ASYMPTOTICS OF RESONANCES INDUCED BY POINT INTERACTIONS

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Abstract. We consider the resonances of the self-adjoint three-dimensional Schrödinger operator with point interactions of constant strength supported on the set $X = \{x_n\}_{n=1}^N$. The size of X is defined by $V_X = \max_{\pi\in\Pi_N}\sum_{n=1}^N |x_n - x_{\pi(n)}|$, where Π_N is the family of all the permutations of the set $\{1, 2, \ldots, N\}$. We prove that the number of resonances counted with multiplicities and lying inside the disc of radius R asymptotically behaves as $\frac{W_X}{\pi}R + \mathcal{O}(1)$ as $R \to \infty$, where $W_X \in [0, V_X]$ is the effective size of X. Moreover, we show that there exist configurations of any number of points such that $W_X = V_X$. Finally, we construct an example for $N = 4$ with $W_X < V_X$, which can be viewed as an analogue of a non-Weyl quantum graph.

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1. Introduction

In this note we discuss the resonances of the three-dimensional Schrödinger operator $\mathsf{H}_{\alpha,X}$ with point interactions of constant strength $\alpha \in \mathbb{R}$ supported on the discrete set $X = \{x_n\}_{n=1}^N \subset \mathbb{R}^3$, $N \geq 2$. The corresponding Hamiltonian $\mathsf{H}_{\alpha,X}$ is associated with the formal differential expression

$$(1.1)\qquad -\Delta + \alpha\sum_{n=1}^N \delta(x - x_n)\,, \qquad \text{on } \mathbb{R}^3\,,$$

where $\delta(\cdot)$ stands for the δ-distribution. The Hamiltonian $\mathsf{H}_{\alpha,X}$ can be rigorously defined as a self-adjoint extension of a certain symmetric operator in the Hilbert space $L^2(\mathbb{R}^3)$; *cf.* Section 3 for details. Resonances of $\mathsf{H}_{\alpha,X}$ were discussed in the monograph [AGHH05] and in several more recent publications *e.g.* [AK17, BFT98, EGST96], see also the review [DFT08] and the references therein.

We define the *size* of X by

$$(1.2)\qquad V_X := \max_{\pi\in\Pi_N}\sum_{n=1}^N |x_n - x_{\pi(n)}|\,,$$

where Π_N is the family of all the permutations of the set $\{1, 2, \ldots, N\}$. This definition of the size is motivated by the condition on resonances for $\mathsf{H}_{\alpha,X}$ given in Section 4.1. As the main result of this note, we prove that the number $\mathcal{N}_{\alpha,X}(R)$ of the resonances of $\mathsf{H}_{\alpha,X}$ lying inside the disc $\{z \in \mathbb{C} : |z| < R\}$ and with multiplicities taken into account asymptotically behaves as

$$(1.3)\qquad \mathcal{N}_{\alpha,X}(R) = \frac{W_X}{\pi}R + \mathcal{O}(1)\,, \qquad R \to \infty\,,$$

where $W_X \in [0, V_X]$ is the effective size of X, which does not depend on α.

In the proof of (1.3) we use that the resonance condition for $\mathsf{H}_{\alpha,X}$ acquires the form of an exponential polynomial, which can be obtained by a direct computation or alternatively using the pseudo-orbit expansion as