Completing Canonical Quantization, and Its Role in Nontrivial Scalar Field Quantization*

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Abstract

The process of canonical quantization is redefined so that the classical and quantum theories coexist when $\hbar > 0$, just as they do in the real world. This analysis not only supports conventional procedures, it also reveals new quantization procedures that, among several examples, permit nontrivial quantization of scalar field models such as ϕ^4_n, for every spacetime dimension $n \geq 2$.

Conventional Quantization

Conventional canonical quantization

The standard recipe for canonical quantization is simply stated. For a single degree of freedom, one version reads:

Classical Theory: Choose canonically conjugate, classical phase space coordinates p and q, along with a classical Hamiltonian $H_c(p, q)$, and adopt

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dynamical equations, i.e., Hamilton's equations of motion, that follow from the stationary variation of a classical (C) action functional given by

$$A_C \equiv \int_0^T [p(t)\,\dot{q}(t) - H_c(p(t), q(t))]\,dt\ . \tag{1}$$

Quantum Theory: Promote the phase space coordinates to Hermitian operators, $p \to P$ and $q \to Q$, that satisfy Heisenberg's commutation relation $[Q, P] = i\hbar\mathbb{1}$. Choose a Hermitian Hamiltonian operator $\mathcal{H}(P, Q)$, with dynamical equations, i.e., Schrödinger's equation and its adjoint, that follow from the stationary variation of a quantum (Q) action functional given by

$$A_Q \equiv \int_0^T \langle\psi(t)|\,[i\hbar\partial/\partial t - \mathcal{H}(P, Q)]\,|\psi(t)\rangle\,dt\ , \tag{2}$$

where $|\psi\rangle$—and its adjoint $\langle\psi|$—denote vectors in a complex Hilbert space.

Comments: Clearly, the classical and quantum theories have several fundamental differences: For a single degree of freedom, classical phase space is two dimensional, while Hilbert space in the quantum theory is infinite dimensional. In the classical theory, one chooses $\hbar = 0$, while in the quantum theory $\hbar \simeq 10^{-27} erg\cdot sec$ as determined by experimental measurement. To relate the classical and quantum models, it is traditional to choose $\mathcal{H}(P, Q) = H_c(P, Q)$, modulo possible $\mathcal{O}(\hbar)$ corrections. While covariance under canonical coordinate transformations leads to many possible choices of coordinates, quantization results are generally better [1] if the original coordinates p and q are chosen as "Cartesian coordinates"—despite the fact that classical phase space is *not* endowed with a metric structure that would permit the identification of such coordinates.

Although canonical quantization as sketched above is highly successful, there are certain cases where the so-defined quantum theory is less than satisfactory. We claim that the triviality of ϕ^4 scalar field models in high enough spacetime dimensions are such cases. We aim to overcome that triviality.

Enhanced canonical quantization

In the real world $\hbar > 0$, and so there must be a formulation of the classical theory that accepts that fact [2]. The action functional for the quantum theory assumes that general variations of the Hilbert space vectors are possible. But suppose that is not the case, and we are able to vary only certain Hilbert space vectors that can be varied *without* disturbing the system. One such variation involves translating the system to a new position, but according to Galilean covariance we can move the observer a corresponding amount instead of moving the system. Likewise we can imagine putting the system into uniform motion with a constant velocity, but again Galilean covariance asserts we can get the same result by putting the observer into uniform motion instead of the system. Thus if we assume that some normalized reference state $|\eta\rangle$ is relevant for our problem, e.g., under appropriate conditions, the ground state for our system, then we can imagine that we can vary the set of states denoted by

$$|p,q\rangle \equiv e^{-iqP/\hbar}\, e^{ipQ/\hbar}|\eta\rangle \;, \tag{3}$$

a set of states recognized as *canonical coherent states* [3], which offer translation of $|\eta\rangle$ to a new position by q as well as to a new velocity as represented by a new momentum p. Here the operators P and Q, which (for the present) we assume to be irreducible, obey the usual commutation relation, and, moreover, are chosen as *self adjoint*—a stronger condition than being simply Hermitian—which is necessary to generate unitary transformations that preserve the normalization of $|\eta\rangle$. It is convenient to choose $|\eta\rangle$ such that $\langle\eta|P|\eta\rangle = \langle\eta|Q|\eta\rangle = 0$, called "physically centered", in which case $\langle p,q|P|p,q\rangle = p$ and $\langle p,q|Q|p,q\rangle = q$, two equations that determine the physical meaning of p and q. Armed with this set of states, we declare as a classical observer that we can only vary p and q in the limited set of states $\{|p,q\rangle\}$, and thus we are led to a restricted (R) quantum action functional given by

$$\begin{aligned} A_{Q(R)} &\equiv \textstyle\int_0^T \langle p(t),q(t)|\,[i\hbar\partial/\partial t - \mathcal{H}(P,Q)]\,|p(t),q(t)\rangle\, dt \\ &= \textstyle\int_0^T [p(t)\,\dot{q}(t) - H(p(t),q(t)]\, dt \;. \end{aligned} \tag{4}$$

Here we have introduced the important relation that

$$\begin{aligned} H(p,q) &\equiv \langle p,q|\,\mathcal{H}(P,Q)\,|p,q\rangle \\ &= \langle\eta|\,\mathcal{H}(P+p,Q+q)\,|\eta\rangle \\ &= \mathcal{H}(p,q) + \mathcal{O}(\hbar;p,q) \;. \end{aligned} \tag{5}$$

Observe that the restricted quantum action functional $A_{Q(R)}$ resembles the classical action functional and differs from it only by the fact that $\hbar > 0$. The "true" classical Hamiltonian may be obtained as

$$H_c(p,q) \equiv \lim_{\hbar\to 0} H(p,q) \;, \tag{6}$$

but that limit may change the character of $H(p,q)$ in unphysical ways.

