EXISTENCE OF STATIONARY SOLUTIONS FOR SOME INTEGRO-DIFFERENTIAL EQUATIONS WITH SUPERDIFFUSION

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Abstract: The work deals with the existence of solutions of an integro-differential equation arising in population dynamics in the case of anomalous diffusion. The proof of existence of solutions relies on a fixed point technique. Solvability conditions for non-Fredholm elliptic operators in unbounded domains are being used.

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1. Introduction

In the present article we study the existence of stationary solutions of the integro-differential equation

$$\frac{\partial u}{\partial t} = -D\sqrt{-\Delta}u + \int_{\mathbb{R}^d} K(x-y)g(u(y,t))dy + f(x), \tag{1.1}$$

which appears in cell population dynamics. The space variable x is correspondent to the cell genotype, $u(x,t)$ stands for the cell density as a function of their genotype and time. The right side of this equation describes the evolution of cell density due to cell proliferation, mutations and cell influx. Namely, the anomalous diffusion term corresponds to the change of genotype via small random mutations, and the nonlocal term describes large mutations. In this context $g(u)$ is the rate of cell birth which depends on u (density dependent proliferation), and the function $K(x-y)$ shows the proportion of newly born cells which change their genotype from y to x.

We assume that it depends on the distance between the genotypes. Finally, the last term in the right-hand side of this equation describes the influx of cells for different genotypes.

The square root of Laplacian in equation (1.1) represents a particular case of superdiffusion intensively studied in relation with various applications in plasma physics and turbulence [11], [12], surface diffusion [13], [14], semiconductors [15] and so on. The physical meaning of superdiffusion is that the random process occurs with longer jumps in comparison with normal diffusion. The moments of jump length distribution is finite in the case of normal diffusion, but this is not the case for superdiffusion. The operator $\sqrt{-\Delta}$ is defined via the spectral calculus. A similar equation in the case with the standard Laplacian in the diffusion term was treated recently in [28].

Further we will set $D = 1$ and will explore the existence of solutions of the equation

$$-\sqrt{-\Delta}u + \int_{\mathbb{R}^d} K(x-y)g(u(y))dy + f(x) = 0. \tag{1.2}$$

Let us consider the case in which the linear part of this operator fails to satisfy the Fredholm property, such that conventional methods of nonlinear analysis may not be applicable. We will use solvability conditions for non Fredholm operators along with the method of contraction mappings.

Consider the problem

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either zero identically or converges to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$ which corresponds to the left side of equation (1.3) contains the origin. Consequently, such operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present work is devoted to the studies of some properties of the operators of this kind. Note that elliptic problems with non Fredholm operators were studied extensively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [2], [3], [4], [5], [6]. The Schrödinger type operators without Fredholm property were treated via the methods of the spectral and the scattering theory in [16], [19], [20], [21], [23]. The Laplacian operator with drift from the point of view of non Fredholm operators was studied in [22] and linearized Cahn-Hilliard equations in [24] and [26]. Nonlinear non Fredholm elliptic problems were treated in [25] and [27]. Important applications to the theory of reaction-diffusion problems were developed in [8], [9]. Non Fredholm operators arise also when studying wave systems with an infinite number of localized traveling waves (see [1]). In particular, when $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the case of $a \neq 0$ is considerably different and the approach developed in these works cannot be applied. Front propagation problems with superdiffusion were studied extensively in recent years (see e.g. [29], [30]).

Let us set $K(x) = \varepsilon \mathcal{K}(x)$, where $\varepsilon \geq 0$ and suppose that the assumption below is fulfilled.

Assumption 1. *Let $f(x) : \mathbb{R}^3 \to \mathbb{R}$ be nontrivial, $f(x) \in L^1(\mathbb{R}^3)$ and $\nabla f(x) \in L^2(\mathbb{R}^3)$. Assume also that $\mathcal{K}(x) : \mathbb{R}^3 \to \mathbb{R}$ and $\mathcal{K}(x) \in L^1(\mathbb{R}^3)$.*

We choose the space dimension $d = 3$, which is related to the solvability conditions for the linear Poisson equation (3.20) discussed in Lemma 5. Our results obtained below can be generalized to $d > 3$. From the perspective of applications, the space dimension is not limited to $d = 3$ due to the fact that the space variable corresponds to cell genotype but not to the usual physical space.

