E, \omega))| < 2$ and, because of (iv),

$$\det \mathcal{B}_E(\mathcal{M}_n(E, \omega)) = \det \mathcal{M}_n(E, \omega) = 1.$$

This means that the matrix $\mathcal{B}_E(\mathcal{M}_n(E, \omega))$ is then quasi-unitary.

Let, under the assumption (11), for an $E \in I$, $\{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis, depending on E, in which the matrix $\tilde{\mathcal{M}}(E)$ and also, because of (v), each $\mathcal{B}_E(\mathcal{M}_n(E,\omega))$, for $n \in \mathbb{N}$, are diagonal. Let $\|\cdot\|_E$ be the norm in this basis, that is, for a vector $\mathbf{v} = x\mathbf{v}_1 + y\mathbf{v}_2$ its norm will be $\|\mathbf{v}\|_E^2 = |x|^2 + |y|^2$, and for a matrix $\mathcal{M}$, $\|\mathcal{M}\|_E = \sup_{\|\mathbf{v}\|_E=1} \|\mathcal{M}(\mathbf{v})\|_E$. In this norm $\mathcal{B}_E(\mathcal{M}_n(E,\omega))$ is unitary for every $n \in \mathbb{N}$ and for every $E \in I$.

Let us denote, for $0 \le l \le n$, $\mathcal{M}_{n+m,l+m}(E,\omega)$ as

$$\mathcal{M}_{l+m}(E,\omega)\ldots\mathcal{M}_{n+m-1}(E,\omega) = \mathcal{P}_{(n+m)q,(l+m)q}(E,\omega).$$

In what follows, we will show that the assumption (11) is eventually fulfilled for a.e. ω and then we can bound the norm of $\mathcal{M}_{n,0}(E,\omega)$ uniformly as $n \to \infty$.

Let, for $E \in I$, $\mathbf{v} \in \mathbb{R}^2$ be an arbitrary vector, such that $\|\mathbf{v}\|_E = 1$. Define

$$\mathbf{w}_l(E,\omega) = \mathcal{B}_E(\mathcal{M}_l(E,\omega)) \cdot \mathcal{C}_E(\mathcal{M}_l(E,\omega)) \cdot \mathcal{M}_{n,l}(E,\omega)\mathbf{v}.$$

Now, denoting $\mathcal{C}_l = \mathcal{C}_E(\mathcal{M}_l(E,\omega))$ and $\mathcal{B}_l = \mathcal{B}_E(\mathcal{M}_l(E,\omega))$, we have

$$\mathbf{w}_l = \mathcal{B}_l\mathcal{C}_l\mathcal{M}_{n,l}\mathbf{v} = \mathcal{B}_l\mathcal{C}_l\mathcal{M}_{l+1}\mathcal{M}_{n,l+1}\mathbf{v} =$$

$$\mathcal{B}_l\mathcal{C}_l\mathcal{C}_{l+1}^{-1}\mathcal{B}_{l+1}\mathcal{C}_{l+1}\mathcal{M}_{n,l+1}\mathbf{v} = \mathcal{B}_l\mathcal{C}_l\mathcal{C}_{l+1}^{-1}\mathbf{w}_{l+1}.$$

Hence,

$$\|\mathbf{w}_l\|_E^2 = \left\|\mathcal{C}_l\mathcal{C}_{l+1}^{-1}\mathbf{w}_{l+1}\right\|_E^2 = \left\|(\mathcal{I} + (\mathcal{C}_l - \mathcal{C}_{l+1})\mathcal{C}_{l+1}^{-1})\mathbf{w}_{l+1}\right\|_E^2 =$$

$$\|\mathbf{w}_{l+1}\|_E^2 + 2\left\langle \mathbf{w}_{l+1}, (\mathcal{C}_l - \mathcal{C}_{l+1})\mathcal{C}_{l+1}^{-1}\mathbf{w}_{l+1} \right\rangle_E + \left\|(\mathcal{C}_l - \mathcal{C}_{l+1})\mathcal{C}_{l+1}^{-1}\mathbf{w}_{l+1}\right\|_E^2.$$

Notice that $\mathcal{C}_l = \mathcal{C}_E(\mathcal{M}_l(E,\omega))$ is analytic as a function of

$$\alpha_{2(m+l)q}(\omega), \ldots, \alpha_{2(m+l+1)q-1}(\omega) \text{ and } E,$$

and, hence, since $\beta_i = \alpha_i - \alpha_{i+2q}$, we can represent, for some matrix functions $\mathcal{D}_i(E,\omega)$, $\mathcal{F}_i(E,\omega)$ and $\mathcal{G}_i(E,\omega)$ that depend only on $\{\beta_j\}_{j=i}^{\infty}$,

$$(\mathcal{C}_l - \mathcal{C}_{l+1})\mathcal{C}_{l+1}^{-1} = \sum_{i=2(m+l)q}^{2(m+l+1)q-1} \beta_i(\omega)\mathcal{G}_i(E,\omega), \tag{12}$$

and, also,

$$(\mathcal{C}_l - \mathcal{C}_{l+1})\mathcal{C}_{l+1}^{-1} = \sum_{i=2(m+l)q}^{2(m+l+1)q-1} \left(\beta_i(\omega)\mathcal{D}_{i+1}(E,\omega) + \beta_i^2(\omega)\mathcal{F}_i(E,\omega)\right). \tag{13}$$