In fact, the relation (5) has *exactly* the feature characterized by the choice of p and q as "Cartesian coordinates". And indeed, we can now show that these are Cartesian coordinates after all. The Hilbert space norm generates a metric for vectors determined by $d(|\psi\rangle,|\phi\rangle)^2 = \|\,|\psi\rangle - |\phi\rangle\,\|^2$. However, in quantum theory the overall phase of a vector carries no physics and we can instead consider the ray (R) metric deetermined by $d_R(|\psi\rangle,|\phi\rangle)^2 = \min_\alpha \|\,|\psi\rangle - e^{i\alpha}|\phi\rangle\,\|^2$. If we apply a rescaled version of the ray metric to two coherent states that are infinitesimally close to each other, we obtain (a.k.a., the Fubini-Study metric)

$$\begin{aligned} d\sigma_R(p,q)^2 &\equiv (2\hbar)\,[\,\|\,d|p,q\rangle\,\|^2 - |\,\langle p,q|\,d|p,q\rangle\,|^2] \\ &= (2/\hbar)\,[\,\langle Q^2\rangle\,dp^2 + \langle PQ + QP\rangle dp\,dq + \langle P^2\rangle\,dq^2\,]\,, \end{aligned} \tag{7}$$

where here $\langle(\cdot)\rangle \equiv \langle\eta|(\cdot)|\eta\rangle$. For a general choice of $|\eta\rangle$, the two-dimensional space $\{p,q\}$ is always flat, and up to a linear coordinate transformation, this metric involves Cartesian coordinates. Moreover, for the common choice where $(\omega Q + iP)\,|\eta\rangle = 0$, i.e., an oscillator ground state, then

$$d\sigma_R(p,q)^2 = \omega^{-1}\,dp^2 + \omega\,dq^2\,, \tag{8}$$

and p and q are indeed Cartesian coordinates according to the Hilbert space metric. Of course, we can now assign that metric to the classical phase space if so desired. It is in this sense that the coordinates p and q are Cartesian.

Conventional canonical quantization is confirmed! *It is important to appreciate that what has been shown so far is equivalent to the standard canonical quantization procedure!* Specifically, we have identified phase-space coordinates p and q that are indeed Cartesian coordinates and the quantum Hamiltonian operator is indeed the same function of the variables as is the classical Hamiltonian, modulo terms of order $\hbar$.

However, there is more to the story.

Reducible canonical operators

Let us consider the example with a classical Hamiltonian given by

$$H_c(\overrightarrow{p},\overrightarrow{q}) = \tfrac{1}{2}(\,\overrightarrow{p}^{\,2} + m_0^2\,\overrightarrow{q}^{\,2}) + \lambda(\overrightarrow{q}^{\,2})^2\,, \tag{9}$$

where $\overrightarrow{p} = \{p_1, p_2, \ldots, p_N\}$, and $\overrightarrow{q} = \{q_1, q_2, \ldots, q_N\}$, with $N \le \infty$. Here, $\overrightarrow{p}^{\,2} \equiv \Sigma_{n=1}^N p_n^2$, $\overrightarrow{q}^{\,2} \equiv \Sigma_{n=1}^N q_n^2$, and for $N = \infty$ we require that $\overrightarrow{p}^{\,2} + \overrightarrow{q}^{\,2} < \infty$.

It is clear that this model is invariant under orthogonal rotations $\overrightarrow{p} \to O\overrightarrow{p}$, $\overrightarrow{q} \to O\overrightarrow{q}$, where $O \in \mathbf{O}(N, \mathbb{R})$, and as a consequence, every solution is equivalent to a solution for $N = 1$ if $\overrightarrow{p} \propto \overrightarrow{q}$ at time $t = 0$, or to a solution for $N = 2$ otherwise. Moreover, solutions for $N = \infty$ may be derived from those for $N < \infty$ by the limit $N \to \infty$, provided we maintain $\overrightarrow{p}^2 + \overrightarrow{q}^2 < \infty$.

