Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems

Vitali Vougalter[1], **Vitaly Volpert**[2]

[1] Department of Mathematics and Applied Mathematics, University of Cape Town
Private Bag, Rondebosch 7701, South Africa
e-mail: Vitali.Vougalter@uct.ac.za
[2] Institute Camille Jordan, UMR 5208 CNRS, University Lyon 1, 69622 Villeurbanne, France
e-mail: volpert@math.univ-lyon1.fr

Abstract. We obtain solvability conditions in H^2 for some elliptic equations in cylindrical domains using the methods of spectral theory and scattering theory for Schrödinger type operators. We prove the existence of standing solitary waves in H^2 for some nonlinear equations. Both linear and nonlinear problems involve second order differential operators without Fredholm property.

Key words: solvability conditions, non Fredholm operators, embedded solitons, elliptic problems
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1. Introduction

The spectral properties of second order differential operators with and without Fredholm property in cylindrical domains were studied extensively in the past in connection with the reaction-diffusion wave propagation phenomena (see e.g. [1], [7], [8], [11]).

In the first part of the present article we simply consider the Schrödinger type operator

$$L_a := -\Delta_x - \Delta_y - a, \tag{1.1}$$

on $L^2(D)$ such that the generalized cylinder $D = \mathbb{R}^d \times \Omega,\ d \in \mathbb{N}$ and $a \geq 0$ is a parameter. The first Laplacian operator here $-\Delta_x$ is on $L^2(\mathbb{R}^d)$ and the second one $-\Delta_y$ on $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^m,\ m \in \mathbb{N}$ is an open domain of finite Lebesgue measure with Dirichlet boundary conditions. It is well known that such a Dirichlet Laplacian possesses a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 <$

$\lambda_3 < ... < \lambda_n < ...$ increasing to infinity. The corresponding eigenfunctions satisfy the equation

$$-\Delta_y \varphi_n^j(y) = \lambda_n \varphi_n^j(y),\ 1 \le j \le m_n,\ \varphi_n^j(y)|_{\partial\Omega} = 0,$$

where m_n is the multiplicity of the eigenvalue $\lambda_n,\ n \in \mathbb{N}$. These eigenfunctions satisfy the orthogonality relations

$$(\varphi_k^j, \varphi_q^l)_{L^2(\Omega)} = \delta_{k,q}\delta_{j,l},\ k, q \in \mathbb{N},\ 1 \le j \le m_k,\ 1 \le l \le m_q,$$

where $\delta_{k,q},\ \delta_{j,l}$ are the Kronecker symbols and form the complete system in $L^2(\Omega)$. The inner product of two functions is being denoted as $(f_1, f_2)_{L^2(A)} := \int_A f_1(x)\bar{f}_2(x)dx$, with a slight abuse of notations when the functions are not square integrable, like for instance the plane waves involved in the orthogonality conditions of Theorem 1 below. The sphere of radius r centered at the origin in the space of d dimensions is being denoted as S_r^d, the unit one as S^d and its Lebesgue measure as $|S^d|$. By means of the Spectral Theorem we have the representation formula

$$-\Delta_y = \sum_{k=1}^{\infty} \lambda_k P_k, \tag{1.2}$$

where $\{P_k\}_{k=1}^{\infty}$ are the projection operators onto the spectral subspaces correspondent to the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$. The total Laplacian operator on $L^2(D)$ here is $\Delta = \Delta_x + \Delta_y$ and the appropriate functional space is

$$H^2(D) := \{u(x, y) : D \to \mathbb{C} \mid u(x, y) \in L^2(D),\ \Delta u(x, y) \in L^2(D)\}.$$

We investigate the solvability conditions in $H^2(D)$ for the nonhomogeneous equation

$$L_a u = f, \tag{1.3}$$

with a square integrable right side $f(x, y),\ x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d,\ y = (y_1, y_2, ..., y_m) \in \Omega$. The result will depend upon how the parameter a is located on the nonnegative semi-axis compared to the spectrum $\{\lambda_n\}_{n=1}^{\infty}$ of our Dirichlet Laplacian.

Theorem 1. *Let* $f(x, y) \in L^2(D)$. *Then*

I) When $0 \le a < \lambda_1$, *problem (1.3) admits a unique solution* $u(x, y) \in H^2(D)$.

II) When $\lambda_N < a < \lambda_{N+1},\ N \in \mathbb{N}$ *and* $|x|^{\frac{\alpha}{2}} f \in L^2(D)$ *for some* $\alpha > d + 2$ *equation (1.3) possesses a unique solution* $u(x, y) \in H^2(D)$ *if and only if*

$$\left(f(x, y), \frac{e^{\pm i\sqrt{a-\lambda_k}x}}{\sqrt{2\pi}}\varphi_k^j(y)\right)_{L^2(D)} = 0, \quad 1 \le j \le m_k, \quad 1 \le k \le N \quad for \quad d = 1, \tag{1.4}$$

$$\left(f(x, y), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\varphi_k^j(y)\right)_{L^2(D)} = 0, \quad p \in S^d_{\sqrt{a-\lambda_k}} \quad a.e., \quad 1 \le j \le m_k, \quad 1 \le k \le N \tag{1.5}$$

for $d \geq 2$.

III) When $a = \lambda_N,\ N \in \mathbb{N}$ *equation (1.3) possesses a unique solution* $u(x,y) \in H^2(D)$ *if and only if*

a) *for* $d = 1$ *provided* $|x|^{\frac{\alpha}{2}} f(x,y) \in L^2(D)$ *with some* $\alpha > 5$

$$\left(f(x,y), \frac{e^{\pm i\sqrt{\lambda_N - \lambda_k} x}}{\sqrt{2\pi}} \varphi_k^j(y) \right)_{L^2(D)} = 0, \quad 1 \leq j \leq m_k, \quad 1 \leq k \leq N-1, \tag{1.6}$$

$$(f(x,y), \varphi_N^j(y))_{L^2(D)} = 0, \quad (f(x,y), \varphi_N^j(y)x)_{L^2(D)} = 0, \quad 1 \leq j \leq m_N. \tag{1.7}$$

b) *for* $d = 2$ *provided* $|x|^{\frac{\alpha}{2}} f(x,y) \in L^2(D)$ *with some* $\alpha > 6$

$$\left(f(x,y), \frac{e^{ipx}}{2\pi} \varphi_k^j(y) \right)_{L^2(D)} = 0, \quad p \in S^2_{\sqrt{\lambda_N - \lambda_k}} \quad a.e., \quad 1 \leq j \leq m_k, \quad 1 \leq k \leq N-1, \tag{1.8}$$

$$(f(x,y), \varphi_N^j(y))_{L^2(D)} = 0, \quad (f(x,y), \varphi_N^j(y)x_k)_{L^2(D)} = 0, \quad 1 \leq j \leq m_N, \quad k = 1, 2. \tag{1.9}$$

c) *for* $d = 3, 4$ *provided* $|x|^{\frac{\alpha}{2}} f(x,y) \in L^2(D)$ *with some* $\alpha > d+2$

$$\left(f(x,y), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}} \varphi_k^j(y) \right)_{L^2(D)} = 0, \quad p \in S^d_{\sqrt{\lambda_N - \lambda_k}} \quad a.e., \quad 1 \leq j \leq m_k, \quad 1 \leq k \leq N-1, \tag{1.10}$$

$$(f(x,y), \varphi_N^j(y))_{L^2(D)} = 0, \quad 1 \leq j \leq m_N. \tag{1.11}$$

d) *for* $d \geq 5$ *provided* $|x|^{\frac{\alpha}{2}} f(x,y) \in L^2(D)$ *with some* $\alpha > d+2$

$$\left(f(x,y), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}} \varphi_k^j(y) \right)_{L^2(D)} = 0, \quad p \in S^d_{\sqrt{\lambda_N - \lambda_k}} \quad a.e., \quad 1 \leq j \leq m_k, \quad 1 \leq k \leq N-1. \tag{1.12}$$

In the second part of the work we incorporate an external shallow, short-range potential in the original problem when $d = 3$, such that

$$H_a := -\Delta_x + V(x) - \Delta_y - a \tag{1.13}$$

considered on $L^2(D)$, with $D = \mathbb{R}^3 \times \Omega$, the Dirichlet Laplacian $-\Delta_y$ is on $L^2(\Omega)$ as before, the parameter $a \geq 0$ and the assumption below is analogous to the one used in [13], [14], [15], [16] and [17].

Assumption 2. *The potential function* $V(x) : \mathbb{R}^3 \to \mathbb{R}$ *satisfies the estimate*

$$|V(x)| \leq \frac{C}{1 + |x|^{3.5+\varepsilon}}$$

with some $\varepsilon > 0$ *and* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ *a.e. such that*

$$4^{\frac{1}{9}} \frac{9}{8} (4\pi)^{-\frac{2}{3}} \|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}} \|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \quad and \quad \sqrt{c_{HLS}} \|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi.$$

Here and below C denotes a finite positive constant and c_{HLS} given on p.98 of [6] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\Big| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x) f_1(y)}{|x-y|^2} dx dy \Big| \le c_{HLS} \|f_1\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^2, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

Here and further down the norm of a function $f_1 \in L^p(\mathbb{R}^d),\ 1 \le p \le \infty,\ d \in \mathbb{N}$ is denoted as $\|f_1\|_{L^p(\mathbb{R}^d)}$.

