THE HVZ THEOREM FOR N-PARTICLE SCHRÖDINGER OPERATORS ON LATTICE

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ABSTRACT. The N-particle Schrödinger operator $H(K)$, $K \in (-\pi,\pi]^d$, K being the total quasi-momentum, with short-range pair potentials on lattice $\mathbb{Z}^d$, $d \geq 1$, is considered. For fixed total quasi-momentum K, the structure of $H(K)$'s essential spectrum is described and the analogue of the Hunziker – van Winter – Zhislin (HVZ) theorem is proved.

INTRODUCTION

In the sixties, fundamental results on the essential spectrum of the many-particle continuous Schrödinger operators were obtained by Hunziker[1], van Winter [11], Zhislin[12]. The theorem describing the essential spectrum for a system of many particles was named *the HVZ theorem* in honor of these three authors. It affirms that the essential spectrum of an N-particle Hamiltonian (after separating off the free center-of-mass motion) is bounded below by the lowest possible energy which two independent subsystems can have. Since then the result was generalized in many ways (see [2, 4, 6, 14, 17, 23] and for more extensive references [16]).

For the multiparticle Chandrasekhar operators the HVZ theorem was proved in [18]. The HVZ theorem for atomic Brown-Ravenhall operators in the Born-Oppenheimer approximation was obtained in [19, 20, 21] in terms of two-cluster decompositions. In [22], the HVZ theorem is proved for a wide class of models which are obtained by projecting of multiparticle Dirac operators to subspaces dependent on the external electromagnetic field.

In the continuous case one method of proving the HVZ theorem is the use of the diagrammatic techniques and the Weinberg-Van Winter equations through the theory of integral equations of Hilbert Schmidt kernel (see e.g. [3, 11, 1, 16]). Another set of equations for the resolvent are the Faddeev-Yakubovky equations [24, 25]. These equations became the base for the creation of new computing techniques in nuclear and atomic physics. Technically simple proofs of the HVZ theorem were given in [6]. The key idea

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here was the use of geometry of configuration space to separate the channels. Moreover this method was applied to prove the finiteness or infinitude of bound states of an N-body quantum systems. Alternative methods for the determination of the essential spectrum of generalized Schrödinger operators involve C^*-algebra techniques [26].

Since the discrete Schrödinger operator is a lattice analogy of the continuous Schrödinger operator (see [8, 7]), we can expect a HVZ type result for the discrete Schrödinger operator associated with the Hamiltonian of the system of N – quantum particles. Furthermore, it may be conjectured that one can use similar methods for the proof.

The kinematics of the system of quantum particles on lattice is quite exotic. Thanks to the translation invariance of the N-particle Hamiltonian $\hat{\mathbf{H}}$ in $\ell_2((\mathbb{Z}^d)^N)$, the study of spectral properties of $\hat{\mathbf{H}}$ is reduced to study of the spectral properties of the family of the N-particle discrete Schrödinger operators $H(K),\ K \in \mathbb{T}^d$, where $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d = (-\pi, \pi]^d$ is the d-dimensional torus.

Namely, the underlying Hilbert space $\ell^2((\mathbb{Z}^d)^N)$ is decomposed to a direct von Neumann integral, associated with the representation of the (discrete) Abelian group $\mathbb{Z}^d$ by shift operators on the lattice and then, the total Hamiltonian $\hat{H}$ is also decomposed into the direct integral

$$H = \int\limits_{\mathbb{T}^d} \oplus H(K) dK,$$

where $K \in \mathbb{T}^d$ – is called the N*-particle quasi-momentum*. In contrast to the continuous case, the corresponding fiber operators, i.e. the N-particle discrete Schrödinger operators $H(K)$, associated with the direct integral decomposition, depend non trivially on the quasi-momentum $K \in \mathbb{T}^d$. As a result, the spectra of the family $H(K)$ turn out to be rather sensitive to a change of the quasi-momentum K.

The physical significance of the study of the N-particle discrete Schrödinger operators was duly stressed in the survey [8]. A review on the Hamiltonians in solid-state physics as multiparticle discrete Schrödinger operators is given in [7].

The essential spectrum of an N-particle additive cluster operator in the subspace of antisymmetric functions of $\ell_2((\mathbb{Z}^d)^N)$ has been studied in [5] and the analogue of the HVZ theorem was proved using the geometric method of Simon.

In [27], for the Hamiltonians of a system of N-identical particles (bosons), interacting via zero-range pair potentials on a lattice, the structure of essential spectrum was studied.

In [28] and [9], the limit operators method has been applied to study the location of the essential spectrum of square-root Klein–Gordon operators, Dirac operators, electromagnetic Schrödinger operators and some classes of the discrete Schrödinger operators under very weak assumptions on the behavior of magnetic and electric potentials at infinity.

For the discrete Schrödinger operators, associated with the Hamiltonians of systems of three quantum particles moving on lattice interacting via zero-range attractive pairs potentials, the location and structure of the essential spectrum has been investigated in [13, 30, 31, 32, 33, 34, 35].

In four particle case, the analogue of the HVZ theorem and Faddeev-Yakubovsky type equations are obtained by Muminov [10], but in this case the structure of the essential spectrum has not been fully understood yet.

In the present paper, we consider the discrete Schrödinger operator $H(K),\ K\in\mathbb{T}^d,$ (see (1.7)) associated with the Hamiltonian of a system of N-particles moving on dN-dimensional lattice $(\mathbb{Z}^d)^N$ and interacting via short-range pair potentials. We prove the analogue of the HVZ theorem using the diagrammatic method of Hunziker for the case when particles have arbitrary bounded *dispersion functions* having not necessarily compact support. Moreover we describe the structure of its essential spectrum by means of two-cluster Hamiltonians (see Theorem 3.1).

Observe that the operator $H(K)$ is bounded, thus, the essential spectrum is no longer the positive real line. More precisely, it consists of the union of closed segments and in turn this may allow the Efimov effect to appear not only at the lower edge of essential spectrum, but at the edges of the gaps between those segments (see [29]).

The present paper is organized as follows. In Section 1, we introduce the N–particle Hamiltonian $\hat{H}$ in both coordinate and momentum representation, introduce the N-particle quasi-momentum, and decompose the energy operator into the von Neumann direct integral of the fiber Hamiltonians $H(K)$, thus providing the reduction to the N-particle discrete Schrödinger operator problem.

The cluster operators and the structure of their spectra are discussed in Section 2. In this section, fixing the quasimomenta of the mass centers $k_1,\dots,k_\ell$ of the clusters specified by the cluster decomposition $D=\{D_1,\dots,D_\ell\},$ we prove the analogous breakup of cluster operator $H^D(K)$ into the sum of the Hamiltonians of clusters corresponding to the cluster decomposition D (see (2.4) in Section 2).

Finally, in Section 4 the analogue of the HVZ theorem is proved in terms of graphs.

Some notations: as in [14] we equip $\mathbb{R}^d$ with the norm:

$$|x|_+=|x^{(1)}|+\dots+|x^{(d)}|,\quad x=(x^{(1)},\dots,x^{(d)})\in\mathbb{R}^d;$$

and hence for $x = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N$:

$$|x| = |x_1|_+ + \dots + |x_N|_+$$

where $|x_i|_+$ is norm of $x_i = (x_i^{(1)}, \dots, x_i^{(d)}) \in \mathbb{R}^d$.

Let $\mathbb{Z}^d$ be d-dimensional lattice and $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d = (-\pi, \pi]^d$ be the d-dimensional torus (the first Brillouin zone, i.e. the dual group of $\mathbb{Z}^d$) equipped with Haar measure. Let $\ell_2((\mathbb{Z}^d)^m)$ (resp. $L_2((\mathbb{T}^d)^m)$) denote the Hilbert space of square-summable (resp. square-integrable) functions defined on $(\mathbb{Z}^d)^m$ (resp. $(\mathbb{T}^d)^m$).

For $s \in \mathbb{Z}^d$ and $p \in \mathbb{T}^d$, $(s, p) = s^{(1)}p^{(1)} + \ldots + s^{(d)}p^{(d)}$.

We denote by $\sigma(A)$ the spectrum of a self-adjoint A and by $\sigma_{\text{ess}}(A)$ its essential spectrum.

1. N-PARTICLE SCHRÖDINGER OPERATOR ON LATTICE $\mathbb{Z}^d$

1.1. **Coordinate representation of N-particle Hamiltonian.** In coordinate representation, the total Hamiltonian $\hat{\mathbf{H}}$ of the system of N-particles moving on d-dimensional lattice $\mathbb{Z}^d$, with the real-valued pair potentials $\hat{\mathbf{V}}_{ij}$, is usually associated with the following self-adjoint (bounded) operator in the Hilbert space $\ell^2((\mathbb{Z}^d)^N)$ of the form

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_0 - \hat{\mathbf{V}},$$

where the unperturbed operator $\hat{\mathbf{H}}_0$, called the free Hamiltonian, is the operator

$$\hat{\mathbf{H}}_0 = \frac{1}{m_1}\hat{\Delta}_{x_1} + \dots + \frac{1}{m_N}\hat{\Delta}_{x_N},$$

with

$$\hat{\Delta}_{x_1} = \hat{\Delta}_1 \otimes \hat{I}_d \otimes \dots \otimes \hat{I}_d, \quad \dots \quad, \hat{\Delta}_{x_N} = \hat{I}_d \otimes \dots \otimes \hat{I}_d \otimes \hat{\Delta}_N,$$

where m_i - the mass of particle i, $\hat{I}_d$ is the identity operator on $\ell^2(\mathbb{Z}^d)$, and $\hat{\Delta}_i$ is the generalized Laplacian, a multidimensional Laurent-Toeplitz-type operator on the Hilbert space $\ell^2(\mathbb{Z}^d)$:

$$\hat{\Delta}_i = \sum_{s \in \mathbb{Z}^d} \hat{\varepsilon}_i(s) T(s), \quad i = 1, \dots, N.$$

Here $T(y)$ is the shift operator by $y \in \mathbb{Z}^d$:

$$(T(y)f)(x) = f(x + y), \quad f \in \ell^2(\mathbb{Z}^d),$$

The function $\hat{\varepsilon}_i : \mathbb{Z}^d \to \mathbb{C}$, so-called *the dispersion function* of the particle i, $i = \overline{1, N}$, is further assumed to satisfy Hypothesis 1.1 below.