A conventional canonical quantization begins with $\overrightarrow{p} \to \overrightarrow{P}$, $\overrightarrow{q} \to \overrightarrow{Q}$, which are irreducible operators that obey $[Q_l, P_n] = i\hbar\delta_{l.n}\mathbb{1}$ as the only non-vanishing commutation relation. For the free model, with a mass m and $\lambda = 0$, the quantum Hamiltonian $\mathcal{H}_0 = \frac{1}{2} : (\overrightarrow{P}^2 + m^2\overrightarrow{Q}^2) :$, where $: (\cdot) :$ denotes normal ordering, has the feature that the Hamiltonian operator for $N = \infty$ is obtained as the limit of those for which $N < \infty$. Moreover, with the ground state $|0\rangle$ of the Hamiltonian operator chosen as the fiducial vector for canonical coherent states, namely,

$$|\overrightarrow{p}, \overrightarrow{q}\rangle = \exp[-i\overrightarrow{q} \cdot \overrightarrow{P}/\hbar] \exp[i\overrightarrow{p} \cdot \overrightarrow{Q}/\hbar] |0\rangle \,, \tag{10}$$

it follows that

$$\langle\overrightarrow{p}, \overrightarrow{q}| \tfrac{1}{2} : (\overrightarrow{P}^2 + m^2\overrightarrow{Q}^2) : |\overrightarrow{p}, \overrightarrow{q}\rangle = \tfrac{1}{2}(\overrightarrow{p}^2 + m^2\overrightarrow{q}^2) = H_0\overrightarrow{p}, \overrightarrow{q}) \tag{11}$$

as desired.

However, canonical quantization of the interacting models with $\lambda > 0$ leads to trivial results for $N = \infty$. To show this we assume that the Schrödinger representation of the ground state of an interacting model is real, unique, and rotationally invariant. As a consequence, the characteristic function (i.e., the Fourier transform) of the ground-state distribution has the form (note: $|f|^2 \equiv \Sigma_{n=1}^N f_n^2$, and $r^2 \equiv \Sigma_{n=1}^N x_n^2$)

$$\begin{aligned} C_N(\overrightarrow{f}) &= \int e^{i\Sigma_{n=1}^N f_n x_n/\hbar}\, \Psi_0(r)^2\, \Pi_{n=1}^N dx_n \\ &= \int e^{i|f|r\,\cos(\theta)/\hbar}\, \Psi_0(r)^2\, r^{N-1}\, dr\, \sin(\theta)^{N-2}\, d\theta\, d\Omega_{N-3} \\ &\simeq M' \int e^{-|f|^2 r^2/2(N-2)\hbar^2}\, \Psi_0(r)^2\, r^{N-1}\, dr\, d\Omega_{N-3} \\ &\to \int_0^\infty e^{-b|f|^2/\hbar}\, w(b)\, db \end{aligned} \tag{12}$$

assuming convergence, where a steepest descent integral has been performed for θ, and in the last line we have taken the limit $N \to \infty$. Additionally, $w(b) \geq 0$, and $\int_0^\infty w(b)\, db = 1$. This is the result based on symmetry.

Uniqueness of the ground state then ensures that $w(b) = \delta(b - 1/4\widetilde{m})$, for some $\widetilde{m} > 0$, implying that the quantum theory is that of a free theory, i.e., *the quantum theory is trivial!*

The way around this unsatisfactory result is to let the representations of $\overrightarrow{P}$ and $\overrightarrow{Q}$ be *reducible*. The weak correspondence principle, namely $H(\overrightarrow{p}, \overrightarrow{q}) \equiv \langle \overrightarrow{p}, \overrightarrow{q} | \mathcal{H} | \overrightarrow{p}, \overrightarrow{q} \rangle$, ensures that the enhanced classical Hamiltonian depends only on the proper variables. A detailed study [4] of the proper reducible representation, still in accord with the argument above that limits the ground-state functional form to a Gaussian, leads to the following formulation. Let $\overrightarrow{R}$ and $\overrightarrow{S}$ represent a new set of operators, independent of the former operators, and which obey the commutation relation $[S_l, R_n] = \imath\hbar\delta_{l,n}\mathbb{1}$. We introduce two Hamiltonian operators:

$$\begin{aligned} \mathcal{H}_{0\,PQ} &\equiv \tfrac{1}{2} : (\overrightarrow{P}^2 + m^2(\overrightarrow{Q} + \zeta \overrightarrow{S})^2) : \\ \mathcal{H}_{0\,RS} &\equiv \tfrac{1}{2} : (\overrightarrow{R}^2 + m^2(\overrightarrow{S} + \zeta \overrightarrow{Q})^2) : , \end{aligned} \tag{13}$$

where $0 < \zeta < 1$. These two operators have a common, unique, Gaussian ground state $|0, 0; \zeta\rangle$. Let new coherent states, which span the Hilbert space of interest, be defined with this ground state as the fiducial vector, as given by

$$|\overrightarrow{p}, \overrightarrow{q}\rangle \equiv \exp[-i\overrightarrow{q} \cdot \overrightarrow{P}/\hbar] \exp[i\overrightarrow{p} \cdot \overrightarrow{Q}/\hbar] |0, 0; \zeta\rangle \ , \tag{14}$$

and it follows that

$$\begin{aligned} &\langle \overrightarrow{p}, \overrightarrow{q} | \mathcal{H}_{0\,PQ} + \mathcal{H}_{0\,RS} + 4v : \mathcal{H}_{0\,RS}^2 : | \overrightarrow{p}, \overrightarrow{q} \rangle \\ &\quad = \tfrac{1}{2}[\overrightarrow{p}^2 + m^2(1 + \zeta^2)\overrightarrow{q}^2] + v\zeta^4 m^4 (\overrightarrow{q}^2)^2 \\ &\quad \equiv \tfrac{1}{2}(\overrightarrow{p}^2 + m_0^2 \overrightarrow{q}^2) + \lambda(\overrightarrow{q}^2)^2 \end{aligned} \tag{15}$$

as required. This example shows that enhanced quantization techniques that make use of reducible kinematical operator representations can lead to a nontrivial and fully satisfactory solution to certain problems.

