A LOWER BOUND ON BLOWUP RATES FOR THE 3D INCOMPRESSIBLE EULER EQUATION AND A SINGLE EXPONENTIAL BEALE-KATO-MAJDA ESTIMATE

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Abstract. We prove a Beale-Kato-Majda criterion for the loss of regularity for solutions of the incompressible Euler equations in $H^s(\mathbb{R}^3)$, for $s > \frac{5}{2}$. Instead of double exponential estimates of Beale-Kato-Majda type, we obtain a single exponential bound on $\|u(t)\|_{H^s}$ involving the dimensionless parameter introduced by P. Constantin in [2]. In particular, we derive lower bounds on the blowup rate of such solutions.

1. Introduction

In this paper, we revisit the Beale-Kato-Majda criterion for the breakdown of smooth solutions to the $3D$ Euler equations.

More precisely, we consider the incompressible Euler equations

$$\partial_t u + (u \cdot \nabla) u + \nabla p \, = \, 0 \tag{1.1}$$

$$\nabla \cdot u \, = \, 0 \tag{1.2}$$

$$u(x,0) \, = \, u_0 \tag{1.3}$$

for an unknown velocity vector $u(x,t) = (u_i(x,t))_{1 \leq i \leq 3} \in \mathbb{R}^3$ and pressure $p = p(x,t) \in \mathbb{R}$, for position $x \in \mathbb{R}^3$ and time $t \in [0,\infty)$.

Existence and uniqueness of local in time solutions to (1.1) – (1.3) in the space

$$C([0,T], H^s) \cap C^1([0,T]; H^{s-1}) \, , \tag{1.4}$$

has long been known for $s > \frac{5}{2}$, see for instance [6]. However, it is an open problem to determine whether such solutions can lose their regularity in finite time. An important result that addresses the question of a possible loss of regularity of solutions to Euler equations (1.1) – (1.3) is the criterion formulated by Beale-Kato-Majda [1] in terms of the L^∞ norm of the vorticity $\omega = \nabla \wedge u$. More precisely, Beale-Kato-Majda in [1] proved the following theorem:

Theorem 1.1. *Let u be a solution to (1.1) – (1.3) in the class (1.4) for $s \geq 3$ integer. Suppose that there exists a time T^* such that the solution cannot be continued in the class (1.4) to $T = T^*$. If T^* is the first such time, then*

$$\int_0^{T^*} \|\omega(\cdot,t)\|_{L^\infty} \, dt = \infty. \tag{1.5}$$

Date: June 17, 2011.

The theorem is proved with a contradiction argument. Under the assumption

$$\int_0^{T^*} \|\omega(\cdot,t)\|_{L^\infty}\, dt < \infty\,,$$

the authors of [1] show that $\|u(\cdot,t)\|_{H^s} \leq C_0$, for all $t < T^*$ contradicting the hypothesis that T^* is the first time such that the solution cannot be continued to $T = T^*$. In particular, Beale-Kato-Majda obtain a double exponential bound for $\|u(\cdot,t)\|_{H^s}$, which follows from the following estimates:

Step 1 An energy-type bound on $\|u\|_{H^s}$ in terms of $\|Du\|_{L^\infty}$, where $Du = [\partial_i u_j]_{ij}$ is a 3×3-matrix valued function. More specifically, one applies the operator D^α to equations (1.1)-(1.2), where α is an integer-valued multi-index with $|\alpha| \leq s$ and uses a certain commutator estimate to derive

$$\frac{d}{dt}\|u(\cdot,t\|_{H^s}^2 \leq 2C\|Du\|_{L^\infty}\|u(\cdot,t)\|_{H^s}^2, \tag{1.6}$$

which via Gronwall's inequality gives the bound:

$$\|u(\cdot,t)\|_{H^s} \;\leq\; \|u_0\|_{H^s}\, \exp\left(C\int_0^t \|Du(\cdot,\tau)\|_{L^\infty}\, d\tau \right). \tag{1.7}$$

Step 2 An estimate on $\|Du(\cdot,t)\|_{L^\infty}$ based on the quantities $\|\omega(\cdot,t)\|_{L^\infty}$, $\|\omega(\cdot,t)\|_{L^2}$, and $\log^+\|u(\cdot,t)\|_{H^3}$, given by

$$\|Du(\cdot,t)\|_{L^\infty} \;\leq\; C\left\{ 1 + \left(1 + \log^+\|u(\cdot,t)\|_{H^3}\right)\|\omega(\cdot,t)\|_{L^\infty} + \|\omega(\cdot,t)\|_{L^2} \right\}, \tag{1.8}$$

where C is a universal constant.

Step 3 The bound on $\|\omega(\cdot,t)\|_{L^2}$ in terms of $\|\omega(\cdot,t)\|_{L^\infty}$ given by

$$\frac{d}{dt}\|\omega(\cdot,t)\|_{L^2}^2 \;\leq\; 2\,D\,\|\omega(\cdot,t)\|_{L^\infty}\,\|\omega(\cdot,t)\|_{L^2}^2\,,$$

which follows from taking the $L^2(\mathbb{R}^3)$-inner product of ω with the equation for vorticity. Then, Gronwall's inequality yields

$$\|\omega(\cdot,t)\|_{L^2} \leq \|\omega(\cdot,0)\|_{L^2}\, \exp\left(D\int_0^t \|\omega(\,\cdot\,,\tau)\|_{L^\infty}\, d\tau\right). \tag{1.9}$$

Consequently, one obtains the double exponential bound

$$\|u(\cdot,t)\|_{H^s} \;\leq\; \|u_0\|_{H^s}\, \exp\left(\exp\left(C\int_0^t \|\omega(\,\cdot\,,\tau)\|_{L^\infty}\, d\tau\right)\right). \tag{1.10}$$

from combining (1.7), (1.8) and (1.9).