The norms of $\mathcal{D}_i(E,\omega)$, $\mathcal{F}_i(E,\omega)$ and $\mathcal{G}_i(E,\omega)$ are (uniformly in n) bounded, provided $\mathcal{M}_n(E,\omega) \in \bar{N}_{\gamma/2}(\tilde{\mathcal{M}}(E))$, $E \in I$.

When we plug (13) into $\left\langle \mathbf{w}_{l+1}, (\mathcal{C}_l - \mathcal{C}_{l+1})\mathcal{C}_{l+1}^{-1}\mathbf{w}_{l+1} \right\rangle_E$ and (12) into $\left\|(\mathcal{C}_l - \mathcal{C}_{l+1})\mathcal{C}_{l+1}^{-1}\mathbf{w}_{l+1}\right\|_E^2$, we see that

$$\|\mathbf{w}_l\|_E^2 < B_l(E,\omega)\,\|\mathbf{w}_{l+1}\|_E^2\,, \tag{14}$$

where

$$B_l(E,\omega) = 1 + \sum_{i=2(m+l)q}^{2(m+l+1)q-1} \left(\beta_i(\omega)D_{i+1}(E,\omega) + \beta_i^2(\omega)F_i(E,\omega)\right)$$

and $D_i(E,\omega)$, $F_i(E,\omega)$ are some scalar (uniformly in n) bounded functions, provided $\mathcal{M}_n(E,\omega) \in \bar{N}_{\gamma/2}(\tilde{\mathcal{M}}(E))$, $E \in I$.

Going from $\mathbf{w}_0$ to $\mathbf{w}_{n-1}$, we get from (14) that $\|\mathcal{M}_{n,0}(E,\omega)\mathbf{v}\|_E^2$ is bounded by

$$\left\|\mathcal{C}_0^{-1}\right\|_E \|\mathcal{C}_n\|_E \prod_{l=0}^{n-1}\left(1 + \sum_{i=2(m+l)q}^{2(m+l+1)q-1} \beta_i(\omega)D_{i+1}(E,\omega) + \beta_i(\omega)^2F_i(E,\omega)\right).$$

Let $B = \max\left\{ \|\mathcal{C}_E(\mathcal{M})\|_E \,\middle|\, E \in I, \mathcal{M} \in \bar{N}_{\gamma/2}(\tilde{\mathcal{M}}(E)) \right\}$. Since, for ω's satisfying the assumption (11), $\mathcal{M}_l(E,\omega)$ lays in $\bar{N}_{\gamma/2}(\tilde{\mathcal{M}}(E))$, we have

$$\|\mathcal{C}_E(\mathcal{M}_l(E,\omega))\|_E \le B$$

for any $E \in I$ and $l \in \mathbb{N}$. In the same way we can assure[2] that

$$\left\|\mathcal{C}_E(\mathcal{M}_n(E,\omega))^{-1}\right\|_E < B,$$
$$\forall i, |D_i(E,\omega)| < B,$$
$$\forall i, |F_i(E,\omega)| < B.$$

For $\{\mathbf{v}_1, \mathbf{v}_2\}$ an orthonormal basis in $\mathbb{R}^2$, for any 2×2 matrix $\mathcal{A}$,

$$\|\mathcal{A}\| \le 2\max\{\|\mathcal{A}\mathbf{v}_1\|, \|\mathcal{A}\mathbf{v}_2\|\}.$$

This, in particular, implies that if for any unit vector $\mathbf{v}$, $\|\mathcal{A}\mathbf{v}\| < B$, then $\|\mathcal{A}\| < 2B$. So, using the fact that $1 + x \le e^x$ for $x \ge 0$,

$$\|\mathcal{M}_{n,0}(E,\omega)\|_E^2 < B\exp\left(\left|\sum_{i=2mq+1}^{2(m+n)q} \beta_i(\omega)D_{i+1}(E,\omega)\right| + B\sum_{i=1}^{\infty}\beta_i(\omega)^2\right).$$

[2]We adopt the convention of denoting different constants by the same letter.