By means of the standard Sobolev inequality (see e.g. p.183 of [10]) under the assumption given above we obtain

$$f(x) \in L^2(\mathbb{R}^3).$$

We consider the Sobolev space

$$H^2(\mathbb{R}^3) := \{u(x) : \mathbb{R}^3 \to \mathbb{C} \mid u(x) \in L^2(\mathbb{R}^3),\ \Delta u \in L^2(\mathbb{R}^3)\}$$

equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R}^3)} := \|u\|^2_{L^2(\mathbb{R}^3)} + \|\Delta u\|^2_{L^2(\mathbb{R}^3)}. \tag{1.4}$$

The Sobolev embedding yields

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq c_e \|u\|_{H^2(\mathbb{R}^3)}, \tag{1.5}$$

where $c_e > 0$ is the constant of the embedding. When the nonnegative parameter ε vanishes, we arrive at the linear Poisson equation (3.20). By means of Lemma 5 below under our Assumption 1 problem (3.20) admits a unique solution $u_0(x) \in H^1(\mathbb{R}^3)$ and no orthogonality relations are required. Lemmas 5 yields that in dimensions $d < 3$ we need specific orthogonality conditions to be able to solve equation (3.20) in $H^1(\mathbb{R}^d)$. Let us not treat the problem in dimensions $d > 3$ to avoid additional technicalities due to the fact that the proof will rely on similar ideas (see Lemma 5). By virtue of Assumption 1, using that

$$\|\Delta u\|^2_{L^2(\mathbb{R}^3)} = \|\nabla f(x)\|^2_{L^2(\mathbb{R}^3)},$$

we obtain for the unique solution of our linear problem (3.20) that $u_0(x) \in H^2(\mathbb{R}^3)$. Let us seek the resulting solution of the nonlinear equation (1.2) as

$$u(x) = u_0(x) + u_p(x). \tag{1.6}$$

Evidently, we obtain the perturbative equation

$$\sqrt{-\Delta}u_p = \varepsilon \int_{\mathbb{R}^3} \mathcal{K}(x-y)g(u_0(y)+u_p(y))dy. \tag{1.7}$$

Let us introduce a closed ball in the Sobolev space

$$B_\rho := \{u(x) \in H^2(\mathbb{R}^3) \mid \|u\|_{H^2(\mathbb{R}^3)} \le \rho\}, \quad 0 < \rho \le 1. \tag{1.8}$$

We seek the solution of (1.7) as the fixed point of the auxiliary nonlinear problem

$$\sqrt{-\Delta}u = \varepsilon \int_{\mathbb{R}^3} \mathcal{K}(x-y)g(u_0(y)+v(y))dy \tag{1.9}$$

in ball (1.8). For a given function $v(y)$ this is an equation with respect to $u(x)$. The left side of (1.9) contains the operator without Fredholm property $\sqrt{-\Delta}$: $H^1(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$, such that this operator has no bounded inverse. The analogous situation appeared in works [25] and [27] but as distinct from the present article, the problems treated there required orthogonality conditions. The fixed point technique was used in [18] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear problem possessed the Fredholm property (see Assumption 1 of [18], also [7]). Let us define the interval on the real line

$$I := [-c_e\|u_0\|_{H^2(\mathbb{R}^3)} - c_e,\ c_e\|u_0\|_{H^2(\mathbb{R}^3)} + c_e]. \tag{1.10}$$

We make the following assumption on the nonlinear part of problem (1.2).

Assumption 2. *Let $g(s) : \mathbb{R} \to \mathbb{R}$, such that $g(0) = 0$ and $g'(0) = 0$. We also assume that $g(s) \in C_2(\mathbb{R})$, such that*

$$a_2 := sup_{s\in I}|g''(s)| > 0.$$

Evidently $a_1 := sup_{s\in I}|g'(s)| > 0$ as well, otherwise the function $g(s)$ will be constant on the interval I and a_2 vanishes. For instance, $g(s) = s^2$ clearly satisfies the assumption above.

Let us introduce the operator T_g, such that $u = T_g v$, where u is a solution of equation (1.9). Our main statement is as follows.

Theorem 3. *Let Assumptions 1 and 2 hold. Then equation (1.9) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon*$ for a certain $\varepsilon* > 0$. The unique fixed point $u_p(x)$ of the map T_g is the only solution of equation (1.7) in B_ρ.*

Apparently the resulting solution of problem (1.2) given by (1.6) will be nontrivial due to the fact that the source term $f(x)$ is nontrivial and $g(0) = 0$ as assumed. We make use of the following elementary lemma.