The next section illustrates yet another procedure that serves to generalize canonical quantization.

Enhanced affine quantization

Affine variables and their algebra

We return to the study of a single degree of freedom. Importantly, the canonical operators P and Q, which have the whole real line for their spectrum and satisfy the Heisenberg commutation rule $[Q,P]=i\hbar\mathbb{1}$, imply a second commutation relation as well. If we multiply the Heisenberg commutator by Q we find $i\hbar Q=[Q,P]Q=[Q,PQ]$, and finally the Lie algebra

$$[Q,D]=i\hbar Q\ , \qquad D\equiv\tfrac{1}{2}(PQ+QP)\ . \tag{16}$$

The variables D and Q are called *affine coordinates* and the commutation relation (16) is called an *affine commutation relation*. Clearly D has the dimensions of $\hbar$, and we will find it convenient to choose Q as dimensionless. If the representation for P and Q was *irreducible*, the representation for D and Q is *reducible*. The irreducible sub-representations of D and Q are one where $Q>0$ and a similar one where $Q<0$; a third representation with $Q=0$ is less important. Let us focus on the irreducible representation where $Q>0$.

If $Q>0$ is a self-adjoint operator, then it follows that P, although Hermitian, can never be self adjoint. Thus P cannot serve as the generator of unitary transformations, and so canonical coherent states do not exist in this case. However, even if P is not self adjoint, it follows that D can be made self adjoint and so we can choose a different algebra made from the affine variables. Again we choose a normalized fiducial vector $|\eta\rangle$ and introduce the set of *affine coherent states* [3], which are defined by

$$|p,q\rangle\equiv e^{ipQ/\hbar}\,e^{-i\ln(q)\,D/\hbar}|\eta\rangle\ , \tag{17}$$

for all $p,q\in\mathbb{R}\times\mathbb{R}^+$, i.e., $q>0$. Whereas a self-adjoint P serves to *translate* Q, e.g., $e^{iqP/\hbar}Q\,e^{-iqP/\hbar}=Q+q\mathbb{1}$, it follows that a self-adjoint D serves to *dilate* Q, e.g., $e^{i\ln(q)D/\hbar}Q\,e^{-i\ln(q)D/\hbar}=qQ$; as already partially noted, it is useful to treat q and Q as dimensionless. If we choose $|\eta\rangle$ so that $[\widetilde{\beta}(Q-1)+i\,D]\,|\eta\rangle=0$—a rough analog of $(\omega Q+iP)\,|\eta\rangle=0$ for the Heisenberg algebra—it follows that $\langle\eta|Q|\eta\rangle=1$ and $\langle\eta|D|\eta\rangle=0$. Moreover, $\langle p,q|Q|p,q\rangle=q$ as well as $\langle p,q|D|p,q\rangle=p\,q$.

It is important to consider reducible affine operator representations as well. In this case we introduce a fiducial vector $|\eta\rangle=|\eta_+\rangle\oplus|\eta_-\rangle$, where $\langle\eta_\pm|Q|\eta_\pm\rangle=\pm1$. We introduce reducible affine coherent states given by

$$|p,q\pm\rangle\equiv|p,q_+\rangle\oplus|p,q_-\rangle\ , \tag{18}$$

where $p\in\mathbb{R}$, $\pm\,q_\pm>0$, $\pm\,Q_\pm>0$, and

$$|p,q_\pm\rangle\equiv e^{ipQ_\pm/\hbar}\,e^{-i\ln(|q|)\,D/\hbar}|\eta_\pm\rangle\ . \tag{19}$$

In particular, the coherent state overlap function is

$$\begin{aligned}&\langle p', q' \pm |p, q\pm\rangle \\ &\qquad = \theta(q'q)\,[\tfrac{1}{2}(\sqrt{q'/q} + \sqrt{q/q'}\,) + i\tfrac{1}{2}\sqrt{q'q}\,(p'-p)/\widetilde{\beta}]^{-2\widetilde{\beta}/\hbar}\;, \end{aligned} \tag{20}$$

where $\theta(y) \equiv 1$ if $y > 0$ and $\theta(y) \equiv 0$ if $y < 0$.