We note that $\sum_{i=1}^{\infty} \beta_i(\omega)^2$ is finite for a.e. ω, from (5), since

$$\int_\Omega \sum_{i=1}^{\infty} \beta_i(\omega)^2 \, dP(\omega) = \sum_{i=1}^{\infty} \int_\Omega \beta_i(\omega)^2 \, dP(\omega) < \infty.$$

To control the rest of the factors in the above estimate and to get along with the assumptions (8) and (11), we will prove the following lemma.

Let $D_l(\omega)$ be an integrable measurable function depending only on $\{\beta_i(\omega)\}_{i=l}^{\infty}$. Remember that $\alpha_n(\omega) = \sum_{i=0}^{\infty} \beta_{2qi+n}(\omega)$ and define

$$S_m^l = \left| \sum_{i=m}^{l-1} \beta_i D_{i+1} \right|.$$

Lemma 1. *For every $\varepsilon > 0$ and $\delta > 0$, there exist $\Omega' \subseteq \Omega$ and $m \in \mathbb{N}$, such that $\mathbf{P}\{\Omega'\} > 1 - \delta$ and for every $\omega \in \Omega'$*

$$\sup_{n>m} |\alpha_n(\omega)| \le \varepsilon, \tag{15}$$

$$\sup_{n>m} S_m^n(\omega) \le 2. \tag{16}$$

Proof. Fix $n > m$ and define

$$\Omega(m,n) = \left\{ \omega \in \Omega \,\middle|\, \max_{l=m,\dots,(n-1)} |\alpha_l| > \varepsilon \;\text{ or }\; \max_{l=(m+1),\dots,n} S_m^l > 2 \right\},$$

$$\Omega_m = \bigcup_{n>m} \Omega(m,n).$$

Note that for $n < n'$, $\Omega(m,n) \subseteq \Omega(m,n')$, so

$$\mathbf{P}\{\Omega_m\} = \lim_{n\to\infty} \mathbf{P}\{\Omega(m,n)\}.$$

The lemma will be proved if we show that $\lim_{m\to\infty} \mathbf{P}\{\Omega_m\} = 0$.

For $m \le l \le n$, $S_m^l \le S_m^n + S_l^n$, so

$$\max_{l=m,\dots,(n-1)} S_l^n \le 1 \implies \max_{l=(m+1),\dots,n} S_m^l \le 2.$$

Hence, for

$$\Omega'(m,n) = \left\{ \omega \in \Omega \,\middle|\, \max_{l=m,\dots,(n-1)} |\alpha_l| > \varepsilon \;\text{ or }\; \max_{l=m,\dots,(n-1)} S_m^l > 1 \right\},$$

$$\Omega(m,n) \subseteq \Omega'(m,n).$$

Now we can proceed as in a standard martingale inequality. Define

$$A_l = \frac{1}{\varepsilon}\sum_{i=l}^{\infty}\beta_i,\ B_l = \sum_{i=l}^{n-1}\beta_i D_{i+1},$$

$$C_j = \{\omega \in \Omega \,|(\forall i > j : |A_i| \le 1, |B_i| \le 1) \text{ and } (|A_j| > 1 \text{ or } |B_j| > 1)\}.$$

Note that for $i<j$, since β_i is independent from β_j, we have, from (6),

$$\int_\Omega \beta_i A_j \chi_{C_j}\, dP(\omega) = \int_\Omega \beta_i\, dP(\omega) \int_\Omega A_j \chi_{C_j}\, dP(\omega) = 0.$$

Also, since D_l depends only on $\{\beta_i\}_{i=l}^{\infty}$, for $i < j$,

$$\int_\Omega \beta_i D_{i+1} B_j \chi_{C_j}\, dP(\omega) = 0.$$

Hence,

$$\int_\Omega A_m^2 \chi_{C_j}\, dP(\omega) \ge \int_\Omega A_j^2 \chi_{C_j}\, dP(\omega),$$

since, as we expand the square $A_m^2 = \left[\frac{1}{\varepsilon}\left(\sum_{i=m}^{j-1}\beta_i\right) + A_j\right]^2$, the expectation of the cross terms vanishes. In the same way,

$$\int_\Omega B_m^2 \chi_{C_j}\, dP(\omega) \ge \int_\Omega B_j^2 \chi_{C_j}\, dP(\omega).$$

Hence

$$\int_\Omega (A_m^2 + B_m^2)\chi_{C_j}\, dP(\omega) \ge \int_\Omega (A_j^2 + B_j^2)\chi_{C_j}\, dP(\omega) \ge \int_\Omega \chi_{C_j}\, dP(\omega).$$

Since $\Omega'(m,n) \subseteq \bigcup_{j=m}^{n-1} C_j$, we have

$$\mathbf{P}\{\Omega(m,n)\} \le \mathbf{P}\{\Omega'(m,n)\} \le \sum_{j=m}^{n-1}\int_\Omega \chi_{C_j}\, dP(\omega)$$

$$\le \sum_{j=m}^{n-1}\int_\Omega (A_m^2 + B_m^2)\chi_{C_j}\, dP(\omega) \le \int_\Omega (A_m^2 + B_m^2)\, dP(\omega).$$