Remark. *By means of Lemma 2.3 of [13] under our Assumption 2 the Schrödinger operator* $-\Delta_x + V(x)$ *is self- adjoint and unitarily equivalent to* $-\Delta_x$ *on* $L^2(\mathbb{R}^3)$ *and therefore, it is non-negative in the sense of quadratic forms. Its functions of the continuous spectrum satisfying the Schrödinger equation*

$$[-\Delta_x + V(x)]\eta_q(x) = q^2 \eta_q(x),\ q \in \mathbb{R}^3,$$

in the integral form the Lippmann-Schwinger equation (see e.g. [10] p.98)

$$\eta_q(x) = \frac{e^{iqx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i|q||x-y|}}{|x-y|} (V\eta_q)(y) dy,$$

the orthogonality conditions $(\eta_q(x), \eta_{q_1}(x))_{L^2(\mathbb{R}^3)} = \delta(q - q_1),\ q, q_1 \in \mathbb{R}^3$*, form the complete system in* $L^2(\mathbb{R}^3)$.

Our goal is to solve the following equation in the Sobolev space $H^2(D)$

$$H_a u = f, \tag{1.14}$$

where the right side $f(x,y) \in L^2(D)$ with $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $y = (y_1, y_2, ..., y_m) \in \Omega$ as before. Our statement is as follows.

Theorem 3. *Let Assumption 2 hold,* $f(x,y) \in L^2(D)$ *and* $|x|^{\frac{\alpha}{2}} f(x,y) \in L^2(D)$ *for some* $\alpha > 5$*. Then*

I) When $0 \le a < \lambda_1$ *equation (1.14) admits a unique solution* $u(x,y) \in H^2(D)$.

II) When $\lambda_N < a < \lambda_{N+1},\ N \in \mathbb{N}$ *problem (1.14) possesses a unique solution* $u(x,y) \in H^2(D)$ *if and only if*

$$(f(x,y), \eta_q(x)\varphi_k^j(y))_{L^2(D)} = 0 \ for \ q \in S^3_{\sqrt{a-\lambda_k}} \ a.e.\ ,\ 1 \le j \le m_k,\ 1 \le k \le N. \tag{1.15}$$

III) When $a = \lambda_N,\ N \in \mathbb{N}$ *problem (1.14) admits a unique solution* $u(x,y) \in H^2(D)$ *if and only if*

$$(f(x,y), \eta_0(x)\varphi_N^j(y))_{L^2(D)} = 0 \ for \ 1 \le j \le m_N, \tag{1.16}$$

$$(f(x,y),\eta_q(x)\varphi_k^j(y))_{L^2(D)} = 0 \ for \ q \in S^3_{\sqrt{\lambda_N-\lambda_k}} \ a.e.\ ,\ 1 \le j \le m_k,\ 1 \le k \le N-1. \quad (1.17)$$

Note that in the theorems above although solvability conditions are similar to the usual Fredholm ones, this similarity is only formal because the operators involved in the left sides of the nonhomogeneous elliptic problems do not satisfy the Fredholm property and their ranges are not closed.

We conclude the article with the studies of the following nonlinear problem in $\mathbb{R}^d,\ 1 \le d \le 3$:

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + \lambda \int_{\mathbb{R}^d} G(x-y)[F(|\psi(y,t)|^2)\psi(y,t) + U(y)\psi(y,t)]dy + h(x)e^{-i\omega t}. \quad (1.18)$$

Although we are not aware of particular practical applications of it, (1.18) is of interest to us due to its resemblance to the forced, nonlocal, Nonlinear Schrödinger (NLS) equation. Here the parameters $\lambda \in \mathbb{R},\ \omega \ge 0$ and the conditions on other terms involved in the problem will be specified below. We seek a solution of (1.18) in the form of a standing solitary wave

$$\psi_s(x,t) = \phi(x)e^{-i\omega t}. \quad (1.19)$$

Note that the sign under the exponent here is negative, which corresponds to the case of so-called embedded solitons (see e.g. [9]), as distinct from the standard situation (see e.g. [2], [12]). Thus we arrive at the following nonlocal elliptic problem

$$-\Delta\phi - \omega\phi + \lambda \int_{\mathbb{R}^d} G(x-y)[F(|\phi(y)|^2)\phi(y) + U(y)\phi(y)]dy + h(x) = 0. \quad (1.20)$$

The real valued parameter λ is assumed to be small in the absolute value and the solvability conditions for equation (1.20) when $\lambda = 0$ are derived in the Appendix. The nonlinear problem above involves the operator $-\Delta - \omega : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ without Fredholm property. We will be using the closed unit ball centered at the origin in the Sobolev space with the norm defined in (4.8):

$$B(H^2(\mathbb{R}^d)) = \{u(x) \in H^2(\mathbb{R}^d) \mid \|u\|_{H^2(\mathbb{R}^d)} \le 1\}, \quad (1.21)$$

as distinct from the results obtained in the whole space in [18].

Theorem 4. *Let $U(x) \in L^\infty(\mathbb{R}^d)$, the function $h(x) \in L^2(\mathbb{R}^d)$ is nontrivial, $F(z) : \mathbb{R}^+ \to \mathbb{R}$ is continuously differentiable, the kernel $G(x) \in L^1(\mathbb{R}^d),\ 1 \le d \le 3$.*

I) *When the dimension $d = 1$ and $\omega > 0$ let $xG(x),\ xh(x) \in L^1(\mathbb{R})$, orthogonality conditions (4.11) hold and*

$$\left(G(x), \frac{e^{\pm i\sqrt{\omega}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0. \quad (1.22)$$

II) *When the dimension $d = 1$ and $\omega = 0$ let $x^2G(x),\ x^2h(x) \in L^1(\mathbb{R})$, orthogonality conditions (4.12) hold and*

$$(G(x), 1)_{L^2(\mathbb{R})} = 0, \quad (G(x), x)_{L^2(\mathbb{R})} = 0. \quad (1.23)$$

III) *When the dimension* $d = 2, 3$ *and* $\omega > 0$ *let* $xG(x),\ xh(x) \in L^1(\mathbb{R}^d)$, *orthogonality conditions (4.15) hold and*

$$\left(G(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0\ for\ p \in S^d_{\sqrt{\omega}}\ a.e. \tag{1.24}$$

IV) *When the dimension* $d = 2, 3$ *and* $\omega = 0$ *let* $x^2G(x) \in L^1(\mathbb{R}^d)$, *for* $d = 2$ *let* $x^2h(x) \in L^1(\mathbb{R}^2)$ *with orthogonality conditions (4.16), for* $d = 3$ *assume* $xh(x) \in L^1(\mathbb{R}^3)$ *with orthogonality conditions (4.17) and*

$$(G(x), 1)_{L^2(\mathbb{R}^d)} = 0, \quad (G(x), x_k)_{L^2(\mathbb{R}^d)} = 0,\ 1 \le k \le d. \tag{1.25}$$

Then there exists $\varepsilon > 0$ *such that for all* $\lambda \in \mathbb{R},\ |\lambda| < \varepsilon$ *equation (1.20) admits a unique nontrivial solution* $\phi(x) \in H^2(\mathbb{R}^d)$.

We start with proving the solvability conditions in the no potential case.

2. The free problem in the cylindrical domain.

Proof of Theorem 1. Note that it is sufficient to study only the solvability conditions in $L^2(D)$ for problem (1.3) since the existence of a square integrable solution $u(x, y)$ for this equation with the square integrable right side will yield $\Delta u(x, y) \in L^2(D)$, where $\Delta = \Delta_x + \Delta_y$. Let us assume that there are two functions $u_{1,2}(x, y) \in L^2(D)$ which solve equation (1.3). Then their difference $v(x, y) := u_1(x, y) - u_2(x, y) \in L^2(D)$ will satisfy the homogeneous equation $L_a v = 0$. Its projection onto the $\varphi_n^j(y),\ n \in \mathbb{N},\ 1 \le j \le m_n$ state will be of the form $c_n^j(x)\varphi_n^j(y)$ with $c_n^j(x) \in L^2(\mathbb{R}^d)$, such that it will satisfy the equation

$$(-\Delta_x + \lambda_n - a)c_n^j(x) = 0.$$

Since the free negative Laplacian on $L^2(\mathbb{R}^d)$ has only the essential spectrum filling the nonnegative semi-axis, $c_n^j(x)$ vanishes, which shows the uniqueness of a square integrable solution of problem (1.3) for an arbitrary nonnegative a.

To prove the statement in case I) we easily establish the lower bound

$$(L_a u, u)_{L^2(D)} \ge ([-\Delta_y - a]u, u)_{L^2(D)} \ge (\lambda_1 - a)\|u\|^2_{L^2(D)},$$

such that the bottom of the spectrum of the operator L_a on $L^2(D)$ is bounded below by the positive constant. Hence equation (1.3) in this case admits a solution $u(x, y) = L_a^{-1} f(x, y)$ and $\|u\|_{L^2(D)} \le \dfrac{1}{\lambda_1 - a}\|f\|_{L^2(D)} < \infty$.

