Asymptotic Analysis of the Fourier Transform of a Probability Measure with Application to Quantum Zeno Effect

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Abstract

Let μ be a probability measure on the set $\mathbb{R}$ of real numbers and $\hat{\mu}(t) := \int_{\mathbb{R}} e^{-it\lambda} d\mu(\lambda)$ $(t \in \mathbb{R})$ be the Fourier transform of μ (i is the imaginary unit). Then, under suitable conditions, asymptotic formulae of $|\hat{\mu}(t/x)|^{2x}$ in $1/x$ as $x \to \infty$ are derived. These results are applied to the so-called quantum Zeno effect to establish asymptotic formulae of its occurrence probability in the inverse of the number N of measurements made in a time interval as $N \to \infty$.

Keywords: quantum Zeno effect, Hamiltonian, probability measure, asymptotic analysis

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1 Introduction

A series of measurements on a quantum system may hinder or inhibit transitions from the initial state to other different states. If such a phenomenon occurs, then it is called quantum Zeno effect (QZE) (see, e.g., [2, 3, 4, 5, 6]). Recently Arai and Fuda [1] reconsidered QZE from mathematical physics points of view and clarified some general mathematical features of it. But, in [1], a problem was left open, which is concerned with asymptotic behaviors of the occurrence probability of QZE in $1/N$ as $N \to \infty$ with N being the number of the measurements made on a quantum system in a time interval. In this paper, we concentrate our attention on this problem and give a complete solution to it.

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To explain the problem concretely, let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ (linear in the second variable) and norm $\|\cdot\|$, and H be a self-adjoint operator on $\mathcal{H}$ with domain $D(H)$. In the context of QZE, $\mathcal{H}$ and H are respectively the Hilbert space of state vectors and the Hamiltonian of the quantum system under consideration. By an axiom of quantum mechanics, the strongly continuous one-parameter unitary group $\{e^{-itH}\}_{t\in\mathbb{R}}$ describes the time development of the quantum system[1]: if the state at time $t=t_0\in\mathbb{R}$ is a unit vector $\Psi\in\mathcal{H}$, then the state at time $t\in\mathbb{R}$ is $\Psi(t):=e^{-i(t-t_0)}\Psi$, provided that no measurement is made during the time interval $(t_0,t]$. Moreover, the probability of finding by measurement a state $\Phi\in\mathcal{H}$ with $\|\Phi\|=1$ at time t is equal to $|\langle\Phi,\Psi(t)\rangle|^2$.

Suppose that, in a time interval $[0,t]$ $(t>0)$, N measurements on the quantum system are made successively at times $t_1=t/N, t_2=2t/N,\cdots,t_j=jt/N,\cdots,t_N=t$ $(j=1,\cdots,N)$ with intial state $\Psi\in\mathcal{H}$, the state at time $t_0=0$, satisfying $\|\Psi\|=1$. Then the probability of finding the state Ψ at each time t_j $(j=1,\cdots,N)$ is given by

$$P_N(\Psi,t):=\prod_{j=1}^{N}\left|\left\langle\Psi,e^{-i(t_j-t_{j-1})H}\Psi\right\rangle\right|^2=|\left\langle\Psi,e^{-itH/N}\Psi\right\rangle|^{2N}. \tag{1.1}$$

It is proved [1, Theorem 2.1] that, if Ψ is in $D(H)$, then

$$\lim_{N\to\infty}P_N(\Psi,t)=1. \tag{1.2}$$

This corresponds to the occurrence of QZE in the present context. In this sense, we call $P_N(\Psi,t)$ the occurrence probability of QZE with respect to the initial state Ψ and the time interval $[0,t]$.

It may be interesting to investigate an asymptotic behavior of $P_N(\Psi,t)$ in $1/N$, i.e.,

$$P_N(\Psi,t)=1+c_1(\Psi,t)\frac{1}{N}+c_2(\Psi,t)\left(\frac{1}{N}\right)^2+\cdots+c_p(\Psi,t)\left(\frac{1}{N}\right)^p+o\left(\frac{1}{N^p}\right)\quad(N\to\infty), \tag{1.3}$$

with some $p\in\mathbb{N}$ (the set of natural numbers), where $c_j(\Psi,t)$ $(j=1,\cdots,p)$ are real numbers to be determined. In [1, Theorem 3.1], it is shown that (1.3) for $p=1$ holds with

$$c_1(\Psi,t)=-t^2(\Delta H)_\Psi^2, \tag{1.4}$$

where

$$(\Delta H)_\Psi:=\|(H-\langle\Psi,H\Psi\rangle)\Psi\|=\sqrt{\|H\Psi\|^2-\langle\Psi,H\Psi\rangle^2}$$

is the uncertainty of H in the state Ψ. But, to find higher order asymptotics of $P_N(\Psi,t)$ was left open. It is the goal of the present paper to derive an asymptotic formula of $P_N(\Psi,t)$ up to an arbitrary order of $1/N$.