Lemma 4. *Consider the function $\varphi(R) := \alpha R + \frac{\beta}{R^2}$ for $R \in (0, +\infty)$, where the constants $\alpha, \beta > 0$. It attains the minimal value at $R^* = \left(\frac{2\beta}{\alpha}\right)^{\frac{1}{3}}$, which is given by $\varphi(R^*) = \frac{3}{2^{\frac{2}{3}}}\alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}$.*

We proceed to the proof of our main result.

2. The existence of the perturbed solution

Proof of Theorem 3. Let us choose arbitrarily $v(x) \in B_\rho$ and denote the term involved in the integral expression in right side of equation (1.9) as

$$G(x) := g(u_0 + v).$$

We apply the standard Fourier transform (3.25) to both sides of problem (1.9) and arrive at

$$\widehat{u}(p) = \varepsilon(2\pi)^{\frac{3}{2}}\frac{\widehat{\mathcal{K}}(p)\widehat{G}(p)}{|p|}.$$

Hence for the norm we obtain

$$\|u\|^2_{L^2(\mathbb{R}^3)} = (2\pi)^3\varepsilon^2\int_{\mathbb{R}^3}\frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}(p)|^2}{p^2}dp. \tag{2.11}$$

As distinct from works [25] and [27] involving the standard Laplacian operator in the diffusion term, here we do not try to control the norm

$$\left\|\frac{\widehat{\mathcal{K}}(p)}{|p|}\right\|_{L^\infty(\mathbb{R}^3)}.$$

Let us estimate the right side of (2.11) using (3.26) with $R > 0$ as

$$(2\pi)^3\varepsilon^2\int_{|p|\le R}\frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}(p)|^2}{p^2}dp + (2\pi)^3\varepsilon^2\int_{|p|>R}\frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}(p)|^2}{p^2}dp \le$$

$$\le \varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)}\left\{\frac{1}{2\pi^2}\|G(x)\|^2_{L^1(\mathbb{R}^3)}R + \frac{1}{R^2}\|G(x)\|^2_{L^2(\mathbb{R}^3)}\right\}. \tag{2.12}$$

Since $v(x) \in B_\rho$, we have

$$\|u_0 + v\|_{L^2(\mathbb{R}^3)} \le \|u_0\|_{H^2(\mathbb{R}^3)} + 1$$

and the Sobolev embedding (1.5) yields

$$|u_0 + v| \le c_e \|u_0\|_{H^2(\mathbb{R}^3)} + c_e.$$

The formula $G(x) = \displaystyle\int_0^{u_0+v} g'(s)ds$ with the interval I defined in (1.10) implies

$$|G(x)| \le sup_{s\in I}|g'(s)||u_0 + v| = a_1|u_0 + v|.$$

Thus

$$\|G(x)\|_{L^2(\mathbb{R}^3)} \le a_1\|u_0 + v\|_{L^2(\mathbb{R}^3)} \le a_1(\|u_0\|_{H^2(\mathbb{R}^3)} + 1).$$

Obviously, $G(x) = \displaystyle\int_0^{u_0+v} ds\Big[\int_0^s g''(t)dt\Big]$. Hence, we arrive at

$$|G(x)| \le \frac{1}{2}sup_{t\in I}|g''(t)||u_0 + v|^2 = \frac{a_2}{2}|u_0 + v|^2,$$

$$\|G(x)\|_{L^1(\mathbb{R}^3)} \le \frac{a_2}{2}\|u_0 + v\|^2_{L^2(\mathbb{R}^3)} \le \frac{a_2}{2}(\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^2.$$

Thus, we obtain the upper bound for the right side of (2.12) as

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)}(\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^2\left\{\frac{a_2^2}{8\pi^2}(\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^2 R + \frac{a_1^2}{R^2}\right\}$$

with $R \in (0, +\infty)$. Lemma 4 yields the minimal value of the expression above. Thus

$$\|u\|^2_{L^2(\mathbb{R}^3)} \le \frac{3}{2^{\frac{2}{3}}4\pi^{\frac{4}{3}}}\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)}(\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^{3\frac{1}{3}}a_1^{\frac{2}{3}}a_2^{\frac{4}{3}}. \tag{2.13}$$