It is an open question whether (1.10) is sharp[1]. While we do not attempt to answer that question itself in this paper, we obtain a single exponential bound on the H^s-norm of solution to Euler equations (1.1) - (1.3) in terms of the quantity

$$\ell_\delta(t) \;=\; \min\left\{ L\,,\ \left(\frac{\|\omega(t)\|_{C^\delta}}{\|u_0\|_{L^2}}\right)^{-\frac{2}{2\delta+5}} \right\}, \tag{1.11}$$

[1]Single exponential bounds have been obtained in other solution spaces than those displayed above, see for instance [7] for such a result in BMO.

where

$$\|\omega\|_{C^\delta} \;=\; \sup_{|x-y|<L} \frac{|\omega(x)-\omega(y)|}{|x-y|^\delta} \tag{1.12}$$

denotes the δ-Holder seminorm, for $L > 0$ fixed, and $\delta > 0$. More precisely, we prove the following theorem:

Theorem 1.2. *Let u be a solution to* (1.1) - (1.3) *in the class* (1.4)*, for $s = \frac{5}{2} + \delta$. Assume that $\ell_\delta(t)$ is defined as above, and that*

$$\int_0^T \left(\ell_\delta(\tau)\right)^{-\frac{5}{2}} \, d\tau \;<\; \infty\,. \tag{1.13}$$

Then, there exists a finite positive constant $C_\delta = O(\delta^{-1})$ independent of u and t such that

$$\|u(\cdot,t)\|_{H^s} \;\leq\; \|u_0\|_{H^s} \, \exp\Big[\, C_\delta \, \|u_0\|_{L^2} \int_0^t \left(\ell_\delta(\tau)\right)^{-\frac{5}{2}} \, d\tau \,\Big]$$

holds for $0 \leq t \leq T$.

The quantity $\ell_\delta(t)$ has the dimension of length, and was introduced by Constantin in [2] (see also the work of Constantin, Fefferman and Majda [4] where a criterion for loss of regularity in terms of the direction of vorticity was obtained), where it was observed that

$$\int_0^T \left(\ell_\delta(t)\right)^{-\frac{5}{2}} \, dt \;=\; \infty \tag{1.14}$$

is a necessary and sufficient condition for blow-up of Euler equations. In particular, the necessity of the condition follows from the inequality obtained in [2]

$$\|\omega(\cdot,t)\|_{L^\infty} \leq \|u(\cdot,t)\|_{L^2} \, \left(\ell_\delta(t)\right)^{-\frac{5}{2}}\,, \tag{1.15}$$

and Theorem 1.1 of Beale-Kato-Majda. This is so because Theorem 1.1 implies that if the solution cannot be continued to some time T, then $\int_0^T \|\omega(\cdot,t)\|_{L^\infty} \, dt = \infty$. As a consequence of (1.15), and conservation of energy

$$\|u(\cdot,t)\|_{L^2} \;=\; \|u_0\|_{L^2}\,, \tag{1.16}$$

this in turn implies (1.14). However, by invoking the result of Beale-Kato-Majda in this argument, one again obtains a double exponential bound on $\|u(\cdot,t)\|_{H^s}$ in terms of $\int_0^T \left(\ell_\delta(t)\right)^{-\frac{5}{2}} dt$. We refer to [3, 5] for recent developments in this and related areas.

In this paper, we observe that one can actually obtain a single exponential bound on the H^s-norm of the solution $u(t)$ in terms of $\int_0^T \left(\ell_\delta(t)\right)^{-\frac{5}{2}} dt$, as stated in Theorem 1.2. This is achieved by avoiding the use of the logarithmic inequality (1.8) from [1]. More precisely, we combine the energy bound (1.6) with a Calderon-Zygmund type bound on the symmetric and antisymmetric parts of Du.

Also, we obtain a lower bound on the blowup rate of solutions in $H^{\frac{5}{2}+\delta}$. Specifically, we prove:

Theorem 1.3. *Let u be a solution to (1.1) – (1.3) in the class*

$$C([0,T];H^{\frac{5}{2}+\delta})\cap C^1([0,T];H^{\frac{3}{2}+\delta}). \tag{1.17}$$

Suppose that there exists a time T^* such that the solution cannot be continued in the class (1.17) to $T=T^*$. If T^* is the first such time then there exists a finite, positive constant $C(\delta,\|u_0\|_{L^2})$ such that

$$\|u(\cdot,t)\|_{H^{\frac{5}{2}+\delta}} \;\geq\; C(\delta,\|u_0\|_{L^2})\left(\frac{1}{T^*-t}\right)^{1+\frac{2}{5}\delta}, \tag{1.18}$$

under the condition that t is sufficiently close to T^* (see the conditions (3.22) and (3.23) below, with $t_0=t$).

The proof of Theorem 1.3 can be outlined as follows. We assume that u is a solution in the class (1.17) that cannot be continued to $T=T^*$, and that T^* is the first such time. Invoking the local in time existence result, we derive a lower bound $T_{loc,t_1}>0$ on the time of existence of solutions to Euler equations in (1.17) for initial data $u(t_1)\in H^{\frac{5}{2}+\delta}$ at an arbitrary time $t_1<T^*$. By definition of T^*, we thus have

$$t_1 \,+\, T_{loc,t_1} \,<\, T^*\,. \tag{1.19}$$

Based on an energy bound on the $H^{\frac{5}{2}+\delta}$-norm of the solution, we obtain in Section 3 an expression for T_{loc,t_1} of the form $\frac{1}{C\|u(\cdot,t_1)\|_{H^{\frac{5}{2}+\delta}}}$, which together with (1.19) implies that

$$\|u(\cdot,t_1)\|_{H^{\frac{5}{2}+\delta}} > \frac{1}{C(T^*-t_1)}\,, \tag{1.20}$$

for all $t_1<T^*$. This is an "a priori" lower bound on the blowup rate. Subsequently, we improve (1.20) by a recursion argument in Theorem 1.3 for times t close to T^*, to yield the stronger bound (1.18).

2. Proof of theorem 1.2

First we recall that the full gradient of velocity Du can be decomposed into symmetric and antisymmetric parts,

$$Du \,=\, Du^+ \,+\, Du^- \tag{2.1}$$

where

$$Du^\pm \,=\, \frac{1}{2}\big(\, Du \pm Du^T \,\big)\,. \tag{2.2}$$

Du^+ is called the deformation tensor.