Recall (see, e.g., [36]) that a Laurent-Toeplitz operator A is a bounded linear operator on $\ell_2(\mathbb{Z}^m)$ with the property that the *matrix entries*

$(A\psi_y, \psi_x)$, $x, y \in \mathbb{Z}^m$, depends only on the difference $x - y$, where $\{\psi_y\}$, $y \in \mathbb{Z}^m$, is the canonical basis in $\ell^2(\mathbb{Z}^m)$:

$$\psi_y(s) = \begin{cases} 1, & \text{if} \quad y = s,\ s \in \mathbb{Z}^m \\ 0, & \text{if} \quad y \neq s,\ s \in \mathbb{Z}^m \end{cases}, \quad y \in \mathbb{Z}^m.$$

In the physical literature, the symbol of the Toeplitz operator Δ_i given by the Fourier series (without the factor $(2\pi)^{\frac{d}{2}}$)

$$\varepsilon_i(p) = \sum_{s \in \mathbb{Z}^d} \hat{\varepsilon}_i(s) e^{\mathrm{i}(s,p)}, \quad p \in \mathbb{T}^d. \tag{1.1}$$

The real-valued continuous function $\varepsilon_i(\cdot)$ on $\mathbb{T}^d$ is called the *dispersion relation of the* i *-th normal mode* associated with the free particle.

The perturbation $\hat{\mathbf{V}}$ of Hamiltonian $\hat{\mathbf{H}}_0$ comes from pair potentials, i.e.

$$\hat{\mathbf{V}} = \sum_{1 \le i < j \le N} \hat{\mathbf{V}}_{ij},$$

where $\hat{\mathbf{V}}_{ij}$ is a multiplication operator by a function $\hat{v}_{ij}(x_i - x_j)$ in $\ell_2((\mathbb{Z}^d)^N)$:

$$(\hat{\mathbf{V}}_{ij} f)(x_1, \dots, x_N) = \hat{v}_{ij}(x_i - x_j) \hat{f}(x_1, \dots, x_N).$$

Further we assume

Hypothesis 1.1. i) $\hat{\varepsilon}_i$, $i = 1, \dots, N$ *satisfy:*

a) *"Regularity" property: the series* $\sum_{y \in \mathbb{Z}^d} |y|_+^\delta \, |\hat{\varepsilon}_i(y)|$ *is convergent for some* $\delta > 0$*;*

b) *"Self-adjointness" property:* $\hat{\varepsilon}_i(y) = \overline{\hat{\varepsilon}_i(-y)}, \quad y \in \mathbb{Z}^d$.

ii) *The functions* $\hat{v}_{ij} : \mathbb{Z}^d \to \mathbb{R}$, $i, j = 1, \dots, N$, $i < j$ *tend to* 0 *at the infinity.*

1.2. **Momentum representation of N-particle Hamiltonian.** Let $\mathcal{F}_m : \ell_2((\mathbb{Z}^d)^m) \to L_2((\mathbb{T}^d)^m)$ be the standard Fourier transform.

In the momentum representation, corresponding Hamiltonian $\mathbf{H}$ has the form

$$\mathbf{H} = \mathbf{H}_0 - \mathbf{V},$$

where $\mathbf{H}_0 = \mathcal{F}_N \hat{\mathbf{H}}_0 \mathcal{F}_N^{-1}, \quad \mathbf{V} = \mathcal{F}_N \hat{\mathbf{V}} \mathcal{F}_N^{-1}$.

Here free Hamiltonian $\mathbf{H}_0$ is a multiplication operator by the function $\mathcal{E} : (\mathbb{T}^d)^N \to \mathbb{R}^1$:

$$(\mathbf{H}_0 f)(p_1, \dots, p_N) = \mathcal{E}(p_1, \dots, p_N) f(p_1, \dots, p_N),$$

$$\mathcal{E}(p_1, \dots, p_N) = \sum_{i=1}^N \varepsilon_i(p_i), \quad p_1, \dots, p_N \in \mathbb{T}^d,$$

and the perturbation operator $\mathbf{V} = \sum_{i<j} \mathbf{V}_{ij}$ is the sum of partial integral operators $\mathbf{V}_{ij}$:

$$(\mathbf{V}_{ij}f)(p_1,\dots,p_N) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int\limits_{(\mathbb{T}^d)^2} v_{ij}(p_i-q_i)\delta(p_i+p_j-q_i-q_j)\times$$
$$\times f(p_1,\dots,q_i,\dots,q_j,\dots,p_N)\,dq_i dq_j,$$

where δ is the Dirac delta function on $\mathbb{T}^d$ and **the kernel function** $v_{ij} = \mathcal{F}_1\hat{v}_{ij} : \mathbb{T}^d \to \mathbb{C}$,

$$v_{ij}(p) = (2\pi)^{-d/2} \sum_{s\in\mathbb{Z}^d} \hat{v}_{ij}(s)\, e^{\mathrm{i}(s,p)}, \quad p \in \mathbb{T}^d$$

belongs to $L_2(\mathbb{T}^d)$.

1.3. **Decomposition of the Hamiltonian H into the direct von Neumann integral.** Let us introduce the unitary representation $\{\hat{\mathbf{U}}_s\}_{s\in\mathbb{Z}^d}$ of the abelian group $\mathbb{Z}^d$ by the shift operators on the Hilbert space $\ell^2((\mathbb{Z}^d)^2)$:

$$(\hat{\mathbf{U}}_s\hat{f})(x_1,\dots,x_N) = \hat{f}(x_1+s,\dots,x_N+s),, \quad x_1,\dots,x_2,s \in \mathbb{Z}^d.$$

Observe that via standard Fourier transform $\mathcal{F}_N$ the family $\{\hat{\mathbf{U}}_s\}_{s\in\mathbb{Z}^d}$ is unitary-equivalent to the family of unitary (multiplication) operators $\{\mathbf{U}_s\}_{s\in\mathbb{Z}^d}$ acting in $L_2((\mathbb{T}^d)^N)$ as

$$(\mathbf{U}_s f)(p_1,\dots,p_N) = \exp(-\mathrm{i}(s,p_1+\dots+p_N))f(p_1,\dots,p_N), \quad f \in L_2((\mathbb{T}^d)^N).$$

Given $K \in \mathbb{T}^d$, we define $\mathbb{F}_K$ as follows

$$\mathbb{F}_K = \{(p_1,\dots,p_N) \in (\mathbb{T}^d)^N : \ p_1+\dots+p_N = K\}.$$

Introducing the mapping $\pi_j : (\mathbb{T}^d)^N \to (\mathbb{T}^d)^{N-1}$,

$$\pi_j((p_1,\dots,p_{j-1},p_j,p_{j+1},\dots,p_N)) = (p_1,\dots,p_{j-1},p_{j+1},\dots,p_N),$$

we denote by π_{jK}, $K \in \mathbb{T}^d$, the restriction of π_j onto $\mathbb{F}_K \subset (\mathbb{T}^d)^N$, that is,

$$\pi_{jK} = \pi_j|_{\mathbb{F}_K}. \tag{1.2}$$

The following lemma shows that $\mathbb{F}_K$, $K \in \mathbb{T}^d$, is a $d \times (N-1)$-dimensional manifold homeomorphic to $(\mathbb{T}^d)^{N-1}$.

Lemma 1.1. *The mapping π_{jK}, $K \in \mathbb{T}^d$, from $\mathbb{F}_K \subset (\mathbb{T}^d)^N$ onto $(\mathbb{T}^d)^{N-1}$ is bijective, with the inverse mapping given by*

$$(\pi_{jK})^{-1}(p_1,\dots,p_{j-1},p_{j+1},\dots,p_N) =$$
$$(p_1,\dots,p_{j-1},K-\sum_{i=1,\,i\neq j}^{N} p_i,p_{j+1},\dots,p_N).$$

The decomposition the space $L^2((\mathbb{T}^d)^N)$ into the direct integral

$$L_2((\mathbb{T}^d)^N) = \int\limits_{K\in\mathbb{T}^d} \oplus L_2(\mathbb{F}_K)dK, \tag{1.3}$$

yields the corresponding decomposition of the unitary representation $\mathbf{U}_s$, $s\in\mathbb{Z}^d$, into the direct integral

$$\mathbf{U}_s = \int_{K\in\mathbb{T}^d} \oplus U_s(K)dK,$$

with

$$U_s(K) = e^{-\mathrm{i}(s,K)} I_{L^2(\mathbb{F}_K)}$$

and $I_{L^2(\mathbb{F}_K)}$ being the identity operator on the Hilbert space $L^2(\mathbb{F}_K)$.