Evidently, (1.9) implies

$$-\Delta u = \varepsilon\sqrt{-\Delta}\int_{\mathbb{R}^3}\mathcal{K}(x-y)G(y)dy$$

and

$$\nabla G(x) = g'(u_0 + v)(\nabla u_0 + \nabla v).$$

We will use the identity

$$g'(u_0 + v) = \int_0^{u_0+v} g''(s)ds.$$

Sobolev embedding (1.5) yields

$$|g'(u_0 + v)| \le sup_{s\in I}|g''(s)||u_0 + v| \le a_2 c_e(\|u_0\|_{H^2(\mathbb{R}^3)} + 1).$$

The following inequality can be trivially obtained using the standard Fourier transform, namely

$$\|\nabla u\|_{L^2(\mathbb{R}^3)} \le \|u\|_{H^2(\mathbb{R}^3)}. \tag{2.14}$$

Then we arrive at

$$\|\Delta u\|_{L^2(\mathbb{R}^3)}^2 \leq \varepsilon^2 \|\mathcal{K}\|_{L^1(\mathbb{R}^3)}^2 a_2^2 c_e^2 (\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^4. \tag{2.15}$$

The definition of the norm (1.4) along with estimates (2.13) and (2.15) imply

$$\|u\|_{H^2(\mathbb{R}^3)} \leq \varepsilon \|\mathcal{K}\|_{L^1(\mathbb{R}^3)} (\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^2 a_2^{\frac{2}{3}} \sqrt{\frac{3}{2^{\frac{2}{3}} 4\pi^{\frac{4}{3}}} a_1^{\frac{2}{3}} + a_2^{\frac{2}{3}} c_e^2} \leq \rho$$

for all positive values of ε small enough. Thus $u(x) \in B_\rho$ as well. If for some $v(x) \in B_\rho$ there exist two solutions $u_{1,2}(x) \in B_\rho$ of problem (1.9), their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^3)$ satisfies

$$\sqrt{-\Delta} w = 0.$$

Since the operator $\sqrt{-\Delta}$ does not have nontrivial square integrable zero modes, $w(x) = 0$ a.e. in $\mathbb{R}^3$. Therefore, equation (1.9) defines a map $T_g : B_\rho \to B_\rho$ for $\varepsilon > 0$ sufficiently small.

The goal is to prove that this map is a strict contraction. We choose arbitrarily $v_{1,2}(x) \in B_\rho$. The argument above yields $u_{1,2} = T_g v_{1,2} \in B_\rho$ as well. (1.9) gives us

$$\sqrt{-\Delta} u_1 = \varepsilon \int_{\mathbb{R}^3} \mathcal{K}(x-y) g(u_0(y) + v_1(y)) dy, \tag{2.16}$$

$$\sqrt{-\Delta} u_2 = \varepsilon \int_{\mathbb{R}^3} \mathcal{K}(x-y) g(u_0(y) + v_2(y)) dy. \tag{2.17}$$

Let us define

$$G_1(x) := g(u_0 + v_1), \quad G_2(x) := g(u_0 + v_2)$$

and apply the standard Fourier transform (3.25) to both sides of problems (2.16) and (2.17). We obtain

$$\widehat{u_1}(p) = \varepsilon (2\pi)^{\frac{3}{2}} \frac{\widehat{\mathcal{K}}(p) \widehat{G_1}(p)}{|p|}, \quad \widehat{u_2}(p) = \varepsilon (2\pi)^{\frac{3}{2}} \frac{\widehat{\mathcal{K}}(p) \widehat{G_2}(p)}{|p|}.$$

Evidently

$$\|u_1 - u_2\|_{L^2(\mathbb{R}^3)}^2 = \varepsilon^2 (2\pi)^3 \int_{\mathbb{R}^3} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G_1}(p) - \widehat{G_2}(p)|^2}{|p|^2} dp.$$

Clearly, it can be bounded from above by means of (3.26) by

$$\varepsilon^2 \|\mathcal{K}\|_{L^1(\mathbb{R}^3)}^2 \left\{ \frac{1}{2\pi^2} \|G_1(x) - G_2(x)\|_{L^1(\mathbb{R}^3)}^2 R + \|G_1(x) - G_2(x)\|_{L^2(\mathbb{R}^3)}^2 \frac{1}{R^2} \right\}$$

with $R \in (0, +\infty)$. Let us use the equality

$$G_1(x) - G_2(x) = \int_{u_0+v_2}^{u_0+v_1} g'(s)ds.$$

Thus

$$|G_1(x) - G_2(x)| \leq sup_{s\in I}|g'(s)||v_1 - v_2| = a_1|v_1 - v_2|.$$

Therefore

$$\|G_1(x) - G_2(x)\|_{L^2(\mathbb{R}^3)} \leq a_1\|v_1 - v_2\|_{L^2(\mathbb{R}^3)} \leq a_1\|v_1 - v_2\|_{H^2(\mathbb{R}^3)}.$$