The method used in [1], which is operator-theoretical, seems to be difficult to extend for higher order asymptotics of $P_N(\Psi,t)$ in $1/N$. This suggests that one has to seek another method. In this paper, we present a new and simple method. The idea of it is as follows.

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[1]We use the physical unit system such that $\hbar=h/2\pi$ (h is the Planck constant) is equal to 1.

We first note that the quantity $\left\langle \Psi, e^{-isH}\Psi \right\rangle$ ($s \in \mathbb{R}$) is written as follows:

$$\left\langle \Psi, e^{-isH}\Psi \right\rangle = \int_{\mathbb{R}} e^{-is\lambda} d\|E_H(\lambda)\Psi\|^2, \tag{1.5}$$

where $E_H(\cdot)$ is the spectral measure of H. The measure

$$\mu_\Psi(\cdot) := \|E_H(\cdot)\Psi\|^2 \tag{1.6}$$

on $\mathbb{R}$ is a probability measure. Putting

$$\hat{\mu}_\Psi(s) := \int_{\mathbb{R}} e^{-is\lambda} d\mu_\Psi(\lambda), \quad s \in \mathbb{R}, \tag{1.7}$$

the Fourier transform of the probability measure μ_Ψ, one has

$$\left\langle \Psi, e^{-isH}\Psi \right\rangle = \hat{\mu}_\Psi(s), \quad s \in \mathbb{R}. \tag{1.8}$$

Hence

$$P_N(\Psi, t) = |\hat{\mu}_\Psi(t/N))|^{2N}. \tag{1.9}$$

Thus the problem may be stated in a general form as follows:

Problem: Let μ be a probability measure on $\mathbb{R}$ and

$$\hat{\mu}(s) := \int_{\mathbb{R}} e^{-is\lambda} d\mu(\lambda), \quad s \in \mathbb{R}. \tag{1.10}$$

Then, for each $t \in \mathbb{R}$, find asymptotic formulae of $|\hat{\mu}(t/x)|^{2x}$ in $1/x$ as $x \to \infty$.

In our method, we first derive asymptotic formulae of $\log |\hat{\mu}(t/x)|^{2x}$ in $1/x$ as $x \to \infty$, instead of $|\hat{\mu}(t/x)|^{2x}$ itself. This is done in Section 2. Then we derive in Section 3 asymptotic formulae of $|\hat{\mu}(t/x)|^{2x}$ in $1/x$ as $x \to \infty$. In the last section we apply the results in Sections 2 and 3 to $P_N(\Psi, t)$ to obtain asymptotic formulae of $\log P_N(\Psi, t)$ and $P_N(\Psi, t)$ in $1/N$ as $N \to \infty$.

2 Asymptotic Formulae of $\log |\hat{\mu}(t/x)|^{2x}$

Let μ be a probability measure on $\mathbb{R}$. For each $k \in \mathbb{N}$, we define

$$M_k := \int_{\mathbb{R}} \lambda^k d\mu(\lambda), \tag{2.1}$$

the k-th moment of the random variable λ, provided that $\int_{\mathbb{R}} |\lambda|^k d\mu(\lambda) < \infty$. With these constants, for each $n \in \mathbb{N}$, we introduce a number a_n by

$$a_n := \sum_{r=1}^{n} \frac{(-1)^{r-1}}{r} \sum_{\substack{k_1+\cdots+k_r=n \\ k_1,\cdots,k_r \geq 1}} \frac{M_{k_1} \cdots M_{k_r}}{k_1! \cdots k_r!}, \tag{2.2}$$

provided that $\int_{\mathbb{R}} |\lambda|^n d\mu(\lambda) < \infty$.