The Hamiltonian $\mathbf{H}$ (in the coordinate representation) obviously commutes with the group of the unitary operators $\mathbf{U}_s$, $s\in\mathbb{Z}^d$. Hence, the operator $\mathbf{H}$ can be decomposed into the direct von Neumann integral

$$\mathbf{H} = \int_{K\in\mathbb{T}^d} \oplus \tilde{H}(K)dK \tag{1.4}$$

associated with the decomposition (1.3).

In the physical literature, the parameter K, $K\in\mathbb{T}^d$, is named the N-*particle quasi-momentum* and the corresponding operators $\widetilde{H}(K)$, $K\in\mathbb{T}^d$, are called the *fiber operators*.

1.4. **The N-particle Schrödinger operators. The momentum representation.** The fiber operator $\widetilde{H}(K) : L_2(\mathbb{F}_K) \to L_2(\mathbb{F}_K)$ is defined as follows:

$$\widetilde{H}(K) = \widetilde{H}_0(K) - \widetilde{V}, \tag{1.5}$$

where

$$\widetilde{H}_0(K)f(p_1,\dots,p_N) = \big(\varepsilon_1(p_1)+\dots+\varepsilon_N(p_N)\big)f(p_1,\dots,p_N),$$

and

$$\widetilde{V} = \sum_{1\le i<j\le N} \widetilde{V}_{ij}$$

with

$$(\widetilde{V}_{ij}f)(p_1,\dots,p_N) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int\limits_{\mathbb{T}^d} v_{ij}(p_i-q_i)f(p_1,\dots,q_i,\dots,p_i+p_j-q_i,\dots,p_N)dq_i.$$

Here $(p_1,\dots,p_N)\in\mathbb{F}_K$.

Using the unitary operator $U_{NK} : L^2(\mathbb{F}_K) \to L^2((\mathbb{T}^d)^{N-1})$, $K\in\mathbb{T}^d$,

$$U_{NK}g = g\circ(\pi_{NK})^{-1},$$

with π_{NK} defined by (1.2), we get the operator $H(K) : L_2((\mathbb{T}^d)^{N-1}) \to L_2((\mathbb{T}^d)^{N-1})$, $H(K) = U_{NK}\widetilde{H}(K)U_{NK}^{-1}$, being unitary equivalent to (1.5), of the form

$$H(K) = H_0(K) - V, \tag{1.6}$$

where the operator $H_0(K)$ is the multiplication operator in $L_2((\mathbb{T}^d)^{N-1})$ by the continuous function $\mathcal{E}_K : (\mathbb{T}^d)^{N-1} \to \mathbb{R}$:

$$\mathcal{E}_K(p_1, \ldots, p_{N-1}) = \varepsilon_1(p_1) + \cdots + \varepsilon_{N-1}(p_{N-1}) + \varepsilon_N(K - p_1 - \ldots - p_{N-1}),$$

and the perturbation $V = \sum\limits_{i<j} V_{ij}$ acts in $L_2((\mathbb{T}^d)^{N-1})$ with

$$(V_{ij}f)(p_1, \ldots, p_{N-1}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int\limits_{\mathbb{T}^d} v_{ij}(p_i - q_i) f(p_1, \ldots, q_i, \ldots, p_i + p_j - q_i, \ldots, p_{N-1}) dq_i$$

for $1 \le i < j < N$ and

$$(V_{iN}f)(p_1, \ldots, p_{N-1}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int\limits_{\mathbb{T}^d} v_{iN}(p_i - q_i) f(p_1, \ldots, q_i, \ldots, p_{N-1}) dq_i$$

for $1 \le i < N$.

Remark 1.1. *Instead of U_{NK} one may use the operator $U_{jK} : L^2(\mathbb{F}_K) \to L^2((\mathbb{T}^d)^{N-1})$, $K \in \mathbb{T}^d$,*

$$U_{jK} g = g \circ (\pi_{jK})^{-1}, \quad j = 1, \ldots, N-1.$$

and obtain the operator $H_j(K) = U_{jK}\widetilde{H}(K)\widetilde{U}_{jK}^*$ which is unitarily equivalent to $H(K)$.

1.5. **The coordinate representation the N-particle Schrödinger operator.** In coordinate representation, the N-particle Schrödinger operator acts in $\ell_2((\mathbb{Z}^d)^{N-1})$ as follows:

$$\hat{H}(K) = \hat{H}_0(K) - \hat{V}. \tag{1.7}$$

Here

$$\hat{H}_0(K) = \hat{\Delta}_{x_1} + \cdots + \hat{\Delta}_{x_{N-1}} + \hat{\Delta}_N(K), \tag{1.8}$$

with

$$\hat{\Delta}_{x_1} = \underbrace{\hat{\Delta}_1 \otimes \hat{I}_d \otimes \cdots \otimes \hat{I}_d}_{N-1\,\text{times}}, \ldots, \hat{\Delta}_{x_{N-1}} = \underbrace{\hat{I}_d \otimes \cdots \otimes \hat{I}_d \otimes \hat{\Delta}_{N-1}}_{N-1\,\text{times}},$$

$$\hat{\Delta}_N(K) = \sum_{s \in \mathbb{Z}^d} e^{isK} \hat{\varepsilon}_N(s) \underbrace{T(s) \otimes \cdots \otimes T(s)}_{N-1\,\text{times}}$$

and

$$\hat{V} = \sum_{i<j} \hat{V}_{ij},$$

where,

$(\hat{V}_{ij}\hat{f})(x_1,\ldots,x_{N-1})=\hat{v}_{ij}(x_i-x_j)\hat{f}(x_1,\ldots,x_{N-1}),\quad 1\le i<j\le N-1,$

$(\hat{V}_{jN}\hat{f})(x_1,\ldots,x_{N-1})=\hat{v}_{jN}(x_j)\hat{f}(x_1,\ldots,x_{N-1}),\quad 1\le j\le N-1,$

$\hat{f}\in\ell^2((\mathbb{Z}^d)^{N-1}).$

Remark 1.2. *In what follows we call* $H(K),\ K\in\mathbb{T}^d$ *– the* N *-particle Schrödinger operator. We abuse this "name" for the other representations* $\hat{H}(K)$ *and* $\widetilde{H}(K)$.

2. Cluster operators

2.1. **Cluster decomposition.**

Definition 2.1. A partition C of the set $\{1,\ldots,N\}$ into nonintersecting subsets $C_1,C_2,...,C_\ell$ is called a *cluster decomposition*. Each C_k is called a *cluster*.

Let $C=\{C_1,C_2,\ldots,C_\ell\}$ be a cluster decomposition, and let the operator I^C be the sum of all potentials V_{ij} with i and j belonging to different clusters. Further we use the following notations: $|C_k|$ denotes the number of elements in C_k, the symbol $ij\in C$ means $i,j\in C_k$ for some $1\le k\le\ell$; analogously, the symbol $ij\notin C$ denotes the situation in which particles i and j are in different clusters (i.e. $i\in C_\alpha$ and $j\in C_\beta$ with $\alpha\neq\beta$) and $\#C$ denotes the number of elements in C, i.e. $\#C=\ell$; besides, set $V^C=\sum\limits_{ij\in C}V_{ij},\ I^C=\sum\limits_{ij\notin C}V_{ij}=V-V^C$.

Definition 2.2. The operator

$$H^C(K)=H(K)+I^C,\quad K\in\mathbb{T}^d$$

acting in $L_2((\mathbb{T}^d)^{N-1})$ is called the *cluster operator* of the corresponding cluster decomposition C.

2.2. **The discrete Schrödinger operator corresponding to a cluster.**

Let $C_\kappa,\ 1\le\kappa\le l$ be a cluster in a decomposition $C=\{C_1,\ldots,C_l\}$ and $m_\kappa=|C_\kappa|$. Suppose $C_\kappa=\{\alpha_1,\ldots,\alpha_{m_\kappa}\}$. For $k\in\mathbb{T}^d$ we define the $d\times(m_\kappa-1)$ -dimensional manifold

$$\mathbb{F}_k^{m_\kappa}=\{q=(q_1,\ldots,q_{m_\kappa})\in(\mathbb{T}^d)^{m_\kappa}:\quad q_1+\ldots q_{m_\kappa}=k\}$$

being homeomorphic to $(\mathbb{T}^d)^{m_\kappa-1}$. Define the operator $\widetilde{h}^{C_\kappa}(k)\ :\ L_2(\mathbb{F}_k^{m_\kappa})\to L_2(\mathbb{F}_k^{m_\kappa})$,

$$\widetilde{h}^{C_\kappa}(k)=\widetilde{h}_0^{C_\kappa}(k)-\widetilde{v}^{C_\kappa},\tag{2.1}$$

where

$$(\widetilde{h}_0^{C_\kappa}(k)f)(p_1,\ldots,p_{m_\kappa}) = \sum_{\alpha_i\in C_\kappa} \varepsilon_{\alpha_i}(p_i)\, f(p_1,\ldots,p_{m_\kappa})$$

and

$$\widetilde{v}^{C_\kappa} = \sum_{\substack{\alpha_i,\alpha_j\in C_\kappa,\\ \alpha_i<\alpha_j}} \widetilde{v}_{\alpha_i\alpha_j}$$

with

$$(\widetilde{v}_{\alpha_i\alpha_j} f)(p_1,\ldots,p_{m_\kappa}) = (2\pi)^{d/2}\int\limits_{\mathbb{T}^d} v_{\alpha_i\alpha_j}(p_i-q_i) f(p_1,\ldots,q_i,\ldots,p_i+p_j-q_i,\ldots,p_{m_\kappa})dq_i.$$

Note that in view of (1.5) and (1.6), the operator defined by (2.1) is the m_κ -particle discrete Schrödinger operator associated with the Hamiltonian of the system of particles C_κ.