In the following lemma we recall important properties of Du^+ and Du^-. For the convenience of the reader, we give proofs of those properties, although some of them are available in the literature, see e.g. [2].

Lemma 2.1. *For both the symmetric and antisymmetric parts Du^+, Du^- of Du, the L^2 bound*

$$\|Du^\pm\|_{L^2} \,\leq\, C\|\omega\|_{L^2}\,. \tag{2.3}$$

holds.

The antisymmetric part Du^- satisfies

$$Du^- v \;=\; \frac{1}{2}\,\omega\wedge v \tag{2.4}$$

for any vector $v\in\mathbb{R}^3$. The vorticity ω satisfies the identity

$$\omega(\xi) \;=\; \frac{1}{4\pi}P.V.\int \sigma(\widehat{y})\,\omega(x+y)\,\frac{dy}{|y|^3}\,, \tag{2.5}$$

("P.V." denotes principal value) where $\sigma(\widehat{y}) = 3\,\widehat{y}\otimes\widehat{y} - \mathbf{1}$, with $\widehat{y}=\frac{y}{|y|}$. Notably,

$$\int_{S^2}\sigma(\widehat{y})\,d\mu_{S^2}(y) \;=\; 0\,, \tag{2.6}$$

where $d\mu_{S^2}$ denotes the standard measure on the sphere S^2.

The matrix components of the symmetric part have the form

$$Du^+_{ij} \;=\; \sum_\ell T^\ell_{ij}(\omega_\ell) \;=\; \sum_\ell \mathcal{K}^\ell_{ij} * \omega_\ell\,, \tag{2.7}$$

where ω_ℓ are the vector components of ω, and where the integral kernels $\mathcal{K}^\ell_{ij}$ have the properties

$$\mathcal{K}^\ell_{ij}(y) \;=\; \sigma^\ell_{ij}(\widehat{y})\,|y|^{-3} \tag{2.8}$$

$$\|\sigma^\ell_{ij}\|_{C^1(S^2)} \;\leq\; C \tag{2.9}$$

$$\int_{S^2}\sigma^\ell_{ij}(\widehat{y})\,d\mu_{S^2}(y) \;=\; 0\,. \tag{2.10}$$

Thus in particular, T^ℓ_{ij} is a Calderon-Zygmund operator, for every $i,j,\ell\in\{1,2,3\}$.

Proof. An explicit calculation shows that the Fourier transform of Du as a function of $\widehat{\omega}$ is given by

$$\widehat{Du}(\xi) \;=\; -[(\partial_i(\Delta^{-1}\nabla\wedge\omega)_j)\widehat{\;}(\xi)]_{i,j} \;=\; \widehat{G}(\xi) \,+\, \widehat{H}(\xi) \tag{2.11}$$

where

$$\widehat{G}(\xi) \;:=\; \frac{1}{2|\xi|^2}\left[\begin{array}{ccc} \xi_1\xi_2\widehat{\omega}_3 - \xi_1\xi_3\widehat{\omega}_2 & -\xi_2\xi_3\widehat{\omega}_2 & \xi_2\xi_3\widehat{\omega}_3 \\ \xi_1\xi_3\widehat{\omega}_1 & \xi_2\xi_3\widehat{\omega}_1 - \xi_1\xi_2\widehat{\omega}_3 & -\xi_1\xi_3\widehat{\omega}_3 \\ -\xi_1\xi_2\widehat{\omega}_1 & \xi_1\xi_2\widehat{\omega}_2 & \xi_1\xi_3\widehat{\omega}_2 - \xi_2\xi_3\widehat{\omega}_1 \end{array}\right] \tag{2.12}$$

and

$$\widehat{H}(\xi) \;:=\; \frac{1}{2|\xi|^2}\left[\begin{array}{ccc} 0 & \xi_2^2\widehat{\omega}_3 & -\xi_3^2\widehat{\omega}_2 \\ -\xi_1^2\widehat{\omega}_3 & 0 & \xi_3^2\widehat{\omega}_1 \\ \xi_1^2\widehat{\omega}_2 & -\xi_2^2\widehat{\omega}_1 & 0 \end{array}\right]\,, \tag{2.13}$$

using the notation $\widehat{\omega}_j\equiv\widehat{\omega}_j(\xi)$ for brevity.

Clearly, every component of G is given by a sum of Fourier multiplication operators with symbols of the form $\frac{\xi_i\xi_j}{|\xi|^2}$, $i\neq j$, applied to a component of ω. For instance,

$$G_{21}(x) \;=\; const.\ P.V.\int \widehat{y}_1\widehat{y}_3\,\omega_1(x+y)\,\frac{dy}{|y|^3} \tag{2.14}$$

corresponds to the component G_{21}. It is easy to see that every component G_{ij} is a sum of Calderon-Zygmund operators applied to components of ω, with kernel satisfying the asserted properties (2.8) $\sim$ (2.10). The same is true for the symmetric part, $G^+ = \frac{1}{2}(G + G^T)$.

The symmetric part of $\widehat{H}(\xi)$ is given by

$$\widehat{H}^+(\xi) \;=\; \frac{1}{2|\xi|^2}\left[\begin{array}{ccc} 0 & (\xi_2^2-\xi_1^2)\widehat{\omega}_3 & (\xi_1^2-\xi_3^2)\widehat{\omega}_2 \\ (\xi_2^2-\xi_1^2)\widehat{\omega}_3 & 0 & (\xi_3^2-\xi_2^2)\widehat{\omega}_1 \\ (\xi_1^2-\xi_3^2)\widehat{\omega}_2 & (\xi_3^2-\xi_2^2)\widehat{\omega}_1 & 0 \end{array}\right] \tag{2.15}$$

so that each component defines a Fourier multiplication operator with symbol of the form $\frac{\xi_i^2-\xi_j^2}{|\xi|^2}$, $i \neq j$, acting on a component of ω (with associated kernel of the form $\frac{x_i^2-x_j^2}{|x|^{n+2}}$). That is, for instance,

$$H_{12}^+(x) \;=\; const\ P.V. \int (\widehat{y}_2^2 - \widehat{y}_1^2)\,\omega_3(x+y)\frac{dy}{|y|^3}\,. \tag{2.16}$$

The properties (2.8) $\sim$ (2.10) follow immediately.