Evidently,

$$G_1(x) - G_2(x) = \int_{u_0+v_2}^{u_0+v_1} ds\Big[\int_0^s g''(t)dt\Big].$$

We derive the upper bound for $G_1(x) - G_2(x)$ in the absolute value as

$$\frac{1}{2}sup_{t\in I}|g''(t)||(v_1 - v_2)(2u_0 + v_1 + v_2)| = \frac{a_2}{2}|(v_1 - v_2)(2u_0 + v_1 + v_2)|.$$

By means of the Schwarz inequality we estimate the norm $\|G_1(x) - G_2(x)\|_{L^1(\mathbb{R}^3)}$ from above by

$$\frac{a_2}{2}\|v_1 - v_2\|_{L^2(\mathbb{R}^3)}\|2u_0 + v_1 + v_2\|_{L^2(\mathbb{R}^3)} \leq a_2\|v_1 - v_2\|_{H^2(\mathbb{R}^3)}(\|u_0\|_{H^2(\mathbb{R}^3)} + 1).$$

Thus we arrive at the upper bound for the norm $\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^3)}$ given by

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)}\|v_1 - v_2\|^2_{H^2(\mathbb{R}^3)}\Big\{\frac{{a_2}^2}{2\pi^2}(\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^2R + \frac{{a_1}^2}{R^2}\Big\}.$$

Lemma 4 enables us to minimize the expression above over $R > 0$ to obtain that $\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^3)}$ is estimated from above by

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)}\|v_1 - v_2\|^2_{H^2(\mathbb{R}^3)}\frac{3}{2^{\frac{4}{3}}}\frac{a_2^{\frac{4}{3}}}{\pi^{\frac{4}{3}}}(\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^{\frac{4}{3}}a_1^{\frac{2}{3}}. \tag{2.18}$$

(2.16) and (2.17) imply that

$$-\Delta(u_1 - u_2) = \varepsilon\sqrt{-\Delta}\int_{\mathbb{R}^3}\mathcal{K}(x - y)[g(u_0(y) + v_1(y)) - g(u_0(y) + v_2(y))]dy.$$

Thus

$$\|\Delta(u_1 - u_2)\|^2_{L^2(\mathbb{R}^3)} \leq \varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)}\|\nabla g(u_0 + v_1) - \nabla g(u_0 + v_2)\|^2_{L^2(\mathbb{R}^3)}.$$

We express $\nabla g(u_0 + v_1) - \nabla g(u_0 + v_2)$ as

$$g'(u_0 + v_1)(\nabla u_0 + \nabla v_1) - g'(u_0 + v_2)(\nabla u_0 + \nabla v_2) =$$

$$= (\nabla u_0 + \nabla v_1) \int_{u_0+v_2}^{u_0+v_1} g''(s) ds + (\nabla v_1 - \nabla v_2) \int_0^{u_0+v_2} g''(s) ds.$$

This yields the estimate from above for $|\nabla g(u_0 + v_1) - \nabla g(u_0 + v_2)|$ as

$$sup_{s\in I}|g''(s)||v_1 - v_2||\nabla u_0 + \nabla v_1| + sup_{s\in I}|g''(s)||u_0 + v_2||\nabla v_1 - \nabla v_2|.$$

This expression can be trivially bounded from above by means of the Sobolev embedding (1.5) by

$$a_2 c_e \|v_1 - v_2\|_{H^2(\mathbb{R}^3)} |\nabla u_0 + \nabla v_1| + a_2 c_e \|u_0 + v_2\|_{H^2(\mathbb{R}^3)} |\nabla v_1 - \nabla v_2|.$$

Hence, by virtue of (2.14) for $v_{1,2} \in B_\rho$ we derive the upper bound for the norm $\|\Delta(u_1 - u_2)\|^2_{L^2(\mathbb{R}^3)}$ as

$$4\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)} a_2^2 c_e^2 (\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^2 \|v_1 - v_2\|^2_{H^2(\mathbb{R}^3)}. \tag{2.19}$$