2.3. **Spectrum of cluster operators.** Let $K\in\mathbb{T}^d$ and $C=\{C_1,\ldots,C_l\}$ be a cluster decomposition. Instead $H^C(K)$ we consider the unitary-equivalent fiber cluster operator

$$\widetilde{H}^C(K): L_2(\mathbb{F}_K)\to L_2(\mathbb{F}_K), \quad \widetilde{H}(K)=\widetilde{H}_0(K)+\widetilde{I}^C$$

with $I^C=\sum\limits_{ij\notin C}\widetilde{V}_{ij}$, where $\widetilde{H}_0(K)$ and $\widetilde{V}_{ij}$ are defined in (1.5).

It is easy to see that

$$\mathbb{F}_K = \bigsqcup_{\substack{k_1,\ldots,k_l\in\mathbb{T}^d,\\ k_1+\cdots+k_l=K}} \mathbb{F}_{k_1}^{|C_1|}\times\ldots\times\mathbb{F}_{k_l}^{|C_l|} := \bigsqcup_{\substack{k_1,\ldots,k_l\in\mathbb{T}^d,\\ k_1+\cdots+k_l=K}} \mathbb{F}(k_1,\ldots,k_l).$$

Hence Hilbert space $L_2(\mathbb{F}_K)$ is decomposed into von Neumann direct integral

$$L_2(\mathbb{F}_K) = \int\limits_{k_1+\cdots+k_l=K} \oplus L_2(\mathbb{F}(k_1,\ldots,k_l))dk, \tag{2.2}$$

where dk is a volume element of the manifold $\{(k_1,\ldots,k_l)\in(\mathbb{T}^d)^l : k_1+\cdots+k_l=K\}$.

Since fiber cluster operator $\widetilde{H}^C(K)$ commutes with the decomposable abelian group of multiplication operators by the function $\phi_s:\mathbb{F}_K\to\mathbb{C}$,

$$\phi_s(q)=\exp(\mathrm{i}(s_1,\sum_{\alpha\in C_1}q_\alpha))\times\cdots\times\exp(\mathrm{i}(s_l,\sum_{\alpha\in C_l}q_\alpha)), \quad s=(s_1,\ldots,s_l)\in(\mathbb{Z}^d)^l,$$

where $q=(q_1,\dots,q_N)\in\mathbb{F}_K,$ the decomposition (2.2) yields the decomposition of the operator $\widetilde{H}^C(K)$ into the *direct integral*

$$\widetilde{H}^C(K)=\int\limits_{k_1+\dots+k_l=K}\oplus\widetilde{H}^C(k_1,\dots,k_l)dk, \tag{2.3}$$

where the fiber operator $\widetilde{H}^C(k_1,\dots,k_l),\ (k_1,\dots,k_l)\in(\mathbb{T}^d)^l$ acts in the Hilbert space

$$L_2(\mathbb{F}(k_1,\dots,k_l))=L_2(\mathbb{F}_{k_1}^{|C_1|})\otimes\dots\otimes L_2(\mathbb{F}_{k_l}^{|C_l|})$$

as follows:

$$\begin{aligned}\widetilde{H}(k_1,\dots,k_l)=&\widetilde{h}^{C_1}(k_1)\otimes\widetilde{I}^{C_2}(k_2)\otimes\dots\otimes\widetilde{I}^{C_l}(k_l)+\cdots\\&+\widetilde{I}^{C_1}(k_1)\otimes\widetilde{I}^{C_2}(k_2)\otimes\dots\otimes\widetilde{h}^{C^l}(k_l),\end{aligned}\tag{2.4}$$

where $\widetilde{I}^{C_j}(k_j)$ is identity operator in $L_2(\mathbb{F}_{k_j}^{|C_j|})$ and the operator $\widetilde{h}^{C_j}(k_j):$ $L_2(\mathbb{F}_{k_j}^{|C_j|})\to L_2(\mathbb{F}_{k_j}^{|C_j|})$ is the m_j-particle discrete Schrödinger operator corresponding to the cluster C_j (see previous subsection).

Theorem 2.1. *Let $C=\{C_1,\dots,C_l\}$ be a cluster decomposition with $l=\#C\geq 2$ and $K\in\mathbb{T}^d$. Then for the spectrum $\sigma(H^C(K))$ of the cluster operator $H^C(K)$ the relations*

$$\sigma(H^C(K))=\sigma_{\text{ess}}(H^C(K))=\bigcup_{\substack{k_1,\dots,k_l\in\mathbb{T}^d,\\k_1+\dots+k_l=K}}\{\sigma(\widetilde{h}^{C_1}(k_1))+\dots+\sigma(\widetilde{h}^{C_l}(k_l))\}\tag{2.5}$$

hold, where $A+B=\{x+y:\ x\in A,\ y\in B\}$ for $A,B\subset\mathbb{R}$.

Proof. For any $k_1,\dots,k_l\in\mathbb{T}^d$ using (2.4) and the theorem on the spectra of tensor products of operators [15] we obtain

$$\sigma(\widetilde{H}^C(k_1,\dots,k_l))=\{\sigma(\widetilde{h}^{C_1}(k_1))+\dots+\sigma(\widetilde{h}^{C_l}(k_l))\}.\tag{2.6}$$

Moreover the theorem on the spectrum of decomposable operators (see e.g.[16]) together with (2.6) and (2.3) gives the representation

$$\sigma(\widetilde{H}^C(K))=\bigcup_{\substack{k_1,\dots,k_l\in\mathbb{T}^d,\\k_1+\dots+k_l=K}}\sigma(\widetilde{H}^C(k_1,\dots,k_l)).$$

and the equality $\sigma(\widetilde{H}^C(K))=\sigma_{\text{ess}}(\widetilde{H}^C(K))$.

□

Let $C=\{C_1,\dots,C_m\}$ and $D=\{D_1,D_2,\dots,D_n\}$ be cluster decompositions, $K\in\mathbb{T}^d$.

Definition 2.3. We say C is refinement of D and write $C \triangleright D$ if each D_j is a union of some C_i's.

Theorem 2.2. *Let* $K \in \mathbb{T}^d$. *Assume that* $C \triangleright D$, $C \neq D$. *Then* $\sigma(\hat{H}^C(K)) \subset \sigma_{\text{ess}}(\hat{H}^D(K))$.

Proof of Theorem 2.2. For simplicity we omit the dependence on K : $\hat{H}^L := \hat{H}^L(K)$ and $\hat{H}_0 := \hat{H}_0(K)$ Let $\lambda \in \sigma(\hat{H}^C) = \sigma_{ess}(\hat{H}^C)$. Then by Weil's criterion there exists a sequence $\{\hat{f}_n\} \subset \ell_2((\mathbb{Z}^d)^{N-1})$ weakly converging to 0 such that $\|\hat{f}_n\| = 1$ and $\|(\hat{H}^C - \lambda)\hat{f}_n\| \to 0$ as $n \to \infty$. Now by means of $\hat{f}_n$ we build the sequence $\{\hat{g}_r\} \subset \ell_2((\mathbb{Z}^d)^{N-1})$ such that $\|\hat{g}_r\| = 1$ and $\|(\hat{H}^D - \lambda)\hat{g}_r\| \to 0$ as $r \to \infty$.

Let $\psi : [0, \infty) \to \mathbb{R}$ be a Hölder continuous function of order δ such that $|\psi(t)| \leq 1$ and

$$\psi(t) = \begin{cases} 1, & t \geq 2, \\ 0, & 0 \leq t < 1. \end{cases}$$

Using ψ define the function:

$$\rho : (\mathbb{R}^d)^{N-1} \to \mathbb{R}, \quad \rho(y_1, \ldots, y_{N-1}) = \prod_{ij \notin C} \psi(|y_i - y_j|_+)$$

with $y_N = 0 \in \mathbb{R}^d$. Recall that the states in the range of ρ are such that the distance between any pair $\{i, j\}$ one from C_α and other from C_β, $\alpha \neq \beta$ is larger than 1.

Observe that since ψ is bounded and Hölder continuous of order δ, so is ρ. Let C_ρ be an Hölder constant of ρ.

For $r \in \mathbb{N}$ we define the function

$$\rho_r : (\mathbb{Z}^d)^{N-1} \to \mathbb{R}, \; \rho_r(y) = \rho(y/r).$$

Let $\mathcal{R}_r$ denote the multiplication operator by ρ_r in $\ell_2((\mathbb{Z}^d)^{N-1})$. Observe that $\operatorname{supp}(1 - \rho_r)$ is finite, and, thus, for any $r \in \mathbb{N}$ the operator $I - \mathcal{R}_r$ is compact.