Affine quantization as canonical quantization

Given a quantum action functional, once again we assume that we can only vary a subset of Hilbert space vectors, in particular, the irreducible affine coherent states $\{|p, q\rangle\}$ or the reducible affine coherent states $\{|p, q\pm\rangle\}$. This leads to two versions of the restricted quantum action functional

$$\begin{aligned} A_{Q(R)} &= \textstyle\int_0^T \langle p(t), q(t)|\,[i\hbar\partial/t - \mathcal{H}'(D,Q)]\,|p(t), q(t)\rangle\,dt \\ &= \textstyle\int_o^T [-q(t)\dot{p}(t) - H(p(t), q(t))]\,dt\;, \end{aligned} \tag{21}$$

in which $q(t) > 0$, as well as

$$\begin{aligned} A_{Q(R\pm)} &= \textstyle\int_0^T \langle p(t), q(t) \pm |\,[i\hbar\partial/\partial t - \mathcal{H}'(D,Q)]\,|p(t), q(t)\pm\rangle\,dt \\ &= \textstyle\int_0^T [-q(t)\dot{p}(t) - H(p(t), q(t))]\,dt\;, \end{aligned} \tag{22}$$

where now $|q(t)| > 0$. Note that this latter case is especially useful if the Hamiltonian has a singularity at $q = 0$. *However, the important point is:* **The restricted quantum action functional, based on affine coherent states, is again identical to the form of a canonical classical system, enhanced, because in these equations, $\hbar > 0$! Stated otherwise, enhanced affine quantization effectively serves as enhanced canonical quantization.**

To complete the story we observe that

$$\begin{aligned} H(p,q) &\equiv \langle p, q \pm |\mathcal{H}'(D,Q)\,|p, q\pm\rangle \\ &= \langle \mathcal{H}'(D + p|q|Q, |q|Q)\rangle = \mathcal{H}'(pq, q) + \mathcal{O}(\hbar; p, q)\;, \end{aligned} \tag{23}$$

as well as

$$\begin{aligned} H(p,q) &\equiv \langle p, q \pm |\,\mathcal{H}(P,Q)\,|p, q\pm\rangle \\ &= \langle\,\mathcal{H}(P/|q| + p, |q|Q)\,\rangle = \mathcal{H}(p,q) + \mathcal{O}(\hbar; p, q)\;. \end{aligned} \tag{24}$$

We note without proof that the Fubini-Study metric is *not* flat for the affine coherent states, but describes a space with a constant negative curvature [5].

A simple example

A simple example serves to illustrates the power of enhanced affine quantization. As the classical Hamiltonian, we choose $H_c(p,q) = p^2/2m - e^2/|q|$, which we call a one-dimensional 'hydrogen atom' problem. The classical solutions fall into the singularity in a finite time. For the Hamiltonian operator we choose $\mathcal{H} = P^2/2m - e^2/|Q|$, and thus the enhanced affine classical Hamiltonian is given by

$$H(p,q) = p^2/2m - C/|q| + C'/q^2 \ , \tag{25}$$

where $C \equiv e^2\langle |Q|^{-1}\rangle \propto e^2$ and $C' \equiv \langle P^2\rangle/2m \propto \hbar^2/m$. Observe that $C' \simeq \hbar^2/(me^2)C$, *a ratio that is seen to be the Bohr radius!*. Thus we see that the enhanced affine classical Hamiltonian prevents all singularities and has a stable minimum at a distance from the singularity of approximately the Bohr radius. Since $\hbar > 0$ in the real world, we are led to claim that the enhanced classical Hamiltonian is 'more physical' than the true classical Hamiltonian with which we started, for which $\hbar = 0$. It would be hard to achieve the same features from an enhanced canonical quantization.

It is important to emphasize that for enhanced affine quantization of Hamiltonians of the general form $P^2 + V(Q)$, the enhanced classical Hamiltonian contains a term of the form q^{-2} with a coefficient proportional to $\hbar^2$. This fact anticipates the naturalness of an actual term proportional to $\hbar^2$ and involving inverse squared operators in the quantum Hamiltonian of other models, suggesting it may not be out of place.

These musings provide an important clue for the next topic of discussion.

Scalar Field Quantization

We next consider a special class of infinitely many degrees-of-freedom problems associated with a covariant scalar field. For ϕ^4_n models, standard canonical quantization procedures have obtained self-consistent solutions for spacetime dimensions $n = 2, 3$, e.g., [6], but those same methods have failed to provide suitable results for $n \geq 4$, leading instead to trivial quantum solutions equivalent to (generalized) free theories. It is our believe that for higher spacetime dimensions ($n \geq 4$), we can find nontrivial solutions by choosing affine quantization procedures, and that this technique also leads to new solutions for $n = 2, 3$ as well, which exhibit compatibility of 'mixed models' in ways we will describe. The reason behind this believe is based on the form of the ground-state distribution we are led to, which has integrable singularities when certain field values are zero. As we shall see, this particular form of the ground-state distribution has the decided advantage that a perturbation series for the interaction does *not* exhibit divergences! A ground-state wave function with square-integrable singularities when certain fields vanish leads to terms in the quantum Hamiltonian proportional to $\hbar^2$, and in our case involve inverse squared field operators. Although an affine quantization is in order, the insight outlined above for these models means that we can proceed to develop the theory in a more direct manner [7].