By means of inequalities (2.18) and (2.19) the norm $\|u_1 - u_2\|_{H^2(\mathbb{R}^3)}$ is estimated from above by

$$\varepsilon \|\mathcal{K}\|_{L^1(\mathbb{R}^3)} (\|u_0\|_{H^2(\mathbb{R}^3)} + 1) a_2^{\frac{2}{3}} \Big[\frac{3}{2^{\frac{4}{3}}} \frac{a_1^{\frac{2}{3}}}{\pi^{\frac{4}{3}}} + 4 a_2^{\frac{2}{3}} c_e^2 \Big]^{\frac{1}{2}} \|v_1 - v_2\|_{H^2(\mathbb{R}^3)}.$$

Therefore, the map $T_g : B_\rho \to B_\rho$ defined by equation (1.9) is a strict contraction for all values of $\varepsilon > 0$ sufficiently small. Its unique fixed point $u_p(x)$ is the only solution of problem (1.7) in B_ρ. The resulting $u(x) \in H^2(\mathbb{R}^3)$ given by (1.6) is a solution of equation (1.2). ■

3. Auxiliary results

First we derive the solvability conditions for the following linear Poisson equation

$$\sqrt{-\Delta} u = f(x), \quad x \in \mathbb{R}^d, \quad d \in \mathbb{N}. \tag{3.20}$$

Let us denote the inner product as

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x) \bar{g}(x) dx, \tag{3.21}$$

with a slight abuse of notations when the functions involved in (3.21) are not square integrable, like for instance the ones present in orthogonality relations (3.22) and (3.23) below. Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x)$ is bounded, then the integral in the right side of (3.21) is well defined. Our technical result is as follows.

Lemma 5. *Let* $f(x) \in L^2(\mathbb{R}^d), d \in \mathbb{N}$.

1) When $d = 1$ *and in addition* $|x|f(x) \in L^1(\mathbb{R})$*, equation (3.20) admits a unique solution* $u(x) \in H^1(\mathbb{R})$ *if and only if the orthogonality condition*

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{3.22}$$

holds.

2) When $d = 2$ *and additionally* $|x|f(x) \in L^1(\mathbb{R}^2)$*, problem (3.20) possesses a unique solution* $u(x) \in H^1(\mathbb{R}^2)$ *if and only if the orthogonality relation*

$$(f(x), 1)_{L^2(\mathbb{R}^2)} = 0 \tag{3.23}$$

holds.

3) When $d \geq 3$ *and in addition* $f(x) \in L^1(\mathbb{R}^d)$*, equation (3.20) has a unique solution* $u(x) \in H^1(\mathbb{R}^d)$*.*

Proof. Let us first address the uniqueness of solutions for problem (3.20). Suppose $u_{1,2}(x) \in H^1(\mathbb{R}^d)$ both satisfy equation (3.20). Then their difference $w(x) := u_1(x) - u_2(x)$ solves the homogeneous problem

$$\sqrt{-\Delta} w = 0.$$

Since the operator $\sqrt{-\Delta}$ in the the whole space does not have nontrivial square integrable zero modes, $w(x)$ vanishes a.e. in $\mathbb{R}^d$. Note that it would be sufficient to establish only the square integrability for the solution of (3.20). Indeed, we have a trivial identity

$$\|\sqrt{-\Delta} u\|_{L^2(\mathbb{R}^d)}^2 = \|\nabla u\|_{L^2(\mathbb{R}^d)}^2. \tag{3.24}$$

Since the source term $f(x) \in L^2(\mathbb{R}^d)$ as assumed, we arrive at $\nabla u \in L^2(\mathbb{R}^d)$, such that $u(x) \in H^1(\mathbb{R}^d)$ as well. We will use the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x) e^{-ipx} dx, \quad d \in \mathbb{N}. \tag{3.25}$$

Clearly, we have the estimate

$$\|\widehat{f}(p)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \|f(x)\|_{L^1(\mathbb{R}^d)}. \tag{3.26}$$

Let us apply (3.25) to both sides of equation (3.20). We obtain

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|},$$

such that the norm can be expressed as

$$\|u\|_{L^2(\mathbb{R}^d)}^2 = \int_{|p| \leq 1} \frac{|\widehat{f}(p)|^2}{|p|^2} dp + \int_{|p| > 1} \frac{|\widehat{f}(p)|^2}{|p|^2} dp. \tag{3.27}$$

Evidently, the second term in the right side of (3.27) can be estimated from above by $\|f\|^2_{L^2(\mathbb{R}^d)} < \infty$ as assumed. Let us estimate the first term in the right side of (3.27) in dimension $d = 1$, using the identity