Theorem 2.1 *Assume that, for some $c > 0$,*

$$\int_{\mathbb{R}} e^{c|\lambda|} d\mu(\lambda) < \infty. \tag{2.3}$$

Let

$$K := \left\{ y \in \mathbb{R} | \int_{\mathbb{R}} e^{|y||\lambda|} d\mu(\lambda) < 2 \right\}. \tag{2.4}$$

Then, for all $x \in \mathbb{R} \setminus \{0\}$ and $t \in \mathbb{R}$ satisfying $t/x \in K$,

$$\log |\hat{\mu}(t/x)|^{2x} = 2 \sum_{n=1}^{\infty} (-1)^n a_{2n} t^{2n} \left(\frac{1}{x} \right)^{2n-1}, \tag{2.5}$$

converging absolutely.

Remark 2.2 Under assumption (2.3), for all $k \in \mathbb{N}$, $\int_{\mathbb{R}} |\lambda|^k d\mu(\lambda) < \infty$ and there exists a constant $\varepsilon_0 > 0$ such that $(-\varepsilon_0, \varepsilon_0) \subset K$.

Remark 2.3 In the right hand side on (2.5), only even powers for t and only odd powers for $1/x$ appear. This is natural, because $\log |\hat{\mu}(t/x)|^{2x}$ is even in t and odd in $1/x$.

To prove Theorem 2.1, we first present an elementary lemma. Let

$$u(x) := \int_{\mathbb{R}} (e^{-ix\lambda} - 1) d\mu(\lambda) = \hat{\mu}(x) - 1, \quad x \in \mathbb{R}. \tag{2.6}$$

Lemma 2.4 *Assume (2.3). Then, for all $x \in K$,*

$$u(x) = \sum_{k=1}^{\infty} \frac{(-ix)^k}{k!} M_k. \tag{2.7}$$

where the right hand side is absolutely convergent.

Proof. Let $x \in K$ be fixed. Then we have $u(x) = \int_{\mathbb{R}} \lim_{N\to\infty} g_N(\lambda) d\mu(\lambda)$ with $g_N(\lambda) := \sum_{k=1}^{N} (-ix)^k \lambda^k / k!$, $\lambda \in \mathbb{R}$. It is easy to see that $|g_N(\lambda)| \le e^{|x||\lambda|}$. Since x is in K, the right hand side is integrable independent of N. Hence, by the Lebesgue dominated convergence theorem, we obtain $u(x) = \lim_{N\to\infty} \int_{\mathbb{R}} g_N(\lambda) d\mu(\lambda)$, which gives (2.7). Moreover

$$\sum_{k=1}^{\infty} \frac{|x|^k}{k!} |M_k| \le \sum_{k=1}^{\infty} \int_{\mathbb{R}} \frac{(|x||\lambda|)^k}{k!} d\mu(\lambda) = \int_{\mathbb{R}} e^{|x||\lambda|} d\mu(\lambda) - 1 < \infty.$$

Hence the infinite series on the right hand side of (2.7) is absolutely convergent. ∎

Proof of Theorem 2.1

By replacing t/x in $|\hat{\mu}(t/x)|^{2x}$ by x, we need only to consider the behavior of the function

$$F(x) := |\hat{\mu}(x)|^{2t/x} \tag{2.8}$$

as $x \downarrow 0$. Since $\hat{\mu}(x) - 1 = \int_{\mathbb{R}} (e^{-ix\lambda} - 1) d\mu(\lambda)$ and $|e^{-ix\lambda} - 1| \leq e^{|x||\lambda|} - 1$, $\forall x \in \mathbb{R}$, it follows that, for all $x \in K$,

$$|\hat{\mu}(x) - 1| < 1. \tag{2.9}$$

Hence we can define

$$f(x) := \log \hat{\mu}(x), \quad x \in K. \tag{2.10}$$

We note that $|\hat{\mu}(x)|^2 = \hat{\mu}(x)\hat{\mu}(-x)$. Hence we have

$$\log F(x) = \frac{t}{x}(f(x) + f(-x)), \quad x \in K \setminus \{0\} \tag{2.11}$$

Assumption (2.3) implies that, for all $k \in \mathbb{N}$, $\hat{\mu}$ is k times continuously differentiable on $\mathbb{R}$ with the k-th derivative equal to

$$\hat{\mu}^{(k)}(x) = (-i)^k \int_{\mathbb{R}} \lambda^k e^{-i\lambda x} d\mu(\lambda), \quad x \in \mathbb{R}. \tag{2.12}$$

In particular, we have

$$\hat{\mu}^{(k)}(0) = (-i)^k M_k. \tag{2.13}$$

Hence f also is infinitely differentiable on K.