The Fourier transforms of the integral kernels $\mathcal{K}_{ij}^\ell$ can be read off from the components $\widehat{G}_{ij}^+ + \widehat{H}_{ij}^+$. In position space, one finds that $\sigma_{ij}^\ell(\widehat{y})$ is obtained from a sum of terms proportional to terms of the form $\widehat{y}_{i_1}\widehat{y}_{j_1}$ and $(\widehat{y}_{i_2}^2 - \widehat{y}_{j_2}^2)$.

For the antisymmetric part Du^-, one generally has $Du^- v = \frac{1}{2}(\nabla \wedge u) \wedge v$ for any $v \in \mathbb{R}^3$, and from $u = -\Delta^{-1}\nabla \wedge \omega$, we get $Du^- v = \frac{1}{2}\omega \wedge v$, using that $\nabla \cdot u = 0$.

As a side remark, we note that while H^- does not by itself exhibit the properties (2.8) $\sim$ (2.10), it combines with G^- in a suitable manner to yield the stated properties of Du^-, thanks to the condition $\nabla \cdot \omega = 0$. □

Next, Lemma 2.2 below provides an upper bound in terms of the quantity $\ell_\delta(t)$ on singular integral operators applied to ω of the type appearing in (2.7). We note that similar bounds were used in [2] and [4] for the antisymmetric part Du^-. Here, we observe that they also hold for the symmetric part Du^+. As shown in [4] for Du^-, the proof of such a bound follows standard steps based on decomposing the singular integral into an inner and outer contribution. The inner contribution can be bounded based on a certain mean zero property, while the outer part is controlled via integration by parts.

Lemma 2.2. *For $L > 0$ fixed, and $\delta > 0$, let $\ell_\delta(t)$ be defined as above. Moreover, let ω_ℓ, $\ell = 1, 2, 3$, denote the components of the vorticity vector $\omega(t)$. Then, any singular integral operator*

$$T\omega_\ell(x) \;=\; \frac{1}{4\pi} P.V. \int \sigma_T(\widehat{y})\,\omega_\ell(x+y)\,\frac{dy}{|y|^3} \tag{2.17}$$

with

$$\int_{S^2} \sigma_T(\widehat{y}) d\mu_{S^2}(y) \;=\; 0 \quad , \quad \|\sigma_T\|_{C^1(S^2)} \;<\; C\,, \tag{2.18}$$

satisfies

$$\|T\omega_\ell\|_{L^\infty} \;\leq\; C(\delta)\,\|u_0\|_{L^2}\,\ell_\delta(t)^{-\frac{5}{2}} \tag{2.19}$$

for $\ell \in \{1, 2, 3\}$, for a constant $C(\delta) = O(\delta^{-1})$ independent of u and t.

Proof. Let $\chi_1(x)$ be a smooth cutoff function which is identical to 1 on $[0,1]$, and identically 0 for $x>2$. Moreover, let $\chi_R(x)=\chi_1(x/R)$, and $\chi_R^c = 1-\chi_R$.

We consider

$$\int_{|y|>\epsilon} \sigma_T(\widehat{y})\,\omega_\ell(x+y)\,\frac{dy}{|y|^3} \;=\; (I)\,+(II) \tag{2.20}$$

for $\epsilon>0$ arbitrary, where

$$(I) \;:=\; \int_{|y|>\epsilon} \sigma_T(\widehat{y})\,\omega_\ell(x+y)\,\chi_{\ell_\delta(t)}(|y|)\;\frac{dy}{|y|^3} \tag{2.21}$$

and

$$(II) \;:=\; \int \sigma_T(\widehat{y})\,\omega_\ell(x+y)\,\chi^c_{\ell_\delta(t)}(|y|)\;\frac{dy}{|y|^3}\,. \tag{2.22}$$

From the zero average property (2.18), we find

$$\begin{aligned}
\|(I)\|_{L^\infty} \;&=\; \Big|\int_{|y|>\epsilon} \sigma_T(\widehat{y})\,(\omega_\ell(x+y)-\omega_\ell(x))\,\chi_{\ell_\delta(t)}(|y|)\;\frac{dy}{|y|^3}\Big| \\
&\leq\; \|\omega_\ell\|_{C^\delta}\int_{|y|<2\ell_\delta(t)} \frac{dy}{|y|^{3-\delta}} \\
&\leq\; \frac{C}{\delta}(\ell_\delta(t))^\delta\,\|\omega_\ell\|_{C^\delta} \\
&\leq\; C\,\delta^{-1}\,\|u_0\|_{L^2}\,(\ell_\delta(t))^{-\frac52}
\end{aligned} \tag{2.23}$$

since from the definition of $\ell_\delta(t)$,

$$\|\omega_\ell\|_{C^\delta} \;\leq\; \|u_0\|_{L^2}\,(\ell_\delta(t))^{-\delta-\frac52} \tag{2.24}$$

follows straightforwardly. We can send $\epsilon\searrow 0$, since the estimates are uniform in ϵ.

On the other hand,

$$(II) \;\;=\;\; \int \sigma_T(\widehat{y})\,(\partial_{y_i}u_j-\partial_{y_j}u_i)(x+y)\,\chi^c_{\ell_\delta(t)}(|y|)\;\frac{dy}{|y|^3}\,. \tag{2.25}$$

It suffices to consider one of the terms in the difference,

$$\begin{aligned}
&\Big|\int \sigma_T(\widehat{y})\,\partial_{y_i}u_j(x+y)\,\chi^c_{\ell_\delta(t)}(|y|)\;\frac{dy}{|y|^3}\Big| \\
&\quad=\;\Big|\int dy\,u_j(x+y)\,\partial_{y_i}\Big(\sigma_T(\widehat{y})\,\chi^c_{\ell_\delta(t)}(|y|)\;\frac{1}{|y|^3}\Big)\Big| \\
&\quad\leq\; C\,\|u_j\|_{L^2}\,\Big\|\partial_{y_i}\Big(\sigma_T(\widehat{y})\,\chi^c_{\ell_\delta(t)}(|y|)\;\frac{1}{|y|^3}\Big)\Big\|_{L^2} \\
&\quad\leq\; C\,\|u_0\|_{L^2}\,(\ell_\delta(t))^{-\frac52}
\end{aligned} \tag{2.26}$$

where to obtain the last line we used the conservation of energy (1.16) and the following three bounds:

(i)

$$\begin{aligned}
\Big\|\Big(\partial_{y_i}\chi^c_R(|y|)\Big)\;\frac{\sigma_T(\widehat{y})}{|y|^3}\Big\|^2_{L^2} \;&\leq\; C\,\frac{1}{R^2}\int_{R<|y|<2R}\frac{dy}{|y|^6} \\
&\leq\; C\,R^{-5}\,,
\end{aligned} \tag{2.27}$$

for $R = \ell_\delta(t)$.