To prove the solvability conditions in the situation when the value of the parameter a is attained between two consecutive eigenvalues λ_N and λ_{N+1} of the Dirichlet Laplacian, we introduce the projection operators

$$P^N := \sum_{k=1}^{N} P_k \quad and \quad P^{N+1} := \sum_{k=N+1}^{\infty} P_k, \tag{2.1}$$

where $\{P_k\}_{k=1}^{\infty}$ are used in (1.2) and we have the resolution of the identity $I = P^N + P^{N+1}$ on $L^2(\Omega)$. This enables us to relate problem (1.3) to the system of two equations

$$L_a^N u_N = f_N, \tag{2.2}$$

$$L_a^{N+1} u_{N+1} = f_{N+1}, \tag{2.3}$$

with the operators restricted to the subspaces $L_a^N := P^N(-\Delta_x - \Delta_y - a)P^N$ and $L_a^{N+1} := P^{N+1}(-\Delta_x - \Delta_y - a)P^{N+1}$ acting on functions $u_N := P^N u$ and $u_{N+1} := P^{N+1}u$ respectively, the right sides of the equations above are $f_N := P^N f$ and $f_{N+1} := P^{N+1} f$. For the quadratic form of the operator involved in the left side of (2.3) on $L^2(\mathbb{R}^d) \otimes \text{Ran}P^{N+1}$, where $\text{Ran}P^{N+1}$ denotes the range of the corresponding projection we easily obtain the estimate from below

$$(L_a^{N+1}u, u)_{L^2(D)} \geq (P^{N+1}(-\Delta_y - a)P^{N+1}u, u)_{L^2(D)} \geq (\lambda_{N+1} - a)\|u\|_{L^2(D)}^2.$$

The inverse of this operator acts $(L_a^{N+1})^{-1} : L^2(\mathbb{R}^d) \otimes \text{Ran}P^{N+1} \to L^2(D)$. Thus equation (2.3) possesses a solution $u_{N+1} = (L_a^{N+1})^{-1} f_{N+1}$, such that $\|u_{N+1}\|_{L^2(D)} \leq \dfrac{1}{\lambda_{N+1} - a}\|f\|_{L^2(D)} < \infty$. The remaining equation (2.2) can be trivially related to the following system of N equations

$$L_a^k u_k = f_k, \ 1 \leq k \leq N, \tag{2.4}$$

with $u_k := P_k u_N$ and $f_k := P_k f_N$. The operator restricted to the spectral subspace is $L_a^k := P_k(-\Delta_x - \Delta_y - a)P_k$. Without loss of generality we can assume that $f_k(x, y) = v_k(x)\varphi_k^1(y), \ 1 \leq k \leq N$. Since the free negative Laplacian operator on $L^2(\mathbb{R}^d)$ has only the essential spectrum, we have $u_k(x, y) = c_k(x)\varphi_k^1(y)$, which yields the equation

$$-\Delta_x c_k(x) - (a - \lambda_k)c_k(x) = v_k(x), \ 1 \leq k \leq N. \tag{2.5}$$

The right side of (2.5) is square integrable. Indeed, $\|v_k\|_{L^2(\mathbb{R}^d)}^2 = \|f_k\|_{L^2(D)}^2 \leq \|f\|_{L^2(D)}^2 < \infty$. We have the following estimate by means of the Schwarz inequality

$$\|xv_k(x)\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |x||(f(x, y), \varphi_k^1(y))_{L^2(\Omega)}|dx \leq \int_{\mathbb{R}^d} dx|x|\sqrt{\int_{\Omega} dy|f(x, y)|^2}.$$

This expression can be bounded above via the Schwarz inequality by

$$\sqrt{\int_{\mathbb{R}^d} dx \frac{|x|^2}{1 + |x|^\alpha}}\sqrt{\int_{\mathbb{R}^d} dz(1 + |z|^\alpha)\int_{\Omega} dy|f(z, y)|^2} =$$

$$= \sqrt{\int_0^\infty d|x| \frac{|S^d||x|^{d+1}}{1 + |x|^\alpha}}\sqrt{\|f\|_{L^2(D)}^2 + \||x|^{\frac{\alpha}{2}} f\|_{L^2(D)}^2} < \infty,$$

such that $xv_k(x) \in L^1(\mathbb{R}^d)$. Thus when dimension $d = 1$ by means of Lemma 5 of the Appendix equation (2.5) admits a unique square integrable solution if and only if

$$\left(v_k(x), \frac{e^{\pm i\sqrt{a - \lambda_k}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0, \quad 1 \leq k \leq N,$$

which yields orthogonality relations (1.4). When dimension $d \geq 2$ due to Lemma 6 equation (2.5) possesses a unique solution belonging to $L^2(\mathbb{R}^d)$ if and only if

$$\left(v_k(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a-\lambda_k}} \quad a.e., \quad 1 \leq k \leq N,$$

which implies orthogonality conditions (1.5).

Finally we consider case III) of the theorem, when the parameter a coincides with one of the eigenvalues of the negative Dirichlet Laplacian. Let us use the resolution of identity on $L^2(\Omega)$, namely $I = P^{N-1} + P^0 + P^{N+1}$, with P^{N+1} defined in (2.1), $P^0 := P_N$ and $P^{N-1} := \sum_{k=1}^{N-1} P_k$. By applying these projection operators to equation (1.3) we relate it to the equivalent system of three equations

$$L_a^{N+1} u_{N+1} = f_{N+1}, \tag{2.6}$$

$$L_a^0 u_0 = f_0, \tag{2.7}$$

$$L_a^{N-1} u_{N-1} = f_{N-1}, \tag{2.8}$$

with the operators restricted to the spectral subspaces

$$L_a^{N-1} := P^{N-1}(-\Delta_x - \Delta_y - a)P^{N-1}, \quad L_a^0 := P^0(-\Delta_x - \Delta_y - a)P^0.$$

These operators act on $u_{N-1} := P^{N-1}u$ and $u_0 := P^0 u$ respectively. The right sides of the equations above are $f_{N-1} := P^{N-1}f$ and $f_0 := P^0 f$. Note that the operator L_a^{N+1} along with functions u_{N+1} and f_{N+1} are defined in the proof of part II) of the theorem. For the quadratic form of the operator L_a^{N+1} on $L^2(\mathbb{R}^d) \otimes \text{Ran}P^{N+1}$ we have the lower bound

$$(L_a^{N+1}u, u)_{L^2(D)} \geq (P^{N+1}(-\Delta_y - a)P^{N+1}u, u)_{L^2(D)} \geq (\lambda_{N+1} - \lambda_N)\|u\|^2_{L^2(D)},$$

such that equation (2.6) admits a solution $u_{N+1} = (L_a^{N+1})^{-1} f_{N+1}$ with the operator $(L_a^{N+1})^{-1}$: $L^2(\mathbb{R}^d) \otimes \text{Ran}P^{N+1} \to L^2(D)$ and $\|u_{N+1}\|_{L^2(D)} \leq \frac{1}{\lambda_{N+1} - \lambda_N}\|f\|_{L^2(D)} < \infty$.

To study the solvability conditions for equation (2.8) we relate it to the system of equivalent equations

$$L_a^k u_k = f_k, \quad 1 \leq k \leq N-1$$

with $u_k = P_k u$ and $f_k = P_k f$. This system can be treated analogously to the one studied in case II), which yields orthogonality conditions (1.6), (1.8), (1.10) and (1.12) dependent upon the value of the dimension d.

Finally, we turn our attention to equation (2.7), which is obviously equivalent to

$$P^0(-\Delta_x)P^0 u_0 = f_0.$$

Without loss of generality we can assume that $f_0(x, y) = v_N(x)\varphi_N^1(y)$. Since the free Laplacian on $L^2(\mathbb{R}^d)$ has no nontrivial square integrable zero modes, we have $u_0(x, y) = c_N(x)\varphi_N^1(y)$, such that

$$-\Delta_x c_N(x) = v_N(x). \tag{2.9}$$

The right side of the equation above is square integrable since

$$\|v_N\|^2_{L^2(\mathbb{R}^d)} = \|f_0\|^2_{L^2(D)} \leq \|f\|^2_{L^2(D)} < \infty.$$

Let us first consider the situation when dimension $d = 1$. Then via the Schwarz inequality

$$\|x^2 v_N(x)\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} dx x^2 |(f(x,y), \varphi_N^1(y))_{L^2(\Omega)}| \leq \int_{-\infty}^{\infty} dx x^2 \sqrt{\int_\Omega |f(x,y)|^2 dy}, \qquad (2.10)$$

which can be bounded above by applying again the Schwarz inequality by

$$\sqrt{\int_{-\infty}^{\infty} dx \frac{x^4}{1+|x|^\alpha}} \sqrt{\int_{-\infty}^{\infty} dz (1+|z|^\alpha) \int_\Omega |f(z,y)|^2 dy} =$$

$$= \sqrt{\int_{-\infty}^{\infty} dx \frac{x^4}{1+|x|^\alpha}} \sqrt{\|f\|^2_{L^2(D)} + \||z|^{\frac{\alpha}{2}} f(z,y)\|^2_{L^2(D)}} < \infty$$

since $\alpha > 5$ and $|z|^{\frac{\alpha}{2}} f(z,y) \in L^2(D)$. Hence $x^2 v_N(x) \in L^1(\mathbb{R})$. By means of Lemma 5 applied to the Poisson equation (2.9) we obtain othogonality relations (1.7).

When dimension $d = 2$ we perform the estimate analogous to (2.10) and then via the Schwarz inequality obtain the upper bound for it as

$$\sqrt{\int_{\mathbb{R}^2} dx \frac{|x|^4}{1+|x|^\alpha}} \sqrt{\int_{\mathbb{R}^2} dz (1+|z|^\alpha) \int_\Omega |f(z,y)|^2 dy} =$$

$$= \sqrt{\int_0^{\infty} d|x| \frac{2\pi |x|^5}{1+|x|^\alpha}} \sqrt{\|f\|^2_{L^2(D)} + \||z|^{\frac{\alpha}{2}} f(z,y)\|^2_{L^2(D)}} < \infty$$

due to the fact that $\alpha > 6$ and $|z|^{\frac{\alpha}{2}} f(z,y) \in L^2(D)$. Therefore, $|x|^2 v_N(x) \in L^1(\mathbb{R}^2)$. Via Lemma 6 we arrive at orthogonality conditions (1.9).