Free vs. pseudofree models

It is self evident that $\lim_{g\to 0}(A_0 + g\,A_I) = A_0$, except when it is false. Consider the action functional for an anharmonic oscillator given by

$$A_g = \textstyle\int_0^T \{\, \frac{1}{2}[\dot{y}(t)^2 - y(t)^2] - g\,y(t)^w \,\}\, dt \;, \tag{26}$$

and the domain of functions allowed by this expression. If the exponent $w = +4$, then the limit as $g \to 0$ leads to the free action $A_0 = \int_0^T \frac{1}{2}[\dot{y}(t)^2 - y(t)^2]\, dt$, but if $w = -4$ that is *not* the case. Instead, when $w = -4$, the limit as $g \to 0$ is $A_0' \equiv \int_0^T \frac{1}{2}[\dot{y}(t)^2 - y(t)^2]\, dt$ *supplemented with the requirement that* $\int_0^T y(t)^{-4}\, dt < \infty$, which is a fundamentally different domain from that of the free action functional. We refer to the theory described by A_0' as a *pseudofree theory*. A pseudofree theory is the one that is continuously connected to the interacting theories. The pseudofree theory may coincide with the usual free theory, as is the case when $w = +4$, but when $w = -4$, the pseudofree and the free theories are different. This rule applies to the quantum theories as well. In particular, the free and pseudofree quantum theories are identical for $w = +4$ and distinct for $w = -4$. If one considered making a perturbation analysis of the interacting theory, one would have to start with the pseudofree theory and *not* with the free theory when the two theories differ.

A rather similar situation applies to scalar fields. Consider the classical action given by

$$A_{g_0} = \textstyle\int (\, \frac{1}{2}\{\, [\partial_\mu \phi(t,x)]^2 - m_0^2\, \phi(t,x)^2 \,\} - g_0\, \phi(t,x)^4 \,)\, dt\, d^s\!x \;, \tag{27}$$

where $x \in \mathbb{R}^s$, $s \geq 1$. Provided $m_0 > 0$ and $g_0 > 0$, a multiplicative inequality [8, 9] implies, for $n = s+1$, and now where $x \in \mathbb{R}^n$, that

$$\{\, g_0 \textstyle\int \phi(x)^4\, d^n\!x \}^{1/2} \leq C'' \int \{\, [\nabla\phi(x)^2] + m_0^2\, \phi(x)^2 \,\}\, d^n\!x \;, \tag{28}$$

where $C'' = (4/3)[g_0^{1/2}\, m_0^{(n-4)/2}]$ if $n \leq 4$, while $C'' = \infty$ if $n \geq 5$, which in the latter case means that there are fields for which the left side of (28) diverges, but the right side is finite. [Remark: The divergent cases are *exactly* the *non*-renormalizable models when quantized—and that fact holds true for all the non-renormalizable models of the form ϕ_n^p as well [9]!] Thus, when $n \geq 5$, (28) ensures that the classical pseudofree model is different from the classical free model. We expect that the quantum pseudofree and quantum free models are also different. We will be guided in choosing the pseudofree model by the requirement that it connects smoothly with the interacting models.

Choosing the pseudofree model

Formally, the action functional determines the quantum Hamiltonian, which, in turn, determines the ground-state wave function. The reverse of the previous ordering is also formally true, so let us start by a study of the lattice regularized form of the presumptive ground-state wave function, $\widetilde{\Psi}_0(\phi) \equiv \exp[-U(\phi)/2]$, where, as usual, $U(\phi)$ is determined by the 'large-field' behavior of the potential, and thus $U(\phi)$ is well behaved when ϕ is 'small'. On the lattice, $\phi(x)$, at $t = 0$, is replaced by $\{\phi_k\}$, where $k \in \mathbb{Z}^s$ labels the site on a (spatial) hypercubic lattice with lattice spacing $a > 0$ and a total number of (spatial) sites $L^s \equiv N' < \infty$. The continuum limit arises when $a \to 0$, $L \to \infty$, but $(La)^s = N' a^s$ is fixed and finite, at least initially. Many moments of the ground-state distribution diverge in the continuum limit, such as

$$\int [\Sigma'_k \phi_k^2]^p \, e^{-U(\phi)} \, \Pi'_k \, d\phi_k = \mathcal{O}(N'^p) \,, \tag{29}$$

where the estimated value arises because there are N'^p terms each of $\mathcal{O}(1)$, and as $N' \to \infty$, divergences arise, for all $p \geq 1$. [Remark: In (29) Σ'_k and Π'_k denote a sum and product over all sites in a single spatial slice.] These divergences seem to arise from the fact that the continuum limit involves an infinite number of integration variables, but the continuum limit need not lead to divergences. To understand this remark, let us first change the integration variables from "Cartesian coordinates" to "hyperspherical coordinates" by the transformation $\phi_k \equiv \kappa \eta_k$, where $\kappa^2 \equiv \Sigma'_k \phi_k^2$, $1 \equiv \Sigma'_k \eta_k^2$,

$0 \le \kappa < \infty$, and $-1 \le \eta_k \le 1$, for all k. In the new variables, (29) becomes

$$\int [\kappa^2]^p \, e^{-U(\kappa\eta)} \, \kappa^{N'-1} d\kappa \, 2\delta(1 - \Sigma'_k \eta_k^2) \, \Pi'_k d\eta_k \; . \tag{30}$$