(ii)

$$\Big\| \sigma_T(\widehat{y})\, \chi_R^c(|y|)\ \partial_{y_i} \frac{1}{|y|^3} \Big\|_{L^2}^2 \ \leq\ C \int_{|y|>R} \frac{dy}{|y|^8} \ \leq\ C\, R^{-5}\,. \tag{2.28}$$

(iii)

$$\Big\| \chi_R^c(|y|)\, \frac{1}{|y|^3}\ \partial_{y_i}\, \sigma_T(\widehat{y}) \Big\|_{L^2}^2 \ \leq\ C \int_{|y|>R} \frac{1}{|y|^6}\, \frac{1}{|y|^2}\, dy \ \leq\ C\, R^{-5}\,, \tag{2.29}$$

where we used that

$$\Big|\nabla_y \sigma_T(\widehat{y})\Big| \ =\ \Big| \frac{1}{|y|}\, (\nabla_z \sigma_T(z_1, z_2, z_3))\Big|_{z=\widehat{y}} \Big| \ \leq\ \frac{1}{|y|}\, \|\sigma_T\|_{C^1(S^2)} \tag{2.30}$$

holds.

Summarizing, we arrive at

$$\|T\omega_\ell\|_{L^\infty} \ \leq\ C(\delta)\, \|u_0\|_{L^2}\, \ell_\delta(t)^{-\frac{5}{2}} \tag{2.31}$$

for $C(\delta) = O(\delta^{-1})$, which is the asserted bound. □

The form of the singular integral operator that appears in the statement of Lemma 2.2 is suitable for application to Du^+ and Du^-, as we shall see in the following corollary.

Corollary 2.3. *There exists a finite, positive constant $C_\delta = O(\frac{1}{\delta})$ independent of u and t such that the estimate*

$$\|Du^+\|_{L^\infty} + \|Du^-\|_{L^\infty} \ \leq\ C_\delta\, \|u_0\|_{L^2}\, \ell_\delta(t)^{-\frac{5}{2}} \tag{2.32}$$

holds.

Proof. According to Lemma 2.1, the matrix components of both Du^+ and Du^- have the form (2.17).

Accordingly, Lemma 2.2 immediately implies the assertion. □

Now we are ready to give a proof of Theorem 1.2, which is based on combining an energy estimate for Euler equations with Corollary 2.3.

For $s \geq 3$ integer-valued, the energy bound (1.6)

$$\frac{1}{2}\partial_t \|u(t)\|_{H^s}^2 \ \leq\ \|Du(t)\|_{L^\infty}\, \|u(t)\|_{H^s}^2 \tag{2.33}$$

was proven in [1]. For fractional $s > \frac{5}{2}$, we recall the definitions of the homogenous and inhomogenous Besov norms for $1 \leq p, q \leq \infty$,

$$\|u\|_{\dot{B}^s_{p,q}} \ =\ \Big(\sum_{j\in\mathbb{Z}} 2^{jqs} \|u_j\|_{L^p}^q \Big)^{\frac{1}{q}}\,, \tag{2.34}$$

respectively,

$$\|u\|_{B^s_{p,q}} = \Big(\|u\|^q_{L^p} + \|u\|^q_{\dot B^s_{p,q}} \Big)^{\frac{1}{q}} , \tag{2.35}$$

where $u_j = P_j u$ is the Paley-Littlewood projection of u of scale j. In analogy to (1.6), we obtain the bound on the $B^s_{2,2}$ Besov norm of $u(t)$ given by

$$\frac{1}{2}\partial_t \|u(t)\|^2_{B^s_{2,2}} \le \|Du(t)\|_{L^\infty} \, \|u(t)\|^2_{B^s_{2,2}} , \tag{2.36}$$

from a straightforward application of estimates obtained in [8]; details are given in the Appendix. Accordingly, since the left hand side yields

$$\partial_t \|u(t)\|^2_{B^s_{2,2}} = 2\|u(t)\|_{B^s_{2,2}} \partial_t \|u(t)\|_{B^s_{2,2}} , \tag{2.37}$$

we get

$$\partial_t \|u(t)\|_{B^s_{2,2}} \le \|Du(t)\|_{L^\infty} \, \|u(t)\|_{B^s_{2,2}} . \tag{2.38}$$

However, Corollary 2.3 implies that

$$\begin{aligned} \|Du(t)\|_{L^\infty} &\le \|Du^+(t)\|_{L^\infty} + \|Du^-(t)\|_{L^\infty} \\ &\le C_\delta \, \|u_0\|_{L^2} \, (\ell_\delta(t))^{-\frac{5}{2}} . \end{aligned} \tag{2.39}$$

Therefore, by combining (2.38) and (2.39) we obtain

$$\partial_t \|u(t)\|_{B^s_{2,2}} \le C_\delta \, \|u_0\|_{L^2} \, (\ell_\delta(t))^{-\frac{5}{2}} \, \|u(t)\|_{B^s_{2,2}} ,$$

which implies that

$$\begin{aligned} \|u(t)\|_{H^s} &\sim \|u(t)\|_{B^s_{2,2}} \\ &\le \|u_0\|_{B^s_{2,2}} \exp\Big[C_\delta \, \|u_0\|_{L^2} \int_0^t \ell_\delta(s)^{-\frac{5}{2}} \, ds \Big] \\ &\sim \|u_0\|_{H^s} \exp\Big[C_\delta \, \|u_0\|_{L^2} \int_0^t \ell_\delta(s)^{-\frac{5}{2}} \, ds \Big] , \end{aligned}$$

for $s \ge 0$, where we recall from (2.23) that $C_\delta = O(\delta^{-1})$.