For dimensions $d = 3, 4$ we estimate the norm via the Schwarz inequality

$$\||x| v_N(x)\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} dx |x| |(f(x,y), \varphi_N^1(y))_{L^2(\Omega)}| \leq \int_{\mathbb{R}^d} dx |x| \sqrt{\int_\Omega |f(x,y)|^2 dy}$$

and by applying the Schwarz inequality again obtain the upper bound for it

$$\sqrt{\int_{\mathbb{R}^d} dx \frac{|x|^2}{1+|x|^\alpha}} \sqrt{\int_{\mathbb{R}^d} dz (1+|z|^\alpha) \int_\Omega |f(z,y)|^2 dy} =$$

$$= \sqrt{\int_0^{\infty} d|x| \frac{|S^d| |x|^{d+1}}{1+|x|^\alpha}} \sqrt{\|f\|^2_{L^2(D)} + \||z|^{\frac{\alpha}{2}} f(z,y)\|^2_{L^2(D)}} < \infty$$

because $\alpha > d+2$ and $|z|^{\frac{\alpha}{2}} f(z,y) \in L^2(D)$. Hence $|x| v_N(x) \in L^1(\mathbb{R}^d)$, $d = 3, 4$. Lemma 6 implies orthogonality condition (1.11).

In dimensions $d \geq 5$ by the same reasoning as above it can be shown that $|x| v_N(x) \in L^1(\mathbb{R}^d)$ since $\alpha > d+2$ and $|x|^{\frac{\alpha}{2}} f(x,y) \in L^2(D)$. No further orthogonality conditions are needed in such high dimensions by means of Lemma 6. ■

3. The problem with an external potential.

Proof of Theorem 3. Note that due to the boundedness of the potential function $V(x)$ it is sufficient to study the solvability conditions for our problem in $L^2(D)$. First we show the uniqueness of a square integrable solution for equation (1.14). Indeed, if there were $u_{1,2}(x,y) \in L^2(D)$ satisfying (1.14), their difference $w(x,y) = u_1(x,y) - u_2(x,y) \in L^2(D)$ would be solving the homogeneous problem $H_a w = 0$. Let us project it onto the state $\varphi_n^j(y),\ n \in \mathbb{N},\ 1 \leq j \leq m_n$. Hence the projection of $w(x,y)$ will be of the form $c_n^j(x)\varphi_n^j(y)$, where $c_n^j(x) \in L^2(\mathbb{R}^3)$. We easily obtain

$$(-\Delta_x + V(x) + \lambda_n - a)c_n^j(x) = 0.$$

The equation above does not have any nontrivial square integrable solutions (see the Remark after Assumption 2), which implies $c_n^j(x) = 0$ in $\mathbb{R}^3$ a.e. and $w(x,y)$ vanishes in D a.e.

Let us first suppose that the parameter $0 \leq a < \lambda_1$. Then we have the lower bound

$$(H_a u, u)_{L^2(D)} \geq ((-\Delta_y - a)u, u)_{L^2(D)} \geq (\lambda_1 - a)\|u\|^2_{L^2(D)}.$$

Thus the operator $H_a \geq \lambda_1 - a > 0$ in the sense of quadratic forms, such that equation (1.14) admits a solution $u = H_a^{-1} f$ and $\|u\|_{L^2(D)} \leq \dfrac{1}{\lambda_1 - a}\|f\|_{L^2(D)} < \infty$.

Then we consider the situation of $\lambda_N < a < \lambda_{N+1},\ N \in \mathbb{N}$. As in the proof of Theorem 1, by using the projection operators (2.1) we relate problem (1.14) to the equivalent system of two equations

$$H_a^N u_N = f_N, \quad H_a^{N+1} u_{N+1} = f_{N+1}, \tag{3.1}$$

where the restricted operators are $H_a^N := P^N H_a P^N$ and $H_a^{N+1} := P^{N+1} H_a P^{N+1}$. The functions involved in the right sides of the equations of the system above are $f_N = P^N f,\ f_{N+1} = P^{N+1} f$ and in the left sides are $u_N = P^N u,\ u_{N+1} = P^{N+1} u$. We estimate from below

$$(H_a^{N+1} u, u)_{L^2(D)} \geq (P^{N+1}(-\Delta_y - a)P^{N+1} u, u)_{L^2(D)} \geq (\lambda_{N+1} - a)\|u\|^2_{L^2(D)}.$$

Hence the operator $H_a^{N+1} \geq \lambda_{N+1} - a > 0$. We easily obtain a solution for the second equation in system (3.1): $u_{N+1} = (H_a^{N+1})^{-1} f_{N+1}$, where $(H_a^{N+1})^{-1} : L^2(\mathbb{R}^3) \otimes \mathrm{Ran} P^{N+1} \to L^2(D)$ and $\|u_{N+1}\|_{L^2(D)} \leq \dfrac{1}{\lambda_{N+1} - a}\|f\|_{L^2(D)} < \infty$. Thus it remains to study the first equation in system (3.1), which can be easily related to the equivalent system of N equations using the projections (1.2):

$$H_a^k u_k = f_k,\ 1 \leq k \leq N, \tag{3.2}$$

where the restricted operators are $H_a^k = P_k H_a P_k$, the functions involed are $f_k = P_k f$ and $u_k = P_k u$. From (3.2) we easily deduce

$$(-\Delta_x + V(x) + \lambda_k - a)u_k = f_k,\ 1 \leq k \leq N.$$

Without loss of generality we can assume that

$$f_k(x,y) = v_k(x)\varphi_k^1(y),\ 1 \leq k \leq N.$$

Seeking a solution of the equation above in the form of $u_k(x,y) = \sum_{j=1}^{m_k} c_k^j(x)\varphi_k^j(y),\ 1 \le k \le N$, we easily arrive at

$$(-\Delta_x + V(x))c_k^j(x) = (a - \lambda_k)c_k^j(x),\ 2 \le j \le m_k.$$

Since the Schrödinger operator $-\Delta_x + V(x)$ is unitarily equivalent to $-\Delta_x$ on $L^2(\mathbb{R}^3)$ (see the Remark after Assumption 2), it does not have any nontrivial square integrable bound states. Therefore, $c_k^j(x) = 0$ for $x \in \mathbb{R}^3$ a.e., $2 \le j \le m_k$ and $u_k(x,y) = c_k(x)\varphi_k^1(y),\ 1 \le k \le N$, which yields

$$(-\Delta_x + V(x) + \lambda_k - a)c_k(x) = v_k(x),\ 1 \le k \le N. \tag{3.3}$$

Clearly, the right side of (3.3) is square integrable, since $\|v_k(x)\|^2_{L^2(\mathbb{R}^3)} = \|f_k(x,y)\|^2_{L^2(D)} \le \|f(x,y)\|^2_{L^2(D)} < \infty$ by our assumption. We estimate the norm $\|xv_k(x)\|_{L^1(\mathbb{R}^3)}$ using the argument analogous to the one of Chapter 2 when the dimension $d = 3$ and show that $\|xv_k(x)\|_{L^1(\mathbb{R}^3)} < \infty$ provided $f(x,y) \in L^2(D)$ and $|x|^{\frac{\alpha}{2}} f(x,y) \in L^2(D)$ with some $\alpha > 5$. Thus by means of Theorem 1.2 of [13] equation (3.3) admits a solution $c_k(x) \in L^2(\mathbb{R}^3)$ if and only if $(v_k(x), \eta_q(x))_{L^2(\mathbb{R}^3)} = 0$ for $q \in S^3_{\sqrt{a-\lambda_k}}$ a.e., $1 \le k \le N$, which yields orthogonality conditions (1.15).