Since f_n weakly converges to zero, there exists $N(r)$ such that $\|(I - \mathcal{R}_r)f_n\| < 1/2$ for all $n > N(r)$. This and the relation

$$1 = \|\hat{f}_n\| \leq \|(I - \mathcal{R}_r)\hat{f}_n\| + \|\mathcal{R}_r\hat{f}_n\|$$

imply that $\|\mathcal{R}_r\hat{f}_n\| \geq 1/2$ for all $n > N(r)$. We can assume that $N(r)$ is monotonously increasing. Now choose the sequence of natural numbers $n_1 < n_2 < \ldots$ such that $n_r > N(r)$ and consider the sequence $g_r = \mathcal{R}_r\hat{f}_{n_r}/\|\mathcal{R}_r\hat{f}_{n_r}\| \in \ell_2((\mathbb{Z}^d)^{N-1})$. Note that for any $\hat{f} \in \ell_2((\mathbb{Z}^d)^{N-1})$ we have

$$|< g_r, f >| \leq \quad 2|< \mathcal{R}_r \hat{f}_{n_r}, \hat{f} >| = 2|< \hat{f}_{n_r}, \mathcal{R}_r \hat{f} >| \leq$$
$$2\|\hat{f}_{n_r}\|\|\mathcal{R}_r \hat{f}\| = 2\|\mathcal{R}_r \hat{f}\| \to 0$$

as $r \to \infty$ and hence g_r weakly converges to 0.

By definition

$$\hat{H}^D = \hat{H}^C - \sum_{ij \in D,\, ij \notin C} \hat{V}_{ij}.$$

Note that $\hat{V}_{ij} \mathcal{R}_r = \mathcal{R}_r \hat{V}_{ij}$ and, so, $\hat{V}^D \mathcal{R}_r = \mathcal{R}_r \hat{V}^D$. Then

$$\|\mathcal{R}_r \hat{f}_{n_r}\|(\hat{H}^D - \lambda)\hat{g}_r = \mathcal{R}_r(\hat{H}^C - \lambda)\hat{f}_{n_r} + [\hat{H}_0, \mathcal{R}_r]\hat{f}_{n_r} -$$
$$- \sum_{ij \in D,\, ij \notin C} \hat{V}_{ij} \mathcal{R}_r \hat{f}_{n_r}$$

and since $\|\mathcal{R}_r \hat{f}_{n_r}\| \geq 1/2$ we have

$$\begin{aligned} \|(\hat{H}^D - \lambda)\hat{g}_r\| \leq & 2\|\mathcal{R}_r(\hat{H}^C - \lambda)\hat{f}_{n_r}\| + \\ & + 2\|[\hat{H}_0, \mathcal{R}_r]\hat{f}_{n_r}\| + \\ & + 2 \sum_{ij \in D,\, ij \notin C} \|\hat{V}_{ij} \mathcal{R}_r \hat{f}_{n_r}\|. \end{aligned} \tag{2.7}$$

Since ρ_r is bounded,

$$\|\mathcal{R}_r(\hat{H}^C - \lambda)\hat{f}_{n_r}\| \leq \|\rho_r\|_\infty \|(\hat{H}^C - \lambda)\hat{f}_{n_r}\|$$

and so

$$\lim_{r \to \infty} \|\mathcal{R}_r(\hat{H}^C - \lambda)\hat{f}_{n_r}\| = 0. \tag{2.8}$$

Further according to Lemma A.1

$$\|[\hat{H}_0, \mathcal{R}_r]\| \leq \frac{C_\rho}{r^\delta} \sum_{i=1}^{N} \left[\sum_y |y|_+^\delta \, |\hat{\varepsilon}_i(y)| \right].$$

and, thus,

$$\lim_{r \to \infty} \|[\hat{H}_0, \mathcal{R}_r]\hat{f}_{n_r}\| = 0. \tag{2.9}$$

Now take any $ij \in D$, $ij \notin C$. By Remark A.1

$$\|\hat{V}_{ij} \mathcal{R}_r\| \leq \|\rho\|_\infty \sup_{|y|_+ \geq r} |\hat{v}_{ij}(y)|.$$

Since $\sup_{|y|_+ \geq r} |\hat{v}_{ij}(y)| \to 0$ as $r \to \infty$,

$$\lim_{r \to \infty} \|\hat{V}_{ij} \mathcal{R}_r \hat{f}_{n_r}\| = 0. \tag{2.10}$$

Now relations (2.7)–(2.10) imply

$$\lim_{r\to\infty} \|(\hat{H}^D - \lambda)\hat{g}_r\| = 0.$$

Since g_r weakly converges to 0, Weil's criterion implies that $\lambda \in \sigma_{\text{ess}}(\hat{H}^D)$. Theorem 2.2 is proved. □

Remark 2.1. *Theorem 2.2 implies that* $\sigma(H^D(K)) \subset \sigma_{\text{ess}}(H(K))$ *for any cluster decomposition* D, $\#D \geq 2$.

3. Main results

The main result of this paper is the following analogue of HVZ theorem.

Theorem 3.1. *Fix* $K \in \mathbb{T}^d$. *Assume Hypothesis 1.1. Then for the essential spectrum* $\sigma_{\text{ess}}(H(K))$ *of the operator* $H(K)$ *the following relation holds:*

$$\sigma_{\text{ess}}(H(K)) = \bigcup_{D\in\Xi,\, \#D\geq 2} \sigma(H^D(K)) = \bigcup_{D\in\Xi,\, \#D=2} \sigma(H^D(K)),$$

where Ξ *is the set of all cluster decompositions.*

Proof of Theorem 3.1. The relation

$$\bigcup_{D\in\Xi,\, \#D\geq 2} \sigma(H^D(K)) = \bigcup_{D\in\Xi,\, \#D=2} \sigma(H^D(K))$$

is a simple consequence of Theorem 2.2. Moreover, due to Remark 2.1

$$\sigma_{\text{ess}}(H(K)) \supset \bigcup_{D\in\Xi,\, \#D\geq 2} \sigma(H^D(K))$$

Now we prove

$$\sigma_{\text{ess}}(H(K)) \subset \sigma_c = \bigcup_{\#D\geq 2} \sigma(H^D). \tag{3.1}$$

It is enough to show that $\sigma(H(K))$ is discrete on the complement of σ_c. The main tool of proving inclusion (3.1) is the functional equation of S. Weinberg and C. van Winter for the resolvent $G(z) = (H(K) - z)^{-1}$.

Let $G_0(z) = (H_0(K) - z)^{-1}$. Observe that for any interaction V_{ij} the relation

$$\lim_{\operatorname{Re} z\to -\infty} \|V_{ij}G_0(z)\| = 0$$

holds.

Let $M < 0$ be such that $\|G_0(z)V_{ij}\| < 2/(N(N-1))$ for all $i < j$ if $\operatorname{Re} z < M$. Thus, for $\operatorname{Re} z < M$, the iteration solution of the resolvent equation

$$G(z) = G_0(z) + G_0(z)VG(z)$$

exists and is given by the series

$$G(z) = \sum_{n=0}^{\infty} \sum_{(i_1j_1),\ldots,(i_nj_n)} G_0(z)V_{i_1j_1}G_0(z)V_{i_2j_2}\ldots G_0(z)V_{i_nj_n}G_0(z), \quad (3.2)$$

which converges absolutely in the sense of operator norm. As in [3], [1] and [16], we define the graph representing the term $G_0(z)V_{i_1j_1}G_0(z)V_{i_2j_2}\ldots G_0(z)V_{i_nj_n}G_0(z)$ of the series (3.2) to consist of N horizontal lines ("particles") and n vertical lines ("interactions") linking just the pairs of particles $i_1j_1,\ldots,i_nj_n$ from left to right (see Figure 1).

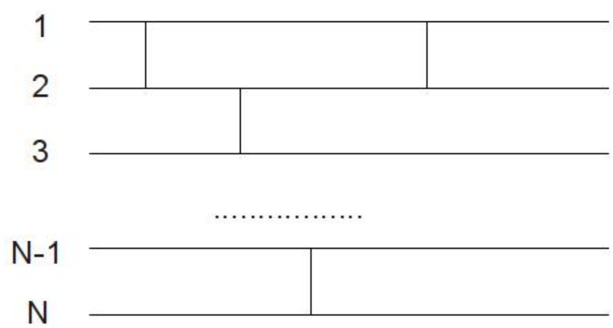

FIGURE 1. Graph representing the term of series (3.2)

Each graph G consists of a certain number k of connected parts $(1 \leq k \leq N)$ – only the endpoints of the interactions counting as connections – and thus defines a cluster decomposition $D(G)$ of length $k:$ two particles belong to the same cluster if their lines belong to the same connected part of G. Recall that $D_k \rhd D_l$ $(k > l)$ means that each cluster in D_l is a union of clusters in D_k. A graph G is called D_l-disconnected if $D_k(G) \not\rhd D_l$ i.e. none of the interactions of G link different clusters of D_l. Identifying graphs with terms in the series (3.2), we find, for $\operatorname{Re} z < M$,

$$\sum(\text{all } D_l\text{-disconnected graphs}) = (H_{D_l} - z)^{-1} = G_{D_l}(z). \quad (3.3)$$

Suppose G has l links. Denote by $G_0,\ldots,G_l$ the diagrams obtained by keeping first $0,\ldots,l$ links counting from left. Then consider distinct cluster decompositions $D_N,\ldots,D_k$ among $D(G_0),\ldots,D(G_l)$. Here D_m has exactly m clusters. We write $S(G)$ for the associated *string* of $G:$

$$S(G) = (D_N, D_{N-1},\ldots,D_k), \quad D_{i+1} \rhd D_i, \quad N \geq k \geq 1, \quad (3.4)$$

where D_k is the decomposition corresponding to the whole graph according to (a). We recall that if $k = 1$ then the string is called *connected* and *disconnected* if $k > 1$. k is called *index* of S and we write $i(S) = k$.