This completes the proof of Theorem 1.2. □

3. Lower bounds on the blowup rate

In this section, we prove Theorem 1.3.

Recalling the energy bound (2.38),

$$\partial_t \|u(t)\|_{B^s_{2,2}} \le \|Du(t)\|_{L^\infty} \, \|u(t)\|_{B^s_{2,2}} , \tag{3.1}$$

we invoke the Sobolev embedding

$$\begin{aligned} \|Du\|_{L^\infty} &\le \|\widehat{Du}\|_{L^1} \\ &\le \Big(\int d\xi \, \langle \xi \rangle^{-3-2\delta} \Big)^{\frac{1}{2}} \|Du\|_{H^{\frac{3}{2}+\delta}} \\ &\le C_\delta \, \|u\|_{H^{\frac{5}{2}+\delta}} \\ &\sim C_\delta \, \|u\|_{B^{\frac{5}{2}+\delta}_{2,2}} , \end{aligned} \tag{3.2}$$

with $C_\delta = O(\delta^{-\frac{1}{2}})$, to get, for $s = \frac{5}{2} + \delta$,

$$\partial_t \|u(t)\|_{B^s_{2,2}} \leq C_\delta \left(\|u(t)\|_{B^s_{2,2}}\right)^2 . \tag{3.3}$$

Straightforward integration implies

$$-\Big(\frac{1}{\|u(t)\|_{B^s_{2,2}}} - \frac{1}{\|u(t_0)\|_{B^s_{2,2}}} \Big) \leq C_\delta (t - t_0) . \tag{3.4}$$

Hence,

$$\begin{aligned} \|u(t)\|_{H^s} &\sim \|u(t)\|_{B^s_{2,2}} \\ &\leq \frac{\|u(t_0)\|_{B^s_{2,2}}}{1 - (t-t_0) C_\delta \|u(t_0)\|_{B^s_{2,2}}} \\ &\sim \frac{\|u(t_0)\|_{H^s}}{1 - (t-t_0) C_\delta \|u(t_0)\|_{H^s}} , \end{aligned} \tag{3.5}$$

where a possible trivial modification of C_δ is implicit in passing to the last line. This implies that the solution $u(t)$ is locally well-posed in H^s, with $s = \frac{5}{2} + \delta$, for

$$t_0 \leq t < t_0 + \frac{1}{C_\delta \|u(t_0)\|_{H^s}} . \tag{3.6}$$

In particular, this infers that if T^* is the first time beyond which the solution u cannot be continued, one necessarily has that

$$T^* > t_0 + \frac{1}{C_\delta \|u(t_0)\|_{H^s}} . \tag{3.7}$$

This in turn implies an a priori lower bound on the blowup rate given by

$$\|u(t)\|_{H^s} > \frac{1}{C_\delta \, (T^* - t)} \tag{3.8}$$

for all $0 \leq t < T^*$. The lower bound on the blowup rate stated in Theorem 1.3 is stronger than this estimate, and we shall prove it in the sequel.

To begin with, we note that

$$\begin{aligned} \|\omega(t)\|_{C^\delta} &\leq C_\delta \|\omega(t)\|_{H^{\frac{3}{2}+\delta}} \\ &\leq C_\delta \|u(t)\|_{H^{\frac{5}{2}+\delta}} \\ &\leq \frac{C_\delta \|u(t_0)\|_{H^{\frac{5}{2}+\delta}}}{1 - (t-t_0) C_\delta \|u(t_0)\|_{H^{\frac{5}{2}+\delta}}} . \end{aligned} \tag{3.9}$$

That is, local well-posedness of u in $H^{\frac{5}{2}+\delta}$ implies δ-Holder continuity of the vorticity.

The parameter L in the definition (1.11) of $\ell_\delta(t)$ is arbitrary. Thus, in view of (3.9), we may now let $L \to \infty$ for convenience. Then,

$$\begin{aligned} \ell_\delta(t)^{-\frac{5}{2}} &= \Big(\frac{\|\omega(t)\|_{C^\delta}}{\|u_0\|_{L^2}} \Big)^{\frac{2}{2\delta+5} \cdot \frac{5}{2}} \\ &\leq \Big(\frac{C_\delta \, \|u(t)\|_{H^{\frac{5}{2}+\delta}}}{\|u_0\|_{L^2}} \Big)^{1-\tilde{\delta}} \\ &\leq \Big(\frac{C_\delta}{\|u_0\|_{L^2}} \Big)^{1-\tilde{\delta}} \Big(\frac{\|u(t_0)\|_{H^s}}{1 - (t-t_0) C_\delta \|u(t_0)\|_{H^s}} \Big)^{1-\tilde{\delta}} , \end{aligned} \tag{3.10}$$

where

$$\tilde{\delta} := \frac{2\delta}{5+2\delta} \text{ and } s = \frac{5}{2} + \delta \,. \tag{3.11}$$

We note that while the right hand side of (3.10) diverges as t approaches

$$t_1 := t_0 + \frac{1}{C_\delta \|u(t_0)\|_{H^s}} \,, \tag{3.12}$$

the integral

$$\begin{aligned}\int_{t_0}^{t_1} \ell_\delta(t)^{-\frac{5}{2}} dt &\leq \Big(\frac{C_\delta}{\|u_0\|_{L^2}}\Big)^{1-\tilde{\delta}} \int_{t_0}^{t_1} \Big(\frac{\|u(t_0)\|_{H^s}}{1-(t-t_0)C_\delta \|u(t_0)\|_{H^s}}\Big)^{1-\tilde{\delta}} dt \\ &=: B_0(\delta) \end{aligned} \tag{3.13}$$

converges for $\delta > 0$ ($\Leftrightarrow \widetilde{\delta} > 0$). This implies that the solution $u(t)$ for $t \in [t_0, t_1)$ can be extended to $t > t_1$.