We complete the proof of the theorem covering the case of $a = \lambda_N,\ N \in \mathbb{N}$. Analogously to the free problem studied in Chapter 2 we relate equation (1.14) to the equivalent system of three equations

$$H_a^{N-1}u_{N-1} = f_{N-1},\ H_a^0 u_0 = f_0 \textit{ and } H_a^{N+1}u_{N+1} = f_{N+1}, \tag{3.4}$$

with the restricted operators

$$H_a^{N-1} = P^{N-1}H_aP^{N-1},\ H_a^0 = P^0H_aP^0 \textit{ and } H_a^{N+1} = P^{N+1}H_aP^{N+1},$$

the right sides

$$f_{N-1} = P^{N-1}f,\ f_0 = P^0 f \textit{ and } f_{N+1} = P^{N+1}f,$$

the functions involved in the left sides of (3.4)

$$u_{N-1} = P^{N-1}u,\ u_0 = P^0 u \textit{ and } u_{N+1} = P^{N+1}u.$$

Let us first analyze the third equation in (3.4). We have a lower bound in the sense of quadratic forms

$$(H_a^{N+1}u, u)_{L^2(D)} \ge (P^{N+1}(-\Delta_y - a)P^{N+1}u, u)_{L^2(D)} \ge (\lambda_{N+1} - \lambda_N)\|u\|^2_{L^2(D)}.$$

Thus $H_a^{N+1} \ge \lambda_{N+1} - \lambda_N > 0$, such that the last equation in (3.4) admits a solution

$$u_{N+1} = (H_a^{N+1})^{-1}f_{N+1},$$

with the operator $(H_a^{N+1})^{-1} : L^2(\mathbb{R}^3) \otimes \mathrm{Ran}P^{N+1} \to L^2(D)$. Thus for the solution above we have a bound

$$\|u_{N+1}\|_{L^2(D)} \le \frac{1}{\lambda_{N+1} - \lambda_N}\|f\|_{L^2(D)} < \infty$$

by the assumption of the theorem. Then we turn our attention to the second equation in (3.4), assuming without loss of generality that its right side $f_0(x,y) = v_N(x)\varphi_N^1(y)$ and looking for a solution in the form $u_0(x,y) = \sum_{j=1}^{m_N} c_N^j(x)\varphi_N^j(y)$. This yields $(-\Delta_x + V(x))c_N^j(x) = 0,\ j = 2, ..., m_N$. Since the Schrödinger operator involved in this equation does not have any nontrivial square integrable zero modes (see the Remark after Assumption 2), we have $c_N^j(x) = 0$ for $x \in \mathbb{R}^3$ a.e. and $j = 2, ..., m_N$ and therefore, $u_0(x,y) = c_N(x)\varphi_N^1(y)$, such that

$$(-\Delta_x + V(x))c_N(x) = v_N(x). \tag{3.5}$$

Clearly, $\|f_0\|^2_{L^2(D)} = \|v_N\|^2_{L^2(\mathbb{R}^3)} \le \|f\|^2_{L^2(D)} < \infty$ by the assumption of the theorem. The norm $\|xv_N(x)\|_{L^1(\mathbb{R}^3)} < \infty$, which can be shown using the argument analogous to that of Chapter 2 in three dimensions provided $f \in L^2(D)$ and $|x|^{\frac{\alpha}{2}}f(x,y) \in L^2(D)$ for some $\alpha > 5$. Therefore, by means of Theorem 1.2 of [13] equation (3.5) possesses a solution in $L^2(\mathbb{R}^3)$ if and only if $(v_N(x), \eta_0(x))_{L^2(\mathbb{R}^3)} = 0$, which implies relations (1.16).

Finally, we study the first equation in (3.4), which can be easily related to the system of equations

$$H_a^k u_k = f_k,\ 1 \le k \le N-1, \tag{3.6}$$

with the restricted operators $H_a^k = P_k H_a P_k$, the right sides $f_k = P_k f$ and functions involved in the left sides $u_k = P_k u$. Without loss of generality we can assume that $f_k(x,y) = v_k(x)\varphi_k^1(y),\ 1 \le k \le N-1$. Let us seek a solution of (3.6) in the form $u_k(x,y) = \sum_{j=1}^{m_k} c_k^j(x)\varphi_k^j(y)$. We easily arrive at $(-\Delta_x + V(x))c_k^j(x) = (\lambda_N - \lambda_k)c_k^j(x),\ j = 2, ..., m_k$. Since the Schrödinger operator does not have nontrivial square integrable bound states (see the Remark after Assumption 2), we have $c_k^j(x) = 0$ for $x \in \mathbb{R}^3$ a.e. and $j = 2, ..., m_k$. Therefore, $u_k(x,y) = c_k(x)\varphi_k^1(y)$, which yields the nonhomogeneous equation

$$(-\Delta_x + V(x) + \lambda_k - \lambda_N)c_k(x) = v_k(x),\ 1 \le k \le N-1. \tag{3.7}$$

Its right side is square integrable since $\|f_k\|^2_{L^2(D)} = \|v_k\|^2_{L^2(\mathbb{R}^3)} \le \|f\|^2_{L^2(D)} < \infty$. The norm $\|xv_k\|_{L^1(\mathbb{R}^3)} < \infty$, which can be shown via the argument analogous to the one used in Chapter 2 in three dimensions provided $f(x,y) \in L^2(D)$ and $|x|^{\frac{\alpha}{2}}f(x,y) \in L^2(D)$ for some $\alpha > 5$. Therefore, by means of Theorem 1.2 of [13] equation (3.7) is solvable in $L^2(\mathbb{R}^3)$ if and only if $(v_k(x), \eta_q(x))_{L^2(\mathbb{R}^3)} = 0$ for $q \in S^3_{\sqrt{\lambda_N - \lambda_k}}$ *a.e.*, $1 \le k \le N-1$, which implies orthogonality relations (1.17). ∎

4. Standing waves of the nonlocal, forced equation

Proof of Theorem 4. Let $\phi_0(x) \in H^2(\mathbb{R}^d),\ 1 \le d \le 3$ be the unique solution of problem (4.9) under the conditions of the theorem, using the results of Lemmas 5 and 6. When the parameter λ is nontrivial, we seek the solution of problem (1.20) in the form $\phi(x) = \phi_0(x) + \eta(x)$ and using (4.9) arrive at

$$-\Delta\eta - \omega\eta + \lambda\int_{\mathbb{R}^d} G(x-y)[F(|\phi_0(y)+\eta(y)|^2)(\phi_0(y)+\eta(y)) + U(y)(\phi_0(y)+\eta(y))]dy = 0. \tag{4.1}$$

Let us introduce the auxiliary equation

$$\Delta\xi + \omega\xi = \lambda \int_{\mathbb{R}^d} G(x-y)[F(|\phi_0(y)+\eta(y)|^2)(\phi_0(y)+\eta(y)) + U(y)(\phi_0(y)+\eta(y))]dy. \quad (4.2)$$

Our goal is to show that for small $|\lambda|$ (4.2) defines a map $T : B(H^2(\mathbb{R}^d)) \to B(H^2(\mathbb{R}^d))$. Let us first suppose that for some $\eta(x) \in B(H^2(\mathbb{R}^d))$ there are two solutions $\xi_{1,2}(x) \in B(H^2(\mathbb{R}^d))$ of problem (4.2). Then the function $\xi(x) := \xi_1(x) - \xi_2(x) \in H^2(\mathbb{R}^d)$ satisfies the equation $-\Delta\xi = \omega\xi$. Since the free Laplacian does not possess any nontrivial square integrable eigenfunctions, we arrive at $\xi(x) = 0$ a.e. in $\mathbb{R}^d$.

Hence consider an arbitrary $\eta(x) \in B(H^2(\mathbb{R}^d))$. By means of the Sobolev embedding theorem we have $\phi_0(x),\ \eta(x) \in L^\infty(\mathbb{R}^d),\ 1 \le d \le 3$, which along with the assumptions of the theorem enables us to estimate the terms of equation (4.2) as

$$|U(\phi_0+\eta)| \le \|U\|_{L^\infty(\mathbb{R}^d)}(|\phi_0| + |\eta|) \in L^2(\mathbb{R}^d) \quad (4.3)$$

and

$$|F(|\phi_0(x)+\eta(x)|^2)(\phi_0(x)+\eta(x))| \le \sup|F(z)|_{z\in[0,\ (\|\phi_0\|_{L^\infty(\mathbb{R}^d)}+\|\eta\|_{L^\infty(\mathbb{R}^d)})^2]}(|\phi_0(x)| + |\eta(x)|), \quad (4.4)$$

which belongs to $L^2(\mathbb{R}^d)$ as well. By applying the standard Fourier transform to equation (4.2) we obtain

$$\widehat{\xi}(p) = \lambda(2\pi)^{\frac{d}{2}} \frac{\widehat{G}(p)}{\omega - p^2}\{\mathcal{F}(p) + \mathcal{G}(p)\}, \quad (4.5)$$

where $\mathcal{F}(p)$ and $\mathcal{G}(p)$ are the transforms of $F(|\phi_0+\eta|^2)(\phi_0+\eta)$ and $U(\phi_0+\eta)$ respectively. By means of (4.3) and (4.4) $\mathcal{F}(p),\ \mathcal{G}(p) \in L^2(\mathbb{R}^d)$. Clearly

$$p^2\widehat{\xi}(p) = \lambda(2\pi)^{\frac{d}{2}} \frac{p^2\widehat{G}(p)}{\omega - p^2}\{\mathcal{F}(p) + \mathcal{G}(p)\}. \quad (4.6)$$

Let us introduce the following quantity

$$N_{\omega,\ d} := \max\left\{\left\|\frac{\widehat{G}(p)}{\omega - p^2}\right\|_{L^\infty(\mathbb{R}^d)},\ \left\|\frac{p^2\widehat{G}(p)}{\omega - p^2}\right\|_{L^\infty(\mathbb{R}^d)}\right\},\ \omega \ge 0,\ 1 \le d \le 3. \quad (4.7)$$