In this way, each graph G uniquely determines a sequence $S(G)$ of type (3.4). To sum up all diagrams, we first sum all diagrams with the same associated string S. Any graph of this class has the form

$$G_0 \prod_{i=N-1}^{k} \left[\begin{pmatrix} \text{any interaction linking different} \\ \text{clusters of } D_{i+1} \text{ but not of } D_i \end{pmatrix} \begin{pmatrix} \text{any } D_i\text{-disconnected} \\ \text{graph} \end{pmatrix} \right] \tag{3.5}$$

where the "factors" are ordered from left to right as i decreases. By (3.5) and (3.3), this yields

$$\begin{aligned} G_S(z) = &\sum (\text{all graphs of class } S) = \\ &= G_0 \prod_{i=N-1}^{k} \left[\begin{pmatrix} \text{sum of all potentials} \\ \text{linking different clusters} \\ \text{of } D_{i+1} \text{ but not of } D_i \end{pmatrix} \begin{pmatrix} \text{sum of all} \\ D_i\text{-disconnected} \\ \text{graphs} \end{pmatrix} \right] = \\ &= G_{D_N}(z) V_{D_N D_{N-1}} G_{D_{N-1}}(z) V_{D_{N-1} D_{N-2}} \dots V_{D_{k+1} D_k} G_{D_k}(z), \end{aligned}$$

where $G_{D_N}(z) = G_0,\ V_{D_i D_{i-1}} = I^{D_i} - I^{D_{i-1}}$ and I^D is defined in subsection 3.1.

The remaining finite sum over associated strings is carried out in two steps: first, we sum over all $S = (D_N, \dots, D_k)$ with $k \geq 2$. This is the sum of all disconnected graphs and defines the disconnected part $D(z)$ of $G(z)$:

$$D(z) = \sum_{\text{all } S \text{ with } k \geq 2} G_S(z).$$

Similarly, we obtain the connected part $C(z)$ by summing over all S with $k = 1$ (sum of all connected graphs):

$$C(z) = \sum_{\text{all } S \text{ with } k = 1} G_S(z).$$

Noting that each term of last sum ends with a factor $G_{D_1}(z) = G(z)$, we finally arrive at

$$G(z) = D(z) + C(z) = D(z) + I(z)G(z), \tag{3.6}$$

$$I(z) = \sum_{\text{all } S \text{ with } k = 1} G_{D_N}(z) V_{D_N D_{N-1}} G_{D_{N-1}}(z) \dots G_{D_2}(z) V_{D_2 D_1}. \tag{3.7}$$

Remark 3.1. *According to Theorem B.1, the operator $G_0(z)V_{i_1 j_1} \dots G_0(z)V_{i_n j_n}$ is compact if and only if the corresponding graph is connected. We recall* *that for* $\operatorname{Re} z < -M$ *the operator*

$$G_{D_N}(z)V_{D_N D_{N-1}}G_{D_{N-1}}(z)\dots G_{D_2}(z)V_{D_2 D_1}$$

is a norm convergent sum of the operators corresponding to the connected graphs, and thus it is a compact operator.

Observe that $I(z)$ and $D(z)$ are defined for all $z \notin \sigma_c$ and are bounded operators, but the functional equation (3.6) for $G(z)$ is established so far only for $\operatorname{Re} z < M$. Since $V_{ij}G_D(z)$ is holomorphic in $\mathbb{C} \setminus \sigma_c$ for all ij and for all cluster decompositions $D \in \Xi$, the operator-valued functions $I(z)$ and $D(z)$ can be analytically extended to $z \in \mathbb{C} \setminus \sigma_c$. Thus, by analytic continuation, (3.6) extends to any $z \notin \sigma_c \cup \sigma(H(K)) = \sigma(H(K))$. By definition of $I(z)$ it is easy to see that

$$\lim_{\operatorname{Re} z \to -\infty} \|I(z)\| = 0.$$

Lemma 3.1. *The operator* $I(z)$ *is compact for all* $z \notin \sigma_c$.

Proof. Since

$$I(z) = \sum_{\text{all } S \text{ with } k=1} G_{D_N}(z)V_{D_N D_{N-1}}G_{D_{N-1}}(z)\dots G_{D_2}(z)V_{D_2 D_1}$$

consists of the sum of finitely many compact operators for $\operatorname{Re} z < -M$. By analyticity $I(z)$ is compact for all $z \notin \sigma_c$ (see Theorem XIII.5 in [16]). $\square$

Since $I(z)$ is a compact-valued operator function on $\mathbb{C} \setminus \sigma_c$ and $I - I(z)$ is invertible if z is real and very negative, the analytic Fredholm theorem implies that there is a discrete set $S \subset \mathbb{C} \setminus \sigma_c$ so that $(I - I(z))^{-1}$ exists and is analytic in $S \subset \mathbb{C} \setminus (\sigma_c \cup S)$ and meromorphic in $S \subset \mathbb{C} \setminus \sigma_c$ with finite rank residues. Thus

$$(I - I(z))^{-1} D(z) \equiv f(z)$$

is analytic in $\mathbb{C} \setminus (\sigma_c \cup S)$ with finite rank residues at points of S. Let $z \notin S$, $\operatorname{Im} z \neq 0$. Then, by (3.6), $f(z) = (H(K) - z)^{-1}$. In particular, $f(z)(H(K) - z)\phi = \phi$ for any $\phi \in L_2((\mathbb{T}^d)^{N-1})$. By analytic continuation, this holds for any $z \notin S \cup \sigma_c$. This proves that $\sigma(H(K)) \setminus \sigma_c$ consists of only isolated points with the edges of segments of σ_c as possible limit points. Finally, since $(H(K) - z)^{-1} = f(z)$ has finite rank residues at any point $\lambda \in S$, we conclude that spectral projection

$$P_\lambda = (-2\pi i)^{-1} \oint_{|z-\lambda|=\varepsilon} (H(K) - z)^{-1} dz$$

is finite dimensional, i.e. any $\lambda \in \sigma(H(K)) \setminus \sigma_c$ is in $\sigma_{\text{disc}}(H(K))$. This concludes the proof of Theorem 3.1. $\square$

Appendix A. Some properties of operators $\hat{H}_0(K)$ and $\hat{V}$

Fix $K \in \mathbb{T}^d$. Let $\phi : (\mathbb{R}^d)^{N-1} \to \mathbb{R}$ be a bounded Hölder continuous function of order δ :

$$|\phi(x) - \phi(y)| \le C|x-y|^{\delta}, \tag{A.1}$$

where $C > 0$ is some constant and $x, y \in (\mathbb{R}^d)^{N-1}$. Define the sequence

$$\phi : (\mathbb{Z}^d)^{N-1} \to \mathbb{R},\ \phi_r(x) = \phi(x/r), \quad r = 1, 2, \dots .$$

Since ϕ_r are bounded, the operators $\Phi_r \hat{H}_0(K)$, and $\hat{H}_0(K)\Phi_r$ are well-defined, where Φ_r is a multiplication operator by ϕ_r.

We recall that according to Hypothesis 1.1 $\sum\limits_y |y|_+^{\delta}\, |\hat{\varepsilon}_j(y)| < \infty$ for all $i = 1, \dots, N$.

Lemma A.1. *For any* $r \in \mathbb{N}$

$$\big\|[\hat{H}_0(K), \Phi_r]\big\| \le \frac{C}{r^{\delta}} \sum_{j=1}^{N} \left[\sum_y |y|_+^{\delta}\, |\hat{\varepsilon}_j(y)| \right].$$

Proof of Lemma A.1. For $y \in \mathbb{Z}^d$ let us introduce

$$\hat{T}_j(y) = \underbrace{I_d \otimes \cdots \otimes I_d}_{j-1\,\text{times}} \otimes T(y) \otimes \underbrace{I_d \otimes \cdots \otimes I_d}_{N-j-1\,\text{times}}, \quad j = 1, \dots, N-1,$$

$$\hat{T}_N(y) = e^{iyK} \underbrace{T(y) \otimes \cdots \otimes T(y)}_{N-1\,\text{times}}.$$

Then

$$\hat{\Delta}_{x_j} = \sum_{y \in \mathbb{Z}^d} \hat{\varepsilon}_j(y) \hat{T}_j(y), \quad j = 1, \dots, N,$$

with $\hat{\Delta}_{x_N} := \hat{\Delta}_N(K)$. By the subadditivity of the norm one has

$$\big\|[\hat{H}_0(K), \Phi_r]\big\| \le \sum_{j=1}^{N} \big\|[\hat{\Delta}_{x_j}, \Phi_r]\big\|.$$

Now it is enough to prove that

$$\big\|[\hat{\Delta}_{x_j}, \Phi_r]\big\| \le \frac{C}{r^{\delta}} \sum_y |y|_+^{\delta}\, |\hat{\varepsilon}_j(y)|. \tag{A.2}$$

Let $\hat{f}, \hat{g} \in \ell_2$. Note that

$$[T_j(y), \Phi_r(x)]\hat{f}(x) = (T_j(y)\phi_r(x) - \phi_r(x)) \cdot T_j(y)\hat{f}(x).$$

Then

$$|\langle[\hat{\Delta}_{x_j},\Phi_r]\hat{f},\hat{g}\rangle| \le \sum_{x\in(\mathbb{Z}^d)^{N-1}} |\hat{g}(x)|\times$$
$$\times \sum_y |\hat{\varepsilon}_j(y)|\cdot|T_j(y)\phi_r(x)-\phi_r(x)|\cdot|T_j(y)\hat{f}(x)|.$$