In particular, we obtain that

$$\begin{aligned}\|u(t_1)\|_{H^{\frac{5}{2}+\delta}} &\leq \|u(t_0)\|_{H^{\frac{5}{2}+\delta}} \exp\Big(C_\delta \,\|u_0\|_{L^2} \int_{t_0}^{t_1} (\ell_\delta(t))^{-\frac{5}{2}} dt \Big) \\ &\leq \|u(t_0)\|_{H^{\frac{5}{2}+\delta}} \exp\Big(C_\delta \,\|u_0\|_{L^2} B_0(\delta) \Big)\end{aligned} \tag{3.14}$$

from Theorem 1.2.

We may now repeat the above estimates with initial data $u(t_1)$ in $H^{\frac{5}{2}+\delta}$, thus obtaining a local well-posedness interval $[t_1, t_2]$. Accordingly, we may set t_2 to be given by

$$t_2 := t_1 + \frac{1}{C_\delta \|u(t_1)\|_{H^s}} \,. \tag{3.15}$$

More generally, we define the discrete times t_j by

$$t_{j+1} := t_j + \frac{1}{C_\delta \|u(t_j)\|_{H^s}} \,. \tag{3.16}$$

We then have

$$\|u(t_{j+1})\|_{H^s} \leq \exp\Big(C_\delta \|u_0\|_{L^2} \, B_j(\delta) \Big) \|u(t_j)\|_{H^s} \,, \tag{3.17}$$

where $B_j(\delta)$ is defined by

$$\begin{aligned} & C_\delta \|u_0\|_{L^2} B_j(\delta) \\ := \; & C_\delta \|u_0\|_{L^2} \Big(\frac{C_\delta}{\|u_0\|_{L^2}}\Big)^{1-\tilde{\delta}} \int_{t_j}^{t_{j+1}} \Big(\frac{\|u(t_j)\|_{H^s}}{1-(t-t_j)C_\delta \|u(t_j)\|_{H^s}}\Big)^{1-\tilde{\delta}} dt \\ = \; & \frac{1}{\tilde{\delta}} C_\delta^{1-\tilde{\delta}} \Big(\frac{\|u_0\|_{L^2}}{\|u(t_j)\|_{H^s}}\Big)^{\tilde{\delta}} \\ =: \; & b_\delta \Big(\frac{\|u_0\|_{L^2}}{\|u(t_j)\|_{H^s}}\Big)^{\tilde{\delta}} . \end{aligned} \tag{3.18}$$

Letting

$$\rho_j := \exp\Big(b_\delta \Big(\frac{\|u_0\|_{L^2}}{\|u(t_j)\|_{H^s}}\Big)^{\tilde{\delta}} \Big) \,, \tag{3.19}$$

we have

$$\|u(t_j)\|_{H^s} \leq \rho_{j-1} \|u(t_{j-1})\|_{H^s} , \tag{3.20}$$

and we remark that $(\rho_j)_j$ satisfy the recursive estimates

$$\begin{aligned} \rho_j &\geq \exp\Big(b_\delta \Big(\frac{\|u_0\|_{L^2}}{\rho_{j-1}\|u(t_{j-1})\|_{H^s}} \Big)^{\tilde{\delta}} \Big) \\ &= (\, \rho_{j-1}\,)^{\rho_{j-1}^{-\tilde{\delta}}} \\ &= \exp\Big(\rho_{j-1}^{-\tilde{\delta}} \ln \rho_{j-1} \Big) . \end{aligned} \tag{3.21}$$

We note that from its definition, $\rho_j > 1$ for all j.

We shall now assume that $T^* > 0$ is the first time beyond which the solution $u(t)$ cannot be continued. Thus, by choosing t_0 close enough to T^*, (3.8) implies that $\|u(t_0)\|_{H^s}$ can be made sufficiently large that the following hold:

(1) The quantity

$$b_\delta \Big(\frac{\|u_0\|_{L^2}}{\|u(t_0)\|_{H^s}} \Big)^{\tilde{\delta}} \ll 1 \tag{3.22}$$

is small.

(2) There is a positive, finite constant $\widetilde{C}$ independent of j such that

$$\|u(t_j)\|_{H^s} \geq \widetilde{C} \, \|u(t_0)\|_{H^s} \tag{3.23}$$

holds for all $j \in \mathbb{N}$. Without any loss of generality (by a redefinition of the constant b_δ if necessary), we can assume that $\widetilde{C} = 1$.

Accordingly, (3.23) with $\widetilde{C} = 1$ implies that $\rho_j \leq \rho_0$ for all j. Then, for any $N \in \mathbb{N}$,

$$\begin{aligned} T^* - t_0 &\geq \sum_{j=0}^{N} (t_{j+1} - t_j) \\ &= \frac{1}{C_\delta} \Big(\frac{1}{\|u(t_0)\|_{H^s}} + \cdots + \frac{1}{\|u(t_N)\|_{H^s}} \Big) \\ &= \frac{1}{C_\delta \|u(t_0)\|_{H^s}} \Big(1 + \frac{\|u(t_0)\|_{H^s}}{\|u(t_1)\|_{H^s}} + \cdots + \frac{\|u(t_0)\|_{H^s}}{\|u(t_N)\|_{H^s}} \Big) \\ &\geq \frac{1}{C_\delta \|u(t_0)\|_{H^s}} \Big(1 + \frac{1}{\rho_0} + \cdots + \frac{1}{\rho_0 \cdots \rho_N} \Big) \\ &\geq \frac{1}{C_\delta \|u(t_0)\|_{H^s}} \Big(1 + \frac{1}{\rho_0} + \cdots + \frac{1}{\rho_0^N} \Big) \end{aligned} \tag{3.24}$$

from $\frac{1}{\rho_j} \geq \frac{1}{\rho_0}$ for all j, and the fact that $\rho_0 > 1$ since the argument in the exponent (3.19) is positive.

Then, letting $N \to \infty$, we obtain

$$\begin{aligned} \frac{1}{T^* - t_0} &\leq C_\delta \|u(t_0)\|_{H^s} \Big(1 - \frac{1}{\rho_0} \Big) \\ &= C_\delta \|u(t_0)\|_{H^s} \Big(1 - \exp \Big(- b_\delta \Big(\frac{\|u_0\|_{L^2}}{\|u(t_0)\|_{H^s}} \Big)^{\tilde{\delta}} \Big) \Big) . \end{aligned} \tag{3.25}$$

Next, we deduce a lower bound on the blowup rate.