By means of orthogonality conditions (1.22)-(1.25) via Lemmas A1 and A2 of [18] we have $N_{\omega,\ d} < \infty$. Therefore,

$$|\widehat{\xi}(p)| \le |\lambda|(2\pi)^{\frac{d}{2}} N_{\omega,\ d}\{|\mathcal{F}|(p) + |\mathcal{G}|(p)\} \in L^2(\mathbb{R}^d),$$

$$|p^2\widehat{\xi}(p)| \le |\lambda|(2\pi)^{\frac{d}{2}} N_{\omega,\ d}\{|\mathcal{F}|(p) + |\mathcal{G}|(p)\} \in L^2(\mathbb{R}^d),$$

such that $\xi \in B(H^2(\mathbb{R}^d))$ when the value of the parameter $|\lambda|$ is small enough and $T\eta = \xi$. Hence it remains to prove that the map $T : B(H^2(\mathbb{R}^d)) \to B(H^2(\mathbb{R}^d))$ is a strict contraction. For that purpose we choose arbitrary $\eta_1(x), \eta_2(x) \in B(H^2(\mathbb{R}^d))$ such that $T\eta_{1,2} = \xi_{1,2} \in B(H^2(\mathbb{R}^d))$ via equation (4.2) for $|\lambda|$ small enough. Clearly, we have

$$\widehat{\xi}_1(p) - \widehat{\xi}_2(p) = \lambda(2\pi)^{\frac{d}{2}} \frac{\widehat{G}(p)}{\omega - p^2} \{\mathcal{F}_1(p) - \mathcal{F}_2(p) + \mathcal{G}_1(p) - \mathcal{G}_2(p)\},$$

where $\mathcal{F}_{1,2}(p)$ and $\mathcal{G}_{1,2}(p)$ are the Fourier images of $F(|\phi_0 + \eta_{1,2}|^2)(\phi_0 + \eta_{1,2})$ and $U(\phi_0 + \eta_{1,2})$ respectively. Hence

$$p^2\widehat{\xi}_1(p) - p^2\widehat{\xi}_2(p) = \lambda(2\pi)^{\frac{d}{2}} \frac{p^2\widehat{G}(p)}{\omega - p^2} \{\mathcal{F}_1(p) - \mathcal{F}_2(p) + \mathcal{G}_1(p) - \mathcal{G}_2(p)\},$$

such that

$$\|\xi_1(x) - \xi_2(x)\|_{L^2(\mathbb{R}^d)} \leq |\lambda|(2\pi)^{\frac{d}{2}} N_{\omega,\, d}\{\|F(|\phi_0 + \eta_1|^2)(\phi_0 + \eta_1) - F(|\phi_0 + \eta_2|^2)(\phi_0 + \eta_2)\|_{L^2(\mathbb{R}^d)} +$$
$$+\|U\eta_1 - U\eta_2\|_{L^2(\mathbb{R}^d)}\}$$

and the analogous upper bound holds for $\|\Delta\xi_1(x) - \Delta\xi_2(x)\|_{L^2(\mathbb{R}^d)}$. We easily estimate

$$\|U(\eta_1 - \eta_2)\|_{L^2(\mathbb{R}^d)} \leq \|U\|_{L^\infty(\mathbb{R}^d)} \|\eta_1 - \eta_2\|_{L^2(\mathbb{R}^d)}.$$

Let us write

$$F(|\phi_0 + \eta_1|^2)(\phi_0 + \eta_1) - F(|\phi_0 + \eta_2|^2)(\phi_0 + \eta_2) = (F(|\phi_0 + \eta_1|^2) - F(|\phi_0 + \eta_2|^2))(\phi_0 + \eta_2) +$$
$$+F(|\phi_0 + \eta_1|^2)(\eta_1 - \eta_2).$$

By means of the Sobolev embedding theorem $\|\eta\|_{L^\infty(\mathbb{R}^d)} \leq c_e \|\eta\|_{H^2(\mathbb{R}^d)},\ 1 \leq d \leq 3$ we have

$$|\phi_0 + \eta_{1,2}| \leq c_e(1 + \|\phi_0\|_{H^2(\mathbb{R}^d)}),$$

where c_e is the constant of the embedding. Therefore,

$$\|F(|\phi_0 + \eta_1|^2)(\eta_1 - \eta_2)\|_{L^2(\mathbb{R}^d)} \leq \sup|F(z)|_{0 \leq z \leq c_e^2(1+\|\phi_0\|_{H^2(\mathbb{R}^d)})^2} \|\eta_1 - \eta_2\|_{L^2(\mathbb{R}^d)}.$$

We will make use of the representation formula

$$F(|\phi_0 + \eta_1|^2) - F(|\phi_0 + \eta_2|^2) = \int_{|\phi_0+\eta_2|^2}^{|\phi_0+\eta_1|^2} F'(z)dz,$$

which by means of the trivial inequality

$$||\phi_0 + \eta_1|^2 - |\phi_0 + \eta_2|^2| \leq 2c_e(1 + \|\phi_0\|_{H^2(\mathbb{R}^d)})|\eta_1 - \eta_2|$$

yields

$$|(F(|\phi_0 + \eta_1|^2) - F(|\phi_0 + \eta_2|^2))(\phi_0 + \eta_2)| \leq 2c_e^2(1 + \|\phi_0\|_{H^2(\mathbb{R}^d)})^2 \times$$

$$\times \text{sup}|F'(s)|_{0\leq s\leq c_e^2(1+\|\phi_0\|_{H^2(\mathbb{R}^d)})^2}|\eta_1-\eta_2|.$$

Hence, we arrive at

$$\|(F(|\phi_0+\eta_1|^2)-F(|\phi_0+\eta_2|^2))(\phi_0+\eta_2)\|_{L^2(\mathbb{R}^d)}\leq 2c_e^2(1+\|\phi_0\|_{H^2(\mathbb{R}^d)})^2\times$$

$$\times \text{sup}|F'(s)|_{0\leq s\leq c_e^2(1+\|\phi_0\|_{H^2(\mathbb{R}^d)})^2}\|\eta_1-\eta_2\|_{L^2(\mathbb{R}^d)}.$$

As a consequence of the estimates above we easily obtain

$$\|T\eta_1-T\eta_2\|_{H^2(\mathbb{R}^d)}\leq\sqrt{2}(2\pi)^{\frac{d}{2}}|\lambda|N_{\omega,\,d}\{\|U\|_{L^\infty(\mathbb{R}^d)}+\text{sup}|F(z)|_{0\leq z\leq c_e^2(1+\|\phi_0\|_{H^2(\mathbb{R}^d)})^2}+$$

$$+2c_e^2(1+\|\phi_0\|_{H^2(\mathbb{R}^d)})^2\text{sup}|F'(z)|_{0\leq z\leq c_e^2(1+\|\phi_0\|_{H^2(\mathbb{R}^d)})^2}\}\|\eta_1-\eta_2\|_{H^2(\mathbb{R}^d)}.$$

Thus, when $|\lambda|$ is small enough, the map $T : B(H^2(\mathbb{R}^d))\to B(H^2(\mathbb{R}^d))$ is a strict contraction, and therefore, it has a unique fixed point $\eta\in B(H^2(\mathbb{R}^d))$. The solution of problem (1.20) does not vanish in $\mathbb{R}^d$ provided $h(x)$ is nontrivial. ■

Appendix

We investigate solvability conditions in $H^2(\mathbb{R}^d),\ d\in\mathbb{N}$ equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R}^d)}:=\|u\|^2_{L^2(\mathbb{R}^d)}+\|\Delta u\|^2_{L^2(\mathbb{R}^d)} \tag{4.8}$$

of the linear equation

$$-\Delta\phi-\omega\phi=-h(x),\ \omega\geq 0 \tag{4.9}$$

with a square integrable right side. Apparently, the uniqueness of solutions for this problem comes from the fact that the free Laplacian operator in the whole space does not have nontrivial square integrable eigenfunctions. Obviously,

$$\widehat{\phi}(p)=-\frac{\widehat{h}(p)}{p^2-\omega},\ p\in\mathbb{R}^d \tag{4.10}$$

with the hat symbol standing for the standard Fourier transform such that

$$\widehat{h}(p):=\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}h(x)e^{-ipx}dx.$$

We have the following statement in one dimension.

Lemma 5. *Let* $h(x)\in L^2(\mathbb{R})$.

a) When $\omega > 0$ and $xh(x) \in L^1(\mathbb{R})$ problem (4.9) admits a unique solution in $H^2(\mathbb{R})$ if and only if

$$\left(h(x), \frac{e^{\pm i\sqrt{\omega}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0. \tag{4.11}$$

b) When $\omega = 0$ and $x^2 h(x) \in L^1(\mathbb{R})$ problem (4.9) admits a unique solution in $H^2(\mathbb{R})$ if and only if

$$(h(x), 1)_{L^2(\mathbb{R})} = 0, \ (h(x), x)_{L^2(\mathbb{R})} = 0. \tag{4.12}$$

Proof. Let us start with case a) and introduce the auxiliary set in the Fourier space

$$A_\delta := [-\sqrt{\omega} - \delta, \ -\sqrt{\omega} + \delta] \cup [\sqrt{\omega} - \delta, \ \sqrt{\omega} + \delta] := A_\delta^- \cup A_\delta^+,$$

with $0 < \delta < \sqrt{\omega}$, such that

$$\widehat{\phi}(p) = -\frac{\widehat{h}(p)}{p^2 - \omega}\chi_{A_\delta} - \frac{\widehat{h}(p)}{p^2 - \omega}\chi_{A_\delta^c}. \tag{4.13}$$

Here and below χ_A stands for the characteristic function of a set A and A^c for its complement. The second term in the right side of (4.13) is not singular and can be easily estimated above in the absolute value by $\frac{|\widehat{h}(p)|}{\delta^2} \in L^2(\mathbb{R})$. To study the behavior of the first term in the right side of (4.13) on A_δ^+ we use the representation formula

$$\widehat{h}(p) = \widehat{h}(\sqrt{\omega}) + \int_{\sqrt{\omega}}^{p} \frac{d\widehat{h}(s)}{ds} ds$$

and $\left|\frac{d\widehat{h}(p)}{dp}\right| \leq \frac{1}{\sqrt{2\pi}} \|xh\|_{L^1(\mathbb{R})}, \ p \in \mathbb{R}$, which yields

$$\left|\frac{\int_{\sqrt{\omega}}^{p} \frac{d\widehat{h}(s)}{ds} ds}{p^2 - \omega}\chi_{A_\delta^+}\right| \leq C\frac{\chi_{A_\delta^+}}{2\sqrt{\omega} - \delta} \in L^2(\mathbb{R}).$$