By (A.1) we get

$$|T_j(y)\phi_r(x)-\phi_r(x)| \le \frac{C|y|_+^\delta}{r^\delta}.$$

Consequently,

$$|\langle[\hat{\Delta}_{x_j},\Phi_r]\hat{f},\hat{g}\rangle| \le \frac{C}{r^\delta}\sum_{x\in(\mathbb{Z}^d)^{N-1}}\sum_y |y|_+^\delta|\hat{\varepsilon}_j(y)|\cdot|T_j(y)\hat{f}(x)|\cdot|\hat{g}(x)| =$$
$$=\frac{C}{r^\delta}\sum_y |\hat{\varepsilon}_j(y)||y|_+^\delta \sum_{x\in(\mathbb{Z}^d)^{N-1}} |T_j(y)\hat{f}(x)|\cdot|\hat{g}(x)|.$$

According to

$$\sum_{x\in(\mathbb{Z}^d)^{N-1}} |T_j(y)\hat{f}(x)|\cdot|\hat{g}(x)| \le \|\hat{f}\|\cdot\|\hat{g}\|,$$

we get

$$|\langle[\hat{\Delta}_{x_j},\Phi_r]\hat{f},\hat{g}\rangle| \le \frac{C}{r^\delta}\left(\sum_y |y|_+^\delta|\hat{\varepsilon}_j(y)|\right)\|\hat{f}\|\,\|\hat{g}\|.$$

Now (A.2) follows from the fact that

$$\|A\| = \sup_{\substack{f,g\in\mathcal{H},\\ ||f||=||g||=1}} |\langle Af,g\rangle|$$

where $A:\mathcal{H}\to\mathcal{H}$ is linear bounded operator.

□

Remark A.1. *Let* $\psi:\mathbb{R}^d\to\mathbb{R}$ *be bounded function with support* $\{x\in\mathbb{R}^d:\ |x|_+\ge 1\}$. *Define the sequence*

$$\psi_r:\mathbb{Z}^d\to\mathbb{R}\quad \psi_r(x)=\psi(x/r),\quad r=1,2,\dots.$$

For fixed $i,j\in\overline{1,N},\ i<j$ *let* Ψ_r *denote the multiplication operator by the function* $\psi_r(y_i-y_j)$ *in* $\ell_2((\mathbb{Z}^d)^{N-1})$, *here* $y_N=0$. *Then*

$$\|V_{ij}\Psi_r\| \le A \sup_{x\in\mathbb{Z}^d,\ |x|_+\ge r} |\hat{v}_{ij}(x)|,$$

where $A=\sup_x|\psi(x)|$.

APPENDIX B. CONNECTEDNESS OF THE GRAPH

We begin with the following proposition.

Proposition B.1. *Let* $E : (\mathbb{T}^d)^{N-1} \to \mathbb{C}$ *be a continuous function with Fourier coefficients* $\hat{E} \in \ell_1((\mathbb{Z}^d)^{N-1})$*. Assume that* G_0 *is multiplication operator by the function* $E(\cdot)$ *in* $L_2((\mathbb{T}^d)^{N-1})$ *and let* $T = V_{\alpha_1\beta_1}G_0 \dots G_0 V_{\alpha_n\beta_n},\ \alpha_j, \beta_j \in \{1, \dots, N\},\ \alpha_j < \beta_j$*. If we formally define the graph* G *corresponding to this operator and if this graph is connected, then* T *is compact operator.*

Proof. In order to show that the operator T is compact it is enough to prove that the operator

$$\hat{T} = \mathcal{F}_{N-1}^{-1} T \mathcal{F}_{N-1} := \hat{V}_{\alpha_1\beta_1}\hat{G}_0 \dots \hat{G}_0 \hat{V}_{\alpha_n\beta_n}$$

is compact, where $\hat{G}_0 = \mathcal{F}_{N-1}^{-1} G_0 \mathcal{F}_{N-1}$.

In order to prove that $\hat{T}$ is compact we show that the image $\hat{T}(B)$ of unit ball $B \subset \ell_2((\mathbb{Z}^d)^{N-1})$ is precompact. So it is enough to prove that for any $\varepsilon > 0$ there exits $R = R_\varepsilon$ such that

$$\Delta_R(\hat{f}) = \sum_{|x|>R} |(\hat{T}\hat{f})(x)|^2 < \varepsilon$$

uniformly in B.

Since $\|\hat{f}\| \le 1$, from the definition of $\hat{G}_0$ and $\hat{V}_{ij}$ one can easily check that

$$\Delta_R(\hat{f}) \le \sup_{|x|>R} \sum_{w^{(1)},\dots,w^{(n-1)}} \prod_{j=1}^{n-1} \hat{E}(w^{(j)})^2 \times$$
$$\times \prod_{j=1}^{n} \hat{v}_{\alpha_j\beta_j}\left(\left(x + \sum_{i=1}^{j-1} w^{(i)} \right)_{\alpha_j} - \left(x + \sum_{i=1}^{j-1} w^{(i)} \right)_{\beta_j} \right)^2 .$$

Let $M = \max_{ij} \sup_x |\hat{v}_{ij}(x)| < \infty$. For $L > 0$ we define

$$B_L^m = \{x = (x_1, \dots, x_m) \in ((\mathbb{Z}^d)^{N-1})^m :\ |x_i|_+ < L,\ i = 1, \dots, n\}.$$

Since $\hat{E} \in \ell_1$, the series

$$A = \sum_{w^{(1)},\dots,w^{(n-1)}} \prod_{j=1}^{n-1} \hat{E}(w^{(j)})^2$$

is convergent. Thus for given $\varepsilon > 0$ there exists $L = L_\varepsilon > 0$ such that

$$\sum_{w\in((\mathbb{Z}^d)^{N-1})^{n-1}\setminus B_L^{n-1}} \prod_{j=1}^{n-1} \hat{E}(w^{(j)})^2 \leq \frac{\varepsilon}{2M^{2n}},$$

here $w = (w^{(1)}, \dots, w^{(n-1)})$. Hence

$$\sum_{w\in((\mathbb{Z}^d)^{N-1})^{n-1}\setminus B_L^{n-1}} \prod_{j=1}^{n-1} \hat{E}(w^{(j)})^2 \times \\ \times \prod_{j=1}^{n} \hat{v}_{\alpha_j\beta_j} \left(\left(x + \sum_{i=1}^{j-1} w^{(i)} \right)_{\alpha_j} - \left(x + \sum_{i=1}^{j-1} w^{(i)} \right)_{\beta_j} \right)^2 \leq \frac{\varepsilon}{2},$$

Now consider the finite sum

$$\sum_{w\in B_L^{n-1}} \prod_{j=1}^{n-1} \hat{E}(w^{(j)})^2 \times \\ \times \prod_{j=1}^{n} \hat{v}_{\alpha_j\beta_j} \left(\left(x + \sum_{i=1}^{j-1} w^{(i)} \right)_{\alpha_j} - \left(x + \sum_{i=1}^{j-1} w^{(i)} \right)_{\beta_j} \right)^2 .$$

First we prove that since G is connected if $x \to \infty$, then at least one of $|x_{\alpha_j} - x_{\beta_j}|_+$ tends to ∞. Assume the converse. Let there exist $R_0 > 0$ such that $|x_{\alpha_j} - x_{\beta_j}|_+ < R_0$ for each $j = 1, \dots, n$. Then since G is connected and $x_N = 0$, one can easily show that $|x_{\alpha_j}|_+ < NR_0$ and so $x \in B_{NR_0}^{N-1}$ which contradicts to $x \to \infty$.

Since $\hat{v}_{\alpha_j\beta_j}(y) \to 0$ as $y \to \infty$ for all $j = 1, \dots, n$, choose $r = r_\varepsilon > 0$ such that for $|y|_+ \geq r$

$$\sup_{|y|_+ > r} |\hat{v}_{\alpha_j\beta_j}(y)|^2 < \frac{\varepsilon}{2AM^{2n-2}}, \; j = 1, \dots, n.$$

Then provided that $|x| > Nr + nL$ (and so $|(x + \sum_{i=1}^{j_0-1} w^{(i)})_{\alpha_{j_0}} - (x + \sum_{i=1}^{j_0-1} w^{(i)})_{\beta_{j_0}}| \geq r$ for some $j_0 = j_0(x)$), we get

$$\sum_{w \in B_L^{n-1}} \prod_{j=1}^{n-1} \hat{E}(w^{(j)})^2 \times$$

$$\times \prod_{j=1}^{n} \hat{v}_{\alpha_j \beta_j} \left(\left(x + \sum_{i=1}^{j-1} w^{(i)} \right)_{\alpha_j} - \left(x + \sum_{i=1}^{j-1} w^{(i)} \right)_{\beta_j} \right)^2 \leq$$

$$\leq \sup_{|x| > NR + nL} \hat{v}_{\alpha_{j_0} \beta_{j_0}} \left(\left(x + \sum_{i=1}^{j-1} w^{(i)} \right)_{\alpha_{j_0}} - \left(x + \sum_{i=1}^{j-1} w^{(i)} \right)_{\beta_{j_0}} \right)^2 \sum_{w \in (\mathbb{Z}^d)^{n-1}} \prod_{j=1}^{n-1} \hat{E}(w^{(j)})^2 \times$$

$$\times \prod_{j=1,\, j \neq j_0}^{n} \hat{v}_{\alpha_j \beta_j} \left(\left(x + \sum_{i=1}^{j-1} w^{(i)} \right)_{\alpha_j} - \left(x + \sum_{i=1}^{j-1} w^{(i)} \right)_{\beta_j} \right)^2 \leq$$

$$\leq \frac{\varepsilon}{2AM^{2n-2}} \cdot AM^{2n-2} = \frac{\varepsilon}{2}.$$

Consequently,

$$\Delta_{Nr+nL}(\hat{f}) \leq \varepsilon$$

uniformly in B. Thus $\hat{T}(B)$ is compact. □

Fix $K \in \mathbb{T}^d$ and $z \notin \sigma_c$.