With u defined by (2.6), we can write

$$f(x) = \log(1 + u(x)).$$

By (2.9), for all $x \in K$, $|u(x)| < 1$. Hence we have

$$f(x) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} u(x)^r, \quad x \in K,$$

where the infinite series is absolutely convergent. By Lemma 2.4, we have for all $x \in K$

$$u(x)^r = \sum_{k_1,\cdots,k_r=1}^{\infty} \frac{(-ix)^{k_1+\cdots+k_r}}{k_1!\cdots k_r!} M_{k_1}\cdots M_{k_r} = \sum_{n=r}^{\infty} (-ix)^n \sum_{\substack{k_1+\cdots+k_r=n \\ k_1,\cdots,k_r \geq 1}} \frac{M_{k_1}\cdots M_{k_r}}{k_1!\cdots k_r!}.$$

Hence, for all $x \in K$

$$f(x) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} \sum_{n=r}^{\infty} (-ix)^n \sum_{\substack{k_1+\cdots+k_r=n \\ k_1,\cdots,k_r \geq 1}} \frac{M_{k_1}\cdots M_{k_r}}{k_1!\cdots k_r!}. \tag{2.14}$$

It is easy to see that, for all $x \in K$,

$$\sum_{r=1}^{\infty} \frac{1}{r} \sum_{n=r}^{\infty} |x|^n \sum_{\substack{k_1+\cdots+k_r=n \\ k_1,\cdots,k_r \geq 1}} \frac{|M_{k_1}|\cdots|M_{k_r}|}{k_1!\cdots k_r!}.$$

conveges. Hence, in (2.14), we can interchange the sums on r and n to obtain

$$f(x) = \sum_{n=1}^{\infty} (-i)^n a_n x^n \tag{2.15}$$

where a_n is given by (2.2). Therefore

$$\log F(x) = 2t \sum_{n=1}^{\infty} (-1)^n a_{2n} x^{2n-1}, \quad x \in K, \tag{2.16}$$

converging absolutely. Replacing x by t/x, we obtain (2.5). ∎

We next consider the case where (2.3) does not necessarily hold. In this case, we have the following result:

Theorem 2.5 *Let $n \in \mathbb{N}$ and suppose that*

$$\int_{\mathbb{R}} |\lambda|^n d\mu(\lambda) < \infty. \tag{2.17}$$

Let

$$p_n := \begin{cases} \dfrac{n}{2} & \text{for } n \geq 2 \text{ even} \\ \dfrac{n-1}{2} & \text{for } n \geq 2 \text{ odd} \end{cases} \tag{2.18}$$

Then

$$\log |\hat{\mu}(t/x)|^{2x} = 2 \sum_{k=1}^{p_n} (-1)^k a_{2k} t^{2k} \left(\frac{1}{x}\right)^{2k-1} + o\left(\frac{1}{x^{2p_n-1}}\right) \quad (x \to \infty). \tag{2.19}$$

Proof. Since $\hat{\mu}(0) = 1$ and $\hat{\mu}$ is continuous on $\mathbb{R}$, there exists a constant $\delta > 0$ such that, for all $x \in I_\delta := (-\delta, \delta)$, inequality (2.9) holds. Hence we can define $g : I_\delta \to \mathbb{R}$ by

$$g(x) := \log \hat{\mu}(x), \quad x \in I_\delta.$$

Then we have

$$F(x) = \frac{t}{x}(g(x) + g(-x)), \quad x \in I_\delta \setminus \{0\}. \tag{2.20}$$

Under the present assumption, $\hat{\mu}$ is n times continuously differentiable on $\mathbb{R}$. Hence so is g on I_δ with derivative g' satisfying

$$g'\hat{\mu} = \hat{\mu}'. \tag{2.21}$$

By Taylor's theorem, we have

$$g(x) = \sum_{k=1}^{n} \frac{g^{(k)}(0)}{k!} x^k + o(x^n) \quad (x \to 0).$$

Differentiating the both sides of (2.21) $(k-1)$ times and applying the Leibniz formula, we obtain the following recursion relation on $g^{(j)}(0)$:

$$g'(0) = -iM_1, \quad g^{(k)}(0) = (-i)^k \left(M_k - \sum_{j=1}^{k-1} {}_{k-1}C_{j-1} i^j M_{k-j} g^{(j)}(0) \right) \quad (k = 2, \cdots, n), \tag{2.22}$$

where ${}_mC_l := m!/[(m-l)!l!]$ $(m, l \in \{0\} \cup \mathbb{N}, m \geq l)$.