Similarly near the negative singularity

$$\widehat{h}(p) = \widehat{h}(-\sqrt{\omega}) + \int_{-\sqrt{\omega}}^{p} \frac{d\widehat{h}(s)}{ds} ds,$$

such that

$$\left|\frac{\int_{-\sqrt{\omega}}^{p} \frac{d\widehat{h}(s)}{ds} ds}{p^2 - \omega}\chi_{A_\delta^-}\right| \leq C\frac{\chi_{A_\delta^-}}{2\sqrt{\omega} - \delta} \in L^2(\mathbb{R}).$$

Therefore, it remains to investigate the square integrability of the sum of the two terms

$$\frac{\widehat{h}(\sqrt{\omega})}{p^2 - \omega}\chi_{A_\delta^+} + \frac{\widehat{h}(-\sqrt{\omega})}{p^2 - \omega}\chi_{A_\delta^-},$$

for which the square of the $L^2(\mathbb{R})$ norm can be easily bounded below by

$$\frac{1}{(2\sqrt{\omega}+\delta)^2}\left[\int_{-\sqrt{\omega}-\delta}^{-\sqrt{\omega}+\delta}\frac{|\widehat{h}(-\sqrt{\omega})|^2}{(p+\sqrt{\omega})^2}dp+\int_{\sqrt{\omega}-\delta}^{\sqrt{\omega}+\delta}\frac{|\widehat{h}(\sqrt{\omega})|^2}{(p-\sqrt{\omega})^2}dp\right].$$

The expression above is finite if and only if $\widehat{h}(\pm\sqrt{\omega})$ vanish which is equivalent to orthogonality relations (4.11). Then using formula (4.10) we easily obtain

$$p^2\widehat{\phi}(p) = -\widehat{h}(p)+\omega\widehat{\phi}(p)\in L^2(\mathbb{R})$$

under the conditions of the lemma such that $\phi(x)\in H^2(\mathbb{R})$. In the case when parameter ω vanishes we write

$$\widehat{\phi}(p) = -\frac{\widehat{h}(p)}{p^2}\chi_{\{p\in\mathbb{R}:\ |p|\le 1\}}-\frac{\widehat{h}(p)}{p^2}\chi_{\{p\in\mathbb{R}:\ |p|>1\}}. \tag{4.14}$$

The second term in the right side of (4.14) can be bounded above in the absolute value by $|\widehat{h}(p)|\in L^2$, which will be true in higher dimensions studied in the following lemma as well. Let us expand the Fourier transform

$$\widehat{h}(p)=\widehat{h}(0)+\frac{d\widehat{h}}{dp}(0)p+\int_0^p\left(\int_0^s\frac{d^2\widehat{h}(q)}{dq^2}dq\right)ds$$

with the second derivative $\left|\frac{d^2\widehat{h}(q)}{dq^2}\right|\le\frac{1}{\sqrt{2\pi}}\|x^2h\|_{L^1(\mathbb{R})}<\infty,\ q\in\mathbb{R}$. Hence we estimate

$$\left|\frac{\int_0^p\left(\int_0^s\frac{d^2\widehat{h}(q)}{dq^2}dq\right)ds}{p^2}\chi_{\{p\in\mathbb{R}:\ |p|\le 1\}}\right|\le C\chi_{\{p\in\mathbb{R}:\ |p|\le 1\}}\in L^2(\mathbb{R}).$$

The remaining sum of the two terms

$$\frac{\widehat{h}(0)}{p^2}\chi_{\{p\in\mathbb{R}:\ |p|\le 1\}}+\frac{\frac{d\widehat{h}}{dp}(0)}{p}\chi_{\{p\in\mathbb{R}:\ |p|\le 1\}}$$

is square integrable if and only if both $\widehat{h}(0)$ and $\frac{d\widehat{h}}{dp}(0)$ vanish which yields orthogonality relations (4.12). Clearly $p^2\widehat{\phi}(p)=-\widehat{h}(p)\in L^2(\mathbb{R})$ which completes the proof of the lemma in case b). ■

Remark. *The proof of the fact that $\Delta\phi$ is square integrable, $\omega\ge 0$ given above is independent of the dimension and therefore, will be omitted in the proof of Lemma 6 below.*

Then we turn our attention to the solvability conditions for equation (4.9) in higher dimensions. Note that the orthogonality relations derived below will be dependent upon the value of $d\ge 2$.

Lemma 6. *Let $h(x) \in L^2(\mathbb{R}^d),\ d \geq 2$.*

a) When $\omega > 0$ and $xh(x) \in L^1(\mathbb{R}^d)$ problem (4.9) admits a unique solution in $H^2(\mathbb{R}^d)$ if and only if

$$\left(h(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0,\ p \in S^d_{\sqrt{\omega}}\ a.e.,\ d \geq 2. \tag{4.15}$$

b) When $\omega = 0$ and $|x|^2 h(x) \in L^1(\mathbb{R}^2)$ problem (4.9) admits a unique solution in $H^2(\mathbb{R}^2)$ if and only if

$$(h(x), 1)_{L^2(\mathbb{R}^2)} = 0,\ (h(x), x_k)_{L^2(\mathbb{R}^2)} = 0,\ 1 \leq k \leq 2. \tag{4.16}$$

c) When $\omega = 0$ and $|x|h(x) \in L^1(\mathbb{R}^d),\ d = 3, 4$ problem (4.9) admits a unique solution in $H^2(\mathbb{R}^d)$ if and only if

$$(h(x), 1)_{L^2(\mathbb{R}^d)} = 0,\ d = 3, 4. \tag{4.17}$$

d) When $\omega = 0$ and $|x|h(x) \in L^1(\mathbb{R}^d),\ d \geq 5$ problem (4.9) possesses a unique solution in $H^2(\mathbb{R}^d)$.

Proof. We start with the case of $\omega > 0$ and introduce the spherical layer set in the space of d dimensions

$$B_\delta := \{p \in \mathbb{R}^d \mid \sqrt{\omega} - \delta \leq |p| \leq \sqrt{\omega} + \delta\},\ 0 < \delta < \sqrt{\omega}.$$

Thus

$$\widehat{\phi}(p) = -\frac{\widehat{h}(p)}{p^2 - \omega}\chi_{B_\delta} - \frac{\widehat{h}(p)}{p^2 - \omega}\chi_{B_\delta^c}. \tag{4.18}$$

The second term in the right side of (4.18) can be easily estimated above in the absolute value by $\frac{|\widehat{h}(p)|}{\delta\sqrt{\omega}} \in L^2(\mathbb{R}^d)$. To study the singular part of the expression above we will use the representation formula

$$\widehat{h}(p) = \widehat{h}(\sqrt{\omega}, \sigma) + \int_{\sqrt{\omega}}^{|p|} \frac{\partial \widehat{h}(|s|, \sigma)}{\partial |s|} d|s|,$$

where σ denotes the variables on the sphere. Clearly, $\left|\frac{\partial \widehat{h}}{\partial |p|}\right| \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \|xh(x)\|_{L^1(\mathbb{R}^d)} < \infty$ by the assumption of the lemma. This yields

$$\left|\frac{\int_{\sqrt{\omega}}^{|p|} \frac{\partial \widehat{h}}{\partial |s|}(|s|, \sigma) d|s|}{p^2 - \omega}\chi_{B_\delta}\right| \leq \frac{C}{|p| + \sqrt{\omega}}\chi_{B_\delta} \in L^2(\mathbb{R}^d).$$

Thus it remains to estimate the norm

$$\left\|\frac{\widehat{h}(\sqrt{\omega}, \sigma)}{p^2 - \omega}\chi_{B_\delta}\right\|^2_{L^2(\mathbb{R}^d)} = \int_{S^d} d\sigma \int_{\sqrt{\omega}-\delta}^{\sqrt{\omega}+\delta} \frac{|\widehat{h}(\sqrt{\omega}, \sigma)|^2}{(p^2 - \omega)^2} |p|^{d-1} d|p| \geq$$

$$\geq \frac{(\sqrt{\omega}-\delta)^{d-1}}{(2\sqrt{\omega}+\delta)^2}\int_{\sqrt{\omega}-\delta}^{\sqrt{\omega}+\delta}\frac{d|p|}{(|p|-\sqrt{\omega})^2}\int_{S^d} d\sigma|\widehat{h}(\sqrt{\omega},\sigma)|^2,$$

which is finite if and only if the Fourier image $\widehat{h}(p)$ vanishes a.e. on the sphere $S^d_{\sqrt{\omega}}$. This is equivalent to orthogonality relations (4.15).