Theorem B.1. *Let* $T = G_0(z) V_{\alpha_1 \beta_1} G_0(z) \ldots G_0(z) V_{\alpha_n \beta_n}$, $\alpha_j, \beta_j \in \{1, \ldots, N\}$, $\alpha_j < \beta_j$, *be the operator corresponding to the graph* G*. Then* T *is compact if and only if* G *is connected.*

Proof. Assume that G is not connected and corresponding cluster decomposition is $D(G) = \{D_1, \ldots, D_l\}$ with $l \geq 2$. Without loss of generality we may assume that $N \in D_\ell$. We define the abelian group of unitary operators U_s, $s \in \mathbb{Z}^d$ in $L_2((\mathbb{T}^d)^{N-1})$ as follows:

$$(U_s f)(p) = \exp(\mathrm{i}(s, \sum_{\alpha \in D_1} p_\alpha)) f(p).$$

It is easy to see that the operators $V_{\alpha_j \beta_j}$, $j = 1, \ldots, n$ and $G_0(z)$ commute with U_s. Hence T commutes with U_s. Moreover, Riemann-Lebesgue theorem implies that $U_s \to 0$ weakly as $s \to \infty$. Take $f_0 \in L_2((\mathbb{T}^d)^{N-1})$ such that $\|f_0\| = 1$ and $T(z) f_0 \neq 0$. Then $U_s f_0 \to 0$ weakly as $s \to \infty$.

But since U_s unitary and commutes with T we have

$$\|T(U_s f_0)\| = \|U_s(T f_0)\| = \|T f_0\| \not\to 0.$$

Hence T is not compact.

Now assume that G is connected. Set $E = (\mathcal{E}_K - z)^{-1}$. Since E is continuous on $(\mathbb{T}^d)^{N-1}$, according to [37] there exists a sequence of trigonometric polynomials $P_s : (\mathbb{T}^d)^{N-1} \to \mathbb{C}$, $s = (s_1, \ldots, s_{N-1}) \in \mathbb{N}^d$ such that P_s converges uniformly to E as $s \to \infty$. Clearly, $\hat{P}_s = \mathcal{F}_{N-1}^{-1} P_s \in \ell_1((\mathbb{Z}^d)^{N-1})$. Let G_0^s denote the multiplication operator by the function P_s. Note that G_0^s converges to $G_0(z)$ in operator norm.

Define the sequence of operators

$$T_s = G_0^s V_{\alpha_1\beta_1} G_0^s \ldots V_{\alpha_n\beta_n} G_0^s.$$

According to proposition B.1, T_s is compact. Since $G_0(z)$ and V_{ij} are bounded operators and n is finite, $T_s \to T$ in operator norm, implying T is also compact.

□

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REFERENCES

[1] Hunziker, M.: On the spectra of Schrödinger multiparticle Hamiltonians. Helv. Phys. Acta **39**, 451-462 (1966).

[2] Balslev, E.: Spectral Theory of Schrödinger Operators of Many-Body Systems with Permutations and Rotation symmetries. Ann. of Phys. **73**, 49-107 (1972).

[3] Weinberg, S.: Systematic Solutions of Multiparticle Scattering Problems. Phys. Rev. **133**, B232-B256 (1964).

[4] Enss, V.: A Note on Hunziker's Theorem. Commun. Math. Phys. **52**, 233-238 (1977).

[5] Zoladek, H.: Essential spectrum of an N-particle additive cluster operator. Teoret. Mat. Fiz. **53**, 216-226 (1982); English Transl. in Theor. and Math. Phys. **53** (1982).

[6] Simon, B.: Geometric Methods in Multiparticle Quantum Systems, Commun. Math. Phys. **55**, 259-274 (1977).

[7] Mogilner, A.: Hamiltonians in Solid-State Physics as Multiparticle Discrete Schrodinger Operators: Problems and Results. Adv. in Sov. Math. **5**, 139-194 (1991).

[8] Mattis, D.: The few-body problem on a lattice, Rev. Mod. Phys. **58:2**, 361-379 (1986).

[9] Rabinovich, V.S., Roch, S.: The essential spectrum of Schrodinger operators on lattices. J. Phys. A: Math. Gen. **39**, 8377-8394 (2006).

[10] Muminov, M.E.: A Hunziker-Van Winter-Zhislin theorem for a four-particle lattice Schrödinger operator. Theor. Math. Phys. **148:3**, 1236-1250 (2006).

[11] Van Winter, C.: Theory of finite systems of particles, I. Mat.-Fys. Skr. Danske Vid. Selsk **1**, 1-60 (1960).

[12] Zhislin, G.M.: Investigation of the spectrum of the Schrodinger operator for a many particle system. Trudy Moskov. Mat. Ob-va **9**, 81-120 (1960).

[13] Albeverio, S., Lakaev, S.N., Muminov, Z.I.: On the structure of the essential spectrum for the three-particle Schrodinger operators on lattices. Math. Nachr. **280**, No. 7, 699-716 (2007).

[14] Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B.: Schrödinger operators with application to quantum mechanics and global geometry. Springer, Berlin (1987).

[15] Reed, M., Simon, B.: Methods of Modern Mathematical Physics. Vol. I: Functional Analysis. Academic Press, New York (1972).

[16] Reed, M., Simon, B.: Methods of Modern Mathematical Physics. Vol. IV: Analysis of Operators. Academic Press, New York (1978).

[17] Jörgens, K., Weidmann, J.: Spectral properties of Hamiltonian Operators. Lecture Notes in Mathematics **313**, Berlin-Heidelberg-New York: Springer (1973).

[18] Lewis, R.T., Siedentop, H., Vugalter, S.: The essential spectrum of relativistic multi-particle operators. Ann. Inst. Henri Poincaré: Phys. Theor. **67(1)**, 1-28 (1997).

[19] Jakubassa-Amundsen, D.H.: Localization of the essential spectrum for relativistic N-electron ions and atoms. Doc. Math. **10**, 417-445 (2005).

[20] Jakubassa-Amundsen, D.H.: The HVZ theorem for a pseudo- relativistic operator. Ann. Henri Poincaré **8(2)**, 337-360 (2007).

[21] Morozov, S., Vugalter, S.: Stability of atoms in the Brown- Ravenhall model. Ann. Henri Poincaré **7(4)**, 661-687 (2006).

[22] Matte, O., Stockmeyer, E.: Spectral theory of no-pair Hamiltonians. Rev. Math. Phys. **22:1**, 1-53 (2010).

[23] Simon, B.: Quantum Mechanics for Hamiltonians defined as quadratic forms. Princeton Univ. Press, Princeton, New Jersey (1972).

[24] Faddeev, L.D.: Mathematical questions in the quantum theory of scattering for a system of three particles. Trudy Mat. Inst. Steklov. **69** (1963); Transl. Israel Program for Scientific Translations (1965).

[25] Yakubovsky, O.A.: On the integral equations in the theory of N -particle scattering. Sov. Journ. Nucl. Phys. **5**, 937-942 (1967).

[26] Georgescu, V., Iftimovici, A.: Crossed products of C^* -algebras and spectral analysis of quantum Hamiltonians. Commun. Math. Phys. **228**, 519-560 (2002).

[27] Mogilner, A.I.: Ph. D. Thesis, Sverdlovsk University, Sverdlovsk (1989).

[28] Rabinovich, V.S.: Essential spectrum of perturbed pseudodifferential operators. Applications to Schrödinger, KleinGordon, and Dirac operators. Russ. J. Math. Phys. **12**, 62-80 (2005).

[29] Muminov, M.I.: The infiniteness of the number of eigenvalues in the gap in the essential spectrum for the three-particle Schrödinger operator on a lattice. Teoret. Mat. Fiz. **159:2**, 299-317 (2009).

[30] Lakaev, S.N.: On an infinite number of three-particle bound states of a system of quantum lattice particles. Theor. and Math. Phys. **89**, No.1, 1079-1086 (1991).

[31] Lakaev, S.N.: The Efimov's Effect of a system of Three Identical Quantum lattice Particles. Funkcionalnii analiz i ego priloj., **27**, No.3, 15-28, translation in Funct. Anal. Appl. (1993).

[32] Lakaev, S.N., Abdullaev J.I.: Finiteness of the discrete spectrum of the three-particle Schrödinger operator on a lattice. Theor. Math. Phys. **111**, 467-479 (1997).

[33] Lakaev, S.N., Samatov, S.M.: On the finiteness of the discrete spectrum of the Hamiltonian of a system of three arbitrary particles on a lattice. Teoret. Mat. Fiz. **129**, No. 3, 415–431 (2001).

[34] Lakaev, S.N., Abdullaev, J.I.: The spectral properties of the three-particle difference Schrödinger operator. Funct. Anal. Appl. **33**, No. 2, 84-88 (1999).

[35] Lakaev, S.N., Abdullaev, Zh.I.: The spectrum of the three-particle difference Schrödinger operator on a lattice, Math. Notes, **71**, No. 5-6, 624-633 (2002).

[36] Gohberg, I., Goldberg, S., Kaashoek, M.A.: Basic Classes of Linear Operators. Birkhäuser Verlag, Basel, 2003.

[37] Zygmund, A.: Trigonometric series: Volumes I & II Combined. Cambridge University Press, 1988.

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