No longer do we have N'^p terms of order $\mathcal{O}(1)$, but it is the power $N' - 1$ of the hyperspherical radius κ that leads to divergences as $N' \to \infty$. Moreover, a steepest descent analysis as $N' \to \infty$—which makes the support of the measure in (30) *disjoint*, at least partially, for a change of parameters in U—leads to divergences in any perturbation analysis. However, if the ground-state distribution contained an additional factor, namely, $\kappa^{-(N'-R)}$, where $R > 0$ is fixed and finite, it would effectively change the κ-factor from $\kappa^{N'-1}$ to κ^{R-1}—a procedure we call "*measure mashing*"—and, as a result, the divergences would disappear as $N' \to \infty$! Stated otherwise, measure mashing would nullify the steepest descent argument, and thus a perturbation analysis would not involve divergences. In summary, the 'trick' in securing a finite version of scalar field quantization—one that also smoothly passes to its own pseudofree theory—is a direct result of mashing the measure.

To achieve measure mashing, we now assume the ground-state distribution has the form

$$\Psi_0(\phi)^2 \equiv \{\Pi'_k [\Sigma'_l J_{k,l} \, \phi_l^2]^{-(1-2ba^s)/2}\} \, e^{-U'(\phi)} \; , \tag{31}$$

where $J_{k,l} \equiv 1/(2s+1)$ for $l = k$ and for l one of the $2s$ spatially nearest neighbors of k; $J_{k,l} \equiv 0$ otherwise. Specifically, $R \equiv 2ba^s N'$, where b has the dimensions $(\text{length})^{-s}$ to make R dimensionless. In turn, the functional form of the lattice ground state determines the functional form of the lattice-regularized Hamiltonian operator. Roughly speaking, the denominator term fixes the small-field dependence of the potential, while $U'(\phi)$ leads to the large-field behavior of the potential. However, to fix the Hamiltonian, we let the chosen denominator (involving $J_{k,l}$) of the ground-state wave function determine the small-field potential, but for the large-field behavior, we choose the form of the Hamiltonian itself. This leads us to the lattice regularized form of the quantum Hamiltonian operator given by

$$\begin{aligned} \mathcal{H} = &-\tfrac{1}{2} a^{-2s} \hbar^2 \textstyle\sum'_k \frac{\partial^2}{\partial \phi_k^2} \, a^s + \tfrac{1}{2} \textstyle\sum'_{k,k^*} (\phi_{k^*} - \phi_k)^2 \, a^{s-2} + \tfrac{1}{2} m_0^2 \textstyle\sum'_k \phi_k^2 \, a^s \\ &+ g_0 \textstyle\sum'_k \phi_k^4 \, a^s + \tfrac{1}{2} \hbar^2 \textstyle\sum'_k \mathcal{F}_k(\phi) \, a^s - E_0 \; , \end{aligned} \tag{32}$$

where k^* denotes each of the s nearest neighbors to k in the positive sense, and the all-important counterterm $\mathcal{F}_k(\phi)$ is given by

$$\begin{aligned}\mathcal{F}_k(\phi) &= \tfrac{1}{4}(1-2ba^s)^2\,a^{-2s}\left(\textstyle\sum'_t \dfrac{J_{t,k}\,\phi_k}{[\Sigma'_m J_{t,m}\,\phi^2_m]}\right)^2\\ &\quad -\tfrac{1}{2}(1-2ba^s)\,a^{-2s}\textstyle\sum'_t \dfrac{J_{t,k}}{[\Sigma'_m J_{t,m}\,\phi^2_m]}\\ &\quad +(1-2ba^s)\,a^{-2s}\textstyle\sum'_t \dfrac{J^2_{t,k}\,\phi^2_k}{[\Sigma'_m J_{t,m}\,\phi^2_m]^2}\;. \end{aligned} \tag{33}$$

Although $\mathcal{F}_k(\phi)$ does not depend only on ϕ_k, it nevertheless becomes a local potential in the formal continuum limit. The constant E_0 is chosen to ensure that $\mathcal{H}\Psi_0(\phi) = 0$. Note that local field powers do not involve normal ordering, but instead, they are defined by an operator product expansion, realized effectively by a multiplicative renormalization of the parameters.

At this point, one may be wondering what is the relation of enhanced quantization, the topic in the first part of this article, and the approach taken to describe scalar field quantization in the second part of this article. In the first part we introduced coherent states, both canonical and affine. Coherent states of any kind are based on a fiducial vector, and for a field theory with an infinite number of degrees of freedom, the usual fiducial vectors (e.g., Gaussian functions) are generally inappropriate. A safe fiducial vector to use is the ground state of the associated Hamiltonian, and for the scalar field case under discussion, the ground-state wave function, as given by the square root of (31), leads to affine coherent states being appropriate [10], and thus our scalar field analysis is a form of affine quantization.