Invoking (3.22), we obtain

$$\begin{aligned} \frac{1}{T^* - t_0} &\leq C_\delta \|u(t_0)\|_{H^s} \Big(1 - \exp \Big(- b_\delta \Big(\frac{\|u_0\|_{L^2}}{\|u(t_0)\|_{H^s}} \Big)^{\tilde{\delta}} \Big) \Big) \\ &\approx C_\delta \|u(t_0)\|_{H^s} b_\delta \Big(\frac{\|u_0\|_{L^2}}{\|u(t_0)\|_{H^s}} \Big)^{\tilde{\delta}} \\ &= C_\delta \, b_\delta \|u_0\|_{L^2}^{\tilde{\delta}} \|u(t_0)\|_{H^s}^{1-\tilde{\delta}} . \end{aligned} \tag{3.26}$$

This implies a lower bound on the blowup rate of the form

$$\begin{aligned} \|u(t_0)\|_{H^{\frac{5}{2}+\delta}} &\geq C(\delta, \|u_0\|_{L^2}) \Big(\frac{1}{T^* - t_0} \Big)^{\frac{1}{1-\tilde{\delta}}} \\ &= C(\delta, \|u_0\|_{L^2}) \Big(\frac{1}{T^* - t_0} \Big)^{\frac{2\delta+5}{5}} , \end{aligned} \tag{3.27}$$

under the condition that (3.22) and (3.23) hold.

This concludes our proof of Theorem 1.3. □

Appendix A. Proof of inequality (2.38) for $s > \frac{5}{2}$

In this Appendix, we prove (2.38) which follows from (2.36),

$$\frac{1}{2} \partial_t \|u(t)\|_{B^s_{2,2}}^2 \lesssim \|Du(t)\|_{L^\infty} \, \|u(t)\|_{B^s_{2,2}}^2 , \tag{A.1}$$

for $s > \frac{5}{2}$. We invoke Eq. (26) in the work [8] of F. Planchon, which is valid for $s > 1 + \frac{n}{2}$ in n dimensions (thus, $s > \frac{5}{2}$ in our case of $n = 3$), for parameter values $p = q = 2$ in the notation of that paper. It yields

$$\begin{aligned} \frac{1}{2} \partial_t 2^{2js} \|u_j\|_{L^2}^2 &\lesssim 2^{2js} \sum_{k \sim j} \|S_{j+1} Du\|_{L^\infty} \, \|u_k\|_{L^2} \, \|u_j\|_{L^2} \\ &\quad + 2^{2js} \sum_{j \lesssim k \sim k'} \|u_k\|_{L^2} \, \|u_{k'}\|_{L^2} \, \|Du_j\|_{L^\infty} \end{aligned} \tag{A.2}$$

where $u_k = P_k u$ is the Paley-Littlewood projection of u at scale k, and $S_j = \sum_{j' \leq j} P_{j'}$ is the Paley-Littlewood projection to scales $\leq j$.

Summing over j,

$$\begin{aligned}\frac{1}{2}\partial_t \sum_j 2^{2js}\|u_j\|_{L^2}^2 &\lesssim \sup_j \|S_{j+1}Du\|_{L^\infty}\Big(\sum_j 2^{2js}\sum_{k\sim j}\|u_k\|_{L^2}\,\|u_j\|_{L^2} \\ &\qquad + \sum_j \sum_{k\sim k'\gtrsim j} 2^{2s(j-k)}2^{ks}\,\|u_k\|_{L^2}\,2^{k's}\,\|u_{k'}\|_{L^2}\Big) \\ &\lesssim \|Du\|_{L^\infty}\Big(\sum_j 2^{2js}\|u_j\|_{L^2}^2 \\ &\qquad + \sum_k \Big(\sum_{j\lesssim k} 2^{2s(j-k)}\Big)2^{ks}\,\|u_k\|_{L^2}^2\Big) \\ &\lesssim \|Du\|_{L^\infty}\sum_j 2^{2js}\|u_j\|_{L^2}^2\,. \end{aligned} \tag{A.3}$$

To pass to the second inequality, we used that

$$\|S_{j+1}Du\|_{L^\infty} \;=\; \|m_{j+1} * Du\|_{L^\infty} \;\lesssim\; \|Du\|_{L^\infty}\,\|m_{j+1}\|_{L^1}\,, \tag{A.4}$$

where $\widehat{m_j}$ is the symbol of the Fourier multiplication operator S_j, and the fact that $\|m_j\|_{L^1}\sim 1$ uniformly in j. Accordingly, we get

$$\frac{1}{2}\partial_t\|u(t)\|_{\dot B^s_{2,2}}^2 \;\lesssim\; \|Du(t)\|_{L^\infty}\,\|u(t)\|_{\dot B^s_{2,2}}^2\,. \tag{A.5}$$

From

$$\|u(t)\|_{B^s_{2,2}}^2 \;=\; \|u(t)\|_{L^2}^2 \;+\; \|u(t)\|_{\dot B^s_{2,2}}^2\,, \tag{A.6}$$

and energy conservation, $\partial_t\|u(t)\|_{L^2}^2 = 0$, we obtain

$$\begin{aligned}\frac{1}{2}\partial_t\|u(t)\|_{B^s_{2,2}}^2 &= \frac{1}{2}\partial_t\|u(t)\|_{\dot B^s_{2,2}}^2 \\ &\lesssim \|Du(t)\|_{L^\infty}\,\|u(t)\|_{\dot B^s_{2,2}}^2 \\ &\lesssim \|Du(t)\|_{L^\infty}\,\|u(t)\|_{B^s_{2,2}}^2\,. \end{aligned}\tag{A.7}$$

This proves (A.1). □

Acknowledgments. The work of T.C. was supported by NSF grant DMS 1009448. The work of N.P. was supported NSF grant number DMS 0758247 and an Alfred P. Sloan Research Fellowship.

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