It is obvious that the function f in the proof of Theorem 2.1 also satisfies (2.21) with g replaced by f. Hence (2.22) holds with g replaced by f. Therefore $g^{(k)}(0) = f^{(k)}(0), k = 1, \cdots, n$. From the proof of Theorem 3.2, we see that $f^{(k)}(0) = (-i)^k a_k k!$. Hence $g^{(k)}(0) = (-i)^k a_k k!$. Thus

$$g(x) = \sum_{k=1}^{n} (-i)^k a_k x^k + o(x^n) \quad (x \to 0),$$

which implies that

$$F(x) = 2t \sum_{k=1}^{p_n} (-1)^k a_{2k} x^{2k-1} + o(x^{2p_n - 1}).$$

Thus (2.19) holds. ∎

3 Asymptotic Formulae of $|\hat{\mu}(t/x)|^{2x}$

To derive from (2.5) an asymptotic formula of $|\mu(t/x)|^{2x}$ itself in $1/x$, we need only to note an elementary fact:

Lemma 3.1 *Let $\{c_m\}_{m=1}^{\infty}$ be a sequence of complex numbers such that the infinite series $S := \sum_{m=1}^{\infty} c_m$ converges absolutely. Let*

$$\gamma_n := \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{m_1 + \cdots + m_k = n \\ m_1, \cdots, m_k \geq 1}} c_{m_1} \cdots c_{m_k}. \tag{3.1}$$

Then

$$e^S = 1 + \sum_{n=1}^{\infty} \gamma_n, \tag{3.2}$$

converging absolutely.

Proof. An easy exercise. ∎

For each $t \in \mathbb{R}$, we define a sequence $\{\alpha_n(t)\}_{n=1}^{\infty}$ as follows:

$$\alpha_{2n-1}(t) := 2(-1)^n a_{2n} t^{2n}, \quad \alpha_{2n}(t) := 0. \tag{3.3}$$

Theorem 3.2 *Assume (2.3) and let*

$$A_n(t) := \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{m_1+\cdots+m_k=n \\ m_1,\cdots,m_k \geq 1}} \alpha_{m_1}(t)\cdots\alpha_{m_k}(t), \quad n \in \mathbb{N}. \tag{3.4}$$

Then, for all $x \in \mathbb{R}\setminus\{0\}$ and $t \in \mathbb{R}$ satisfying $t/x \in K$,

$$|\hat{\mu}(t/x)|^{2x} = 1 + \sum_{n=1}^{\infty} A_n(t)\left(\frac{1}{x}\right)^n, \tag{3.5}$$

converging absolutely.

Proof. By Theorem 2.1, we have

$$|\hat{\mu}(t/x)|^{2x} = \exp\left(\sum_{m=1}^{\infty} \alpha_m(t) x^{-m}\right).$$

Hence, by Lemma 3.1, we obtain (3.5). ∎

A finite sum version of Lemma 3.1 is given as follows, which also is easy to prove:

Lemma 3.3 *Let c_m, $m = 1,\cdots,p$, be complex numbers, $p \in \mathbb{N}$, and*

$$S_p := \sum_{m=1}^{p} c_m x^m + o(x^p) \quad (x \to 0).$$

Then

$$e^{S_p} = 1 + \sum_{n=1}^{p} \gamma_n x^n + o(x^p) \quad (x \to 0), \tag{3.6}$$

where γ_n is defined by (3.1).

Theorem 3.4 *Assume (2.17). Then, for all $t \in \mathbb{R}$,*

$$|\hat{\mu}(t/x)|^{2x} = 1 + \sum_{n=1}^{2p_n-1} A_n(t)\left(\frac{1}{x}\right)^n + o\left(\frac{1}{x^{2p_n-1}}\right) \quad (x \to \infty). \tag{3.7}$$