When ω vanishes and the problem is in two dimensions we use the formula analogous to (4.14) in which our primary concern will be the first term in the right side. In the polar coordinates $x = (|x|, \theta_x)$ and $p = (|p|, \theta_p)$. We will make use of the expansion

$$\widehat{h}(p) = \widehat{h}(0) + |p|\frac{\partial\widehat{h}}{\partial|p|}(0,\theta_p) + \int_0^{|p|}\left(\int_0^s \frac{\partial^2}{\partial|q|^2}\widehat{h}(|q|,\theta_p)d|q|\right)ds$$

with

$$\widehat{h}(p) = \frac{1}{2\pi}\int_{\mathbb{R}^2} h(x)e^{-i|p||x|cos\theta}dx, \tag{4.19}$$

where θ here and below stands for the angle between vectors x and p in $\mathbb{R}^2$. Thus for the derivatives we have $\frac{\partial\widehat{h}}{\partial|p|}(0,\theta_p) = -\frac{i}{2\pi}\int_{\mathbb{R}^2} h(x)|x|cos\theta dx$ and $\left|\frac{\partial^2}{\partial|p|^2}\widehat{h}(p)\right| \leq \frac{1}{2\pi}\|x^2h(x)\|_{L^1(\mathbb{R}^2)} < \infty$ by the assumption of the lemma. Clearly

$$\left|\frac{\int_0^{|p|}\left(\int_0^s \frac{\partial^2}{\partial|q|^2}\widehat{h}(|q|,\theta_p)d|q|\right)ds}{p^2}\chi_{\{p\in\mathbb{R}^2:|p|\leq 1\}}\right| \leq C\chi_{\{p\in\mathbb{R}^2:|p|\leq 1\}} \in L^2(\mathbb{R}^2)$$

and it remains to estimate the terms

$$-\frac{\widehat{h}(0)}{p^2}\chi_{\{p\in\mathbb{R}^2:|p|\leq 1\}} + \frac{i\int_{\mathbb{R}^2}|x|h(x)cos(\theta_p-\theta_x)dx}{2\pi|p|}\chi_{\{p\in\mathbb{R}^2:|p|\leq 1\}},$$

which can be written as

$$-\frac{\widehat{h}(0)}{p^2}\chi_{\{p\in\mathbb{R}^2:|p|\leq 1\}} + \frac{i}{2\pi}\frac{\sqrt{R_1^2+R_2^2}cos(\theta_p-\beta)}{|p|}\chi_{\{p\in\mathbb{R}^2:|p|\leq 1\}},$$

where $R_k := \int_{\mathbb{R}^2} x_k h(x)dx,\ k = 1,2$ and $tan\beta := \frac{R_2}{R_1}$. Note that the case of $R_1 = 0$ and $R_2 \neq 0$ corresponds to the situation when the argument $\beta = \frac{\pi}{2}$ or $-\frac{\pi}{2}$. Evaluation of the square of the L^2 norm of the sum above yields

$$2\pi|\widehat{h}(0)|^2\int_0^1\frac{d|p|}{|p|^3} + \frac{R_1^2+R_2^2}{4\pi^2}\int_0^1\frac{d|p|}{|p|}\int_0^{2\pi} d\theta_p cos^2(\theta_p-\beta),$$

which is finite if and only if $\widehat{h}(0)$ along with $R_{1,2}$ vanish. This is equivalent to relations (4.16).

When $\omega = 0$ and the equation is studied in $\mathbb{R}^3$ we will use the formula

$$\widehat{h}(p) = \widehat{h}(0) + \int_0^{|p|} \frac{\partial \widehat{h}}{\partial |s|}(|s|, \sigma) d|s|. \tag{4.20}$$

Let us investigate the square integrability of the sum

$$\frac{\widehat{h}(0)}{p^2} \chi_{\{p \in \mathbb{R}^3 \colon \, |p| \le 1\}} + \frac{\int_0^{|p|} \frac{\partial \widehat{h}}{\partial |s|}(|s|, \sigma) d|s|}{p^2} \chi_{\{p \in \mathbb{R}^3 \colon \, |p| \le 1\}}.$$

Using the three dimensional analog of (4.19) we obtain $\left| \dfrac{\partial \widehat{h}}{\partial |p|}(p) \right| \le \dfrac{1}{(2\pi)^{\frac{3}{2}}} \| |x| h(x) \|_{L^1(\mathbb{R}^3)} < \infty$ by the assumption of the lemma. Hence

$$\left| \frac{\int_0^{|p|} \frac{\partial \widehat{h}}{\partial |s|}(|s|, \sigma) d|s|}{p^2} \chi_{\{p \in \mathbb{R}^3 \colon \, |p| \le 1\}} \right| \le \frac{C}{|p|} \chi_{\{p \in \mathbb{R}^3 \colon \, |p| \le 1\}} \in L^2(\mathbb{R}^3). \tag{4.21}$$

The square of the L^2 norm of the remaining term will be given by $4\pi |\widehat{h}(0)|^2 \displaystyle\int_0^1 \frac{d|p|}{|p|^2} < \infty$ if and only if $\widehat{h}(0) = 0$, which is equivalent to relation (4.17) in three dimensions. For $\omega = 0$ in $\mathbb{R}^4$ the argument will be similar to the three dimensional one.

When the parameter ω vanishes and $d \ge 5$ we will make use of the representation formula analogous to (4.20) and the upper bound similar to (4.21). Thus the square of the L^2 norm which remains to estimate will be equal to

$$\int_0^1 \frac{|\widehat{h}(0)|^2}{|p|^4} |S^d| |p|^{d-1} d|p| = |S^d| |\widehat{h}(0)|^2 \int_0^1 |p|^{d-5} d|p| < \infty,$$

which proves that when $\omega = 0$ the orthogonality conditions in dimensions five and higher are not needed for solving equation (4.9). ■

The final proposition of the article is another, even simpler way to look at the solvability conditions for equation (4.9) in higher dimensions.

Lemma 7. *Let $\omega = 0$ and $h(x) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ with $d \ge 5$. Then problem (4.9) admits a unique solution in $H^2(\mathbb{R}^d)$.*

Proof. Obviously,

$$|\widehat{h}(p)| \le \frac{1}{(2\pi)^{\frac{d}{2}}} \|h(x)\|_{L^1(\mathbb{R}^d)} < \infty, \ p \in \mathbb{R}^d.$$

It is sufficient to consider the first term in the right side of the higher dimensional analog of formula (4.14). For it we have the upper bound in the absolute value as

$$\frac{C}{p^2} \chi_{\{p \in \mathbb{R}^d \colon \, |p| \le 1\}} \in L^2(\mathbb{R}^d), \ d \ge 5,$$

which completes the proof of the lemma. ■

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References

[1] J.F. Collet, V.A. Volpert. *Computation of the index of linear elliptic operators in unbounded cylinders*. J. Funct. Anal., 164 (1999), No. 1, 34–59.

[2] S. Cuccagna, D. Pelinovsky, V. Vougalter. *Spectra of positive and negative energies in the linearized NLS problem*. Comm. Pure Appl. Math., 58 (2005), No.1, 1–29.

[3] A. Ducrot, M. Marion, V. Volpert. *Systemes de réaction-diffusion sans propriété de Fredholm*. C.R.Math.Acad.Sci.Paris, 340 (2005), No. 9, 659–664.

[4] A. Ducrot, M. Marion, V. Volpert. *Reaction-diffusion problems with non Fredholm operators*. Adv. Differential Equations, 13 (2008), No. 11-12, 1151–1192.

[5] B.L.G. Jonsson, M. Merkli, I.M. Sigal, F. Ting. Applied Analysis. In preparation.

[6] E. Lieb, M. Loss. Analysis. Graduate studies in Mathematics, 14. American Mathematical Society, Providence, RI, 1997.

[7] S.G. Kryzhevich, V.A. Volpert. *Fredholm conditions and solvability of elliptic problems in unbounded cylinders.* Vestnik St. Petersburg Univ. Math. 37 (2004), No.1, 15–22.

[8] S.G. Kryzhevich, V.A. Volpert. *On solvability of linear elliptic problems in unbounded cylinders.* Vestnik St. Petersburg Univ. Math. 37 (2004), No.3, 22–27.

[9] D.E. Pelinovsky, J. Yang. *A normal form for nonlinear resonance of embedded solitons*. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 458 (2002), No. 2022, 1469–1497.

[10] M. Reed, B. Simon. Methods of modern mathematical physics, Volume III. Scattering theory. Academic Press, 1979.

[11] V. Volpert, B. Kazmierczak, M. Massot, Z.Peradzynski. *Solvability conditions for elliptic problems with non-Fredholm operators*. Appl. Math., 29 (2002), No. 2, 219–238.

[12] V. Vougalter. *On threshold eigenvalues and resonances for the linearized NLS equation*. Math. Model. Nat. Phenom. 5 (2010), No.4, 448–469.

[13] V. Vougalter, V. Volpert. *Solvability conditions for some non Fredholm operators*. Proc. Edinb. Math. Soc. (2), 54 (2011), No. 1, 249–271.

[14] V. Vougalter, V. Volpert. *On the solvability conditions for some non Fredholm operators*. Int. J. Pure Appl. Math., 60 (2010), No. 2, 169–191.

[15] V. Vougalter, V. Volpert. *On the solvability conditions for the diffusion equation with convection terms*. Commun. Pure Appl. Anal., 11 (2012), No.1, 365–373

[16] V. Vougalter, V. Volpert. *Solvability relations for some non Fredholm operators*. Int. Electron. J. Pure Appl. Math., 2 (2010), No. 1, 75–83.

[17] V. Volpert, V. Vougalter. *On the solvability conditions for a linearized Cahn-Hilliard equation*. To appear in Rend. Istit. Mat. Univ. Trieste

[18] V. Vougalter, V. Volpert. *On the existence of stationary solutions for some non-Fredholm integro-differential equations*. Doc. Math., 16 (2011), 561–580.