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^p \frac{d\widehat{f}(s)}{ds} ds.$$

Clearly, via definition (3.25)

$$\left| \frac{d\widehat{f}(p)}{dp} \right| \leq \frac{1}{\sqrt{2\pi}} \||x|f\|_{L^1(\mathbb{R})} < \infty$$

due to one of our assumptions. Therefore,

$$\left| \frac{\int_0^p \frac{d\widehat{f}(s)}{ds} ds}{|p|} \chi_{\{p\in\mathbb{R} \ | \ |p|\leq 1\}} \right| \leq \frac{1}{\sqrt{2\pi}} \||x|f\|_{L^1(\mathbb{R})} \chi_{\{p\in\mathbb{R} \ | \ |p|\leq 1\}} \in L^2(\mathbb{R}).$$

Here and further down χ_A stands for the characteristic function of a set $A \in \mathbb{R}^d$. The remaining term $\dfrac{\widehat{f}(0)}{|p|} \chi_{\{p\in\mathbb{R} \ | \ |p|\leq 1\}}$ belongs to $L^2(\mathbb{R})$ if and only if $\widehat{f}(0)$ vanishes, which gives us orthogonality relation (3.22) in dimension $d = 1$.

Then we turn our attention to the case of dimension $d = 2$. Let us use the formula

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^{|p|} \frac{\partial \widehat{f}}{\partial s}(s,\theta) ds,$$

where θ stands for the angle variable on the circle. Clearly, definition (3.25) yields

$$\left| \frac{\partial \widehat{f}}{\partial |p|} \right| \leq \frac{1}{2\pi} \||x|f\|_{L^1(\mathbb{R}^2)} < \infty$$

as assumed. Thus

$$\left| \frac{\int_0^{|p|} \frac{\partial \widehat{f}(s,\theta)}{\partial s} ds}{|p|} \chi_{\{p\in\mathbb{R}^2 \ | \ |p|\leq 1\}} \right| \leq \frac{1}{2\pi} \||x|f\|_{L^1(\mathbb{R}^2)} \chi_{\{p\in\mathbb{R}^2 \ | \ |p|\leq 1\}} \in L^2(\mathbb{R}^2).$$

Finally, the term $\dfrac{\widehat{f}(0)}{|p|} \chi_{\{p\in\mathbb{R}^2 \ | \ |p|\leq 1\}} \in L^2(\mathbb{R}^2)$ if and only if $\widehat{f}(0) = 0$, such that we obtain orthogonality condition (3.23) for dimension $d = 2$.

To complete the proof of the lemma, it remains to study the case of higher dimensions $d \geq 3$. By virtue of inequality (3.26), we easily estimate the first term in the right side of (3.27) by

$$\frac{1}{(2\pi)^d} \|f(x)\|^2_{L^1(\mathbb{R}^d)} |S_d| \frac{1}{d-2} < \infty$$

due to one of our assumptions. Here S_d denotes the unit sphere in the space of d dimensions centered at the origin and $|S_d|$ stands for its Lebesgue measure. ∎

Note that in dimensions $d \geq 3$ under the assumptions given above no orthogonality conditions are needed to solve the linear Poisson equation (3.20) in $H^1(\mathbb{R}^d)$.

Let us show that it is possible to incorporate a shallow, short-range potential into the linear Poisson equation considered above and generalize the result of Lemma 5. No orthogonality relations will be required in Lemma 7 below as well. Consider the following equation

$$\sqrt{-\Delta + V(x)}u = f(x), \quad x \in \mathbb{R}^3, \tag{3.28}$$

with the operator $\sqrt{-\Delta + V(x)}$ well defined via the spectral calculus, since under our assumptions the operator $-\Delta + V(x)$ on $L^2(\mathbb{R}^3)$ is nonnegative as discussed below.