If we expand the squares, using the mutual independence of the β's, (6) and (5), we see that

$$\int_\Omega (A_m^2 + B_m^2)\, dP(\omega) \le B \sum_{i=m}^{\infty}\mathbb{E}\big(\beta_i^2\big) \overset{m\to\infty}{\longrightarrow} 0.$$

This proves the Lemma. $\square$

The Lemma says that for any given $E \in I$, the set of ω's such that

- there exists m such that the assumptions (8) and (11) hold for $n \geq m$ and then
- $\overline{\lim}_{n\to\infty} \|\mathcal{M}_{m+n,m}(E,\omega)\| < \infty$,

is of full measure.

Since I is an arbitrary compact set in the spectrum of $\mathbf{J}_p$, this, along with Theorem 6, proves the first part of our Theorem.

To prove the second part of the Theorem, first note that, when E varies in the compact set I, the matrix $\tilde{\mathcal{M}}(E)$ always has two separate eigenvalues, and hence, we can pick the eigenvectors for it in a continuous manner. This, in particular, will imply that there will be a constant B, such that for any vector $\mathbf{v}$ and a matrix $\mathcal{M}$,

$$1/B\,\|\mathbf{v}\| < \|\mathbf{v}\|_E < B\,\|\mathbf{v}\|\,,\ 1/B\,\|\mathcal{M}\| < \|\mathcal{M}\|_E < B\,\|\mathcal{M}\|\,, \tag{17}$$

where $\|\cdot\|$ is the canonical norm of $\mathbb{R}^2$.

Now the inequality (14) implies that, for $l_- = 2(m+l)q$, $l^+ = 2(m+l+1)q-1$,

$$\|\mathbf{w}_l\|_E^2 < \left(1 + B\sum_{i=l_-}^{l^+}\beta_i^2\right)\|\mathbf{w}_{l+1}\|_E^2 + \sum_{i=l_-}^{l^+}\beta_i\,\langle\,\mathbf{w}_{l+1}, \mathcal{D}_{i+1}(E,\omega)\mathbf{w}_{l+1}\,\rangle_E\,.$$

Squaring this and grouping the similar powers of β's, we get, after some tedious but straightforward calculations, that (remember that B is just a generic name for some constant)

$$\|\mathbf{w}_l\|_E^4 < \left(1 + B\sum_{i=l_-}^{l^+}(\beta_i^2+\beta_i^4)\right)\|\mathbf{w}_{l+1}\|_E^4 + \\ \sum_{i=l_-}^{l^+}\beta_i\,\langle\,\mathbf{w}_{l+1}, \mathcal{D}_{i+1}(E,\omega)\mathbf{w}_{l+1}\,\rangle_E\,\|\mathbf{w}_{l+1}\|_E^2\,.$$

Integrating the last inequality over I, we get

$$\int_I \|\mathbf{w}_l\|_E^4\,dE < \left(1 + \sum_{i=l_-}^{l^+}(\beta_i\tilde{D}_{i+1}(\omega) + B\beta_i^2 + B\beta_i^4)\right)\int_I \|\mathbf{w}_{l+1}\|_E^4\,dE,$$

where

$$\tilde{D}_i(\omega) = \frac{\int_I \langle\,\mathbf{w}_{l+1}(E,\omega), \mathcal{D}_i(E,\omega)\mathbf{w}_{l+1}(E,\omega)\,\rangle_E\,\|\mathbf{w}_{l+1}(E,\omega)\|_E^2\,dE}{\int_I \|\mathbf{w}_{l+1}(E,\omega)\|_E^4\,dE}$$

is an integrable measurable function depending only on $\{\beta_j(\omega)\}_{j=i}^\infty$.

Now we can proceed as in the first part, use the same lemma, this time for $\tilde{D}(\omega)$, and the fact that we now have

$$\int_\Omega \sum_{i=1}^\infty \beta_i(\omega)^4 \, dP(\omega) = \sum_{i=1}^\infty \int_\Omega \beta_i(\omega)^4 \, dP(\omega) < \infty,$$

so $\sum_{i=1}^\infty \beta_i(\omega)^4$ is finite for a.e. ω, to establish that the set of ω's for which

- there exists m such that the assumptions (8) and (11) hold for $n \geq m$ and then
- $\overline{\lim}_{n\to\infty} \int_I \|\mathcal{M}_{m+n,m}(E,\omega)\|^4 \, dE < \infty$,

is of full measure. By the fact that I is an arbitrary compact set in the spectrum of $\mathbf{J}_p$ and Theorem 7, this proves the second part of Theorem 5. □

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