From the Hamiltonian operator we can determine the form of the (Euclidean) lattice action functional as given by

$$I = \tfrac{1}{2}\Sigma_{k,k^*}(\phi_{k^*}-\phi_k)^2\,a^{n-2} + \tfrac{1}{2}m_0^2\Sigma_k\,\phi_k^2\,a^n + g_0\,\Sigma_k\,\phi_k^4\,a^n + \tfrac{1}{2}\hbar^2\,\Sigma_k\mathcal{F}_k(\phi)\,a^n\;, \tag{34}$$

where $n = s+1$, Σ_k signifies a sum over the entire, finite, n-dimensional spacetime lattice, and k^* runs over all n nearest neighbors to k in the positive sense. The Euclidean-spacetime generating functional is given, in turn, by

$$S(h) = M\int e^{Z^{-1/2}\Sigma_k h_k\,\phi_k\,a^n/\hbar \,-\, I/\hbar}\;\Pi_k\,d\phi_k\;, \tag{35}$$

where Z denotes the field strength renormalization constant, Π_k is a product over all sites in the (finite) spacetime lattice, and M is chosen so that $S(0) = 1$. Based on the distribution underlying this integral, preliminary Monte Carlo studies [11] show a positive, non-vanishing renormalized coupling constant vs. the bare coupling constant for ϕ^4_4. This result compares with a vanishing renormalized coupling constant vs. the bare coupling constant that follows from conventional canonical quantization procedures [12].

Discussion of scalar field quantization

The counterterm $\mathcal{F}_k(\phi)$ does not depend on g_0 and thus it remains as $g_0 \to 0$. The result of that limit is the pseudofree model, and it differs from the usual free model. This kind of interacting theory provides a valid quantization of the non-renormalizable models such as ϕ^4_n, $n \geq 5$. However, we can also extend the use of the new Hamiltonian operator to lower dimensions as well, even though, for the classical theory, the free and pseudofree models are the same. The purpose of extending the new form of the Hamiltonian is to ensure the uniqueness of 'mixed models'. For example, consider the case of ϕ^4_3, which is a super-renormalizable model that has been studied perturbatively and non-perturbatively with the same self-consistent results; e.g., see [6]. But, suppose we studied the mixed model given by $g_0\phi^4_3 + g'_0\phi^8_3$, which is a sum of a super-renormalizable and a non-renormalizable model, or the mixed model $g''_0\phi^4_5 + g'''_0\phi^8_5$, which is the sum of two different non-renormalizable models, etc. Conventional canonical quantization procedures would not be able to make any sense of such mixed models, but the new version, which treats each ingredient of such models in the same manner, would make perfectly good sense as the coupling constants are turned on, then off, then on again, etc., in any order. In this regard we are also proposing new and different quantization procedures for models like ϕ^4_2 and ϕ^4_3, and other super-renormalizable models. [Remark: Of course, the original solutions for super-renormalizable models have their own role to play for different physical situations.] On these grounds, we advocate accepting the lattice Hamiltonian (32) and the lattice action (34) for all $n \geq 2$.

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References

[1] P.A.M. Dirac, *The Principles of Quantum Mechanics*, (Clarendon Press, Oxford, 1947), page 114.

[2] J.R. Klauder, "Enhanced Quantization: A Primer", J. Phys. A: Math. Theor. **45**, 285304 (8pp) (2012); arXiv:1204.2870.

[3] J.R. Klauder and B.-S. Skagerstam, *Coherent States: Applications to Physics and Mathematical Physics*, (World Scientific, Singapore, 1985).

[4] J.R. Klauder, "Rotationally-Symmetric Model Field Theories", J. Math. Phys. **6**, 1666-1679 (1965); H.D.I. Abarbanel, J.R. Klauder, and J.G. Taylor, "Green's Functions for Rotationally Symmetric Models", Phys. Rev. **152**, 198-206 (1966).

[5] J.R. Klauder, '*A Modern Approach to Functional Integration*, (Birkhauser, Boston, MA, 2010).

[6] J. Glimm and A. Jaffe, *Quantum Physics*, (Springer Verlag, New York, 1987), Second edition.

[7] J.R. Klauder, "Scalar Field Quantization Without Divergences In All Spacetime Dimensions", J. Phys. A: Math. Theor. 44, 273001 (30pages) (2011); arXiv:1101.1706; J.R. Klauder, "Divergences in Scalar Quantum Field Theory: The Cause and the Cure", Mod. Phys. Lett. A **27**, 1250117 (9pp) (2012); arXiv:1112.0803.

[8] O.A. Ladyzenskaja, V. Solonnikov, and N.N. Ural'ceva, *Linear and Quasi-linear Equations of Parabolic Type*, (Am. Math. Soc., Providence, Vol. 23, 1968).

[9] J.R. Klauder, *Beyond Conventional Quantization*, (Cambridge University Press, Cambridge, 2000 & 2005).

[10] J.R. Klauder, "New Affine Coherent States Based on Elements of Nonrenormalizable Scalar Field Models", Advances in Mathematical Physics, vol. 2010, Article ID 191529, 16 pages, (2010); http://www.hindawi.com/journals/amp/2010/191529.html; J.R. Klauder,"The Utility of Affine Variables and Affine Coherent States", J. Phys. A: Math. Theor. **45**, 24001 (17pp) (2012); arXiv:1108.3380.

[11] J. Stankowicz, private communication.

[12] B. Freedman, P. Smolensky, and D. Weingarten, "Monte Carlo Evaluation of the Continuum Limit of ϕ_4^4 and ϕ_3^4", Phys. Lett. B **113**, 481 − 486 (1982).