Assumption 6. *The potential function $V(x) : \mathbb{R}^3 \to \mathbb{R}$ satisfies the estimate*

$$|V(x)| \leq \frac{C}{1 + |x|^{3.5+\varepsilon}}$$

with some $\varepsilon > 0$ and $x \in \mathbb{R}^3$ a.e. such that

$$4^{\frac{1}{9}} \frac{9}{8} (4\pi)^{-\frac{2}{3}} \|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}} \|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \quad \text{and} \quad \sqrt{c_{HLS}} \|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi.$$

This is analogous to Assumption 1.1 of [20] under which by virtue of Lemma 2.3 of [20] our Schrödinger operator $-\Delta + V(x)$ is self-adjoint and unitarily equivalent to $-\Delta$ on $L^2(\mathbb{R}^3)$ via the wave operators. Thus the essential spectrum of $\sqrt{-\Delta + V(x)} : H^1(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ fills the nonnegative semi-axis $[0, +\infty)$. Hence such operator does not have a bounded inverse and therefore it fails to satisfy the Fredholm property. Here C stands for a finite, positive constant and c_{HLS} for the constant in the Hardy-Littlewood-Sobolev inequality

$$\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x) f_1(y)}{|x-y|^2} dx dy \right| \leq c_{HLS} \|f_1\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^2, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3)$$

given on p.98 of [10]. The functions of the continuous spectrum of our Schrödinger operator satisfy

$$(-\Delta + V(x))\varphi_k(x) = k^2 \varphi_k(x), \quad k \in \mathbb{R}^3,$$

in the integral formulation the Lippmann-Schwinger equation (see e.g. p.98 of [17])

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{3.29}$$

and the orthogonality relations

$$(\varphi_k(x), \varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q), \quad k, q \in \mathbb{R}^3.$$

They form a complete system in $L^2(\mathbb{R}^3)$. Let us denote by tilde the generalized Fourier transform with respect to these functions, such that

$$\tilde{f}(k) := (f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)}, \quad k \in \mathbb{R}^3. \tag{3.30}$$

The integral operator involved in the right side of equation (3.29) is

$$(Q\varphi)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi)(y)dy, \quad \varphi \in L^\infty(\mathbb{R}^3).$$

We consider $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$. By virtue of Lemma 2.1 of [20] under Assumption 6 above on the scalar potential we have $\|Q\|_\infty < 1$. Moreover, this norm is bounded above by the quantity, which is independent of the wave vector k and can be expressed in terms of the appropriate $L^p(\mathbb{R}^3)$ norms of $V(x)$. Corollary 2.2 of [20] yields the estimate

$$|\tilde{f}(k)| \le \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{1-\|Q\|_\infty}\|f\|_{L^1(\mathbb{R}^3)}. \tag{3.31}$$

We have the following statement.

Lemma 7. *Let the potential $V(x)$ satisfy Assumption 6 and $f(x) \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Then equation (3.28) has a unique solution $u(x) \in H^1(\mathbb{R}^3)$.*

Proof. Let us first suppose that problem (3.28) has two solutions $u_{1,2}(x) \in H^1(\mathbb{R}^3)$. Then their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^3)$ solves the homogeneous equation

$$\sqrt{-\Delta + V(x)}w = 0,$$

which cannot have nontrivial square integrable solutions due to the fact that our self-adjoint operator $-\Delta + V(x)$ is unitarily equivalent to $-\Delta$ on $L^2(\mathbb{R}^3)$. Hence $w(x)$ vanishes a.e. in $\mathbb{R}^3$.

Let us apply the generalized Fourier transform (3.30) to both sides of equation (3.28) to obtain

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{|k|}, \quad k \in \mathbb{R}^3.$$

This enables us to express the norm as

$$\|u\|^2_{L^2(\mathbb{R}^3)} = \int_{|k|\le 1} \frac{|\tilde{f}(k)|^2}{k^2} dk + \int_{|k|>1} \frac{|\tilde{f}(k)|^2}{k^2} dk. \tag{3.32}$$

Clearly, the second term in the right side of (3.32) can be bounded from above by $\|f\|^2_{L^2(\mathbb{R}^3)} < \infty$ as assumed. Let us use (3.31) to estimate from above the first term in the right side of (3.32) as

$$\frac{1}{2\pi^2} \frac{1}{(1 - \|Q\|_\infty)^2} \|f\|^2_{L^1(\mathbb{R}^3)} < \infty$$

as well. Hence $u(x) \in L^2(\mathbb{R}^3)$. A trivial calculation using (3.28) yields

$$\|f\|^2_{L^2(\mathbb{R}^3)} = \|\nabla u\|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} V(x)|u(x)|^2 dx.$$

Since $f(x)$ is square integrable and the scalar potential $V(x)$ is bounded as assumed, we have $\nabla u(x) \in L^2(\mathbb{R}^3)$ as well, such that the solution $u(x) \in H^1(\mathbb{R}^3)$. ∎

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