Proof. Similar to the proof of Theorem 3.2. ∎

4 Applications to QZE

To apply the results in Sections 2 and 3 to QZE, for each $k \in \mathbb{N}$ and a unit vector $\Psi \in D(|H|^{k/2})$, we introduce

$$\langle H^k \rangle := \int_{\mathbb{R}} \lambda^k d\|E_H(\lambda)\Psi\|^2, \tag{4.1}$$

the k-th expectation value of the Hamiltonian H in the state Ψ, and, for each $n \in \mathbb{N}$ and a unit vector $\Psi \in D(|H|^{n/2})$, we define

$$b_n(\Psi) := \sum_{r=1}^{n} \frac{(-1)^{r-1}}{r} \sum_{\substack{k_1+\cdots+k_r=n \\ k_1,\cdots,k_r \geq 1}} \frac{\langle H^{k_1}\rangle \cdots \langle H^{k_r}\rangle}{k_1! \cdots k_r!}. \tag{4.2}$$

Theorem 4.1 *Let $t \in \mathbb{R}$ be fixed. Suppose that, for some $c > 0$, $\Psi \in D(e^{c|H|})$ with $\|\Psi\| = 1$ and that N obeys the following condition:*

$$\int_{\mathbb{R}} e^{|t||\lambda|/N} d\|E_H(\lambda)\Psi\|^2 < 2. \tag{4.3}$$

Then

$$\log P_N(\Psi, t) = 2\sum_{k=1}^{\infty}(-1)^k b_{2k}(\Psi) t^{2k} \left(\frac{1}{N}\right)^{2k-1}, \tag{4.4}$$

conveging absolutely.

Proof. Let μ_Ψ be given by (1.6). Then we need only to show that $\mu = \mu_\Psi$ satisfies the assumption of Theorem 2.1. The assumption $\Psi \in D(e^{c|H|})$ is equivalent to that

$$\int_{\mathbb{R}} e^{2c|\lambda|} d\mu_\Psi(\lambda) < \infty.$$

Hence (2.3) holds with $\mu = \mu_\Psi$. In the present case, we have $M_k = \langle H^k \rangle$. Thus (2.5) gives (4.4). ∎

In the case where Ψ is not necessarily in $D(e^{c|H|})$, we have the following result:

Theorem 4.2 *Let $n \in \mathbb{N}$ and suppose that $\Psi \in D(|H|^n)$ with $\|\Psi\| = 1$. Then, for all $t \in \mathbb{R}$,*

$$\log P_N(\Psi, t) = 2\sum_{k=1}^{p_n}(-1)^k b_{2k}(\Psi) t^{2k} \left(\frac{1}{N}\right)^{2k-1} + o(1/N^{2p_n-1}) \quad (N \to \infty). \tag{4.5}$$

Proof. A simple application of Theorem 2.5. ∎

Finally we derive asymptotic formulae of $P_N(\Psi, t)$ itself. For this purpose, we define a sequence $\{\beta_n(\Psi, t)\}_{n=1}^{\infty}$ $(t \in \mathbb{R})$ as follows:

$$\beta_{2n-1}(\Psi, t) := 2(-1)^n b_{2n}(\Psi) t^{2n}, \tag{4.6}$$

$$\beta_{2n}(\Psi, t) := 0. \tag{4.7}$$

Theorem 4.3 *Suppose that the same assumption as in Theorem 4.1 holds. Let*

$$\gamma_n(\Psi, t) := \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{m_1+\cdots+m_k=n \\ m_1,\cdots,m_k \geq 1}} \beta_{m_1}(\Psi, t) \cdots \beta_{m_k}(\Psi, t), \quad n \in \mathbb{N}. \tag{4.8}$$

Then

$$P_N(\Psi, t) = 1 + \sum_{n=1}^{\infty} \gamma_n(\Psi, t) \left(\frac{1}{N}\right)^n, \tag{4.9}$$

converging absolutely.

Proof. A simple application of Theorem 3.2. ∎

Theorem 4.4 *Let $n \in \mathbb{N}$ and suppose that $\Psi \in D(|H|^n)$ with $\|\Psi\| = 1$. Then, for all $t \in \mathbb{R}$,*

$$P_N(\Psi, t) = 1 + \sum_{n=1}^{2p_n - 1} \gamma_n(\Psi, t) \left(\frac{1}{N}\right)^n + o\left(\frac{1}{N^{2p_n - 1}}\right) \quad (N \to \infty). \qquad (4.10)$$

Proof. This follows from an application of Theorem 3.4. ∎

Example 4.5 By direct computations, we have

$$\gamma_1(\Psi, t) = -(\Delta H)_\Psi t^2,$$

which coincides with $c_1(\Psi, t)$ is given by (1.4), and

$$\gamma_2(\Psi, t) = \frac{1}{2}(\Delta H)_\Psi^4 t^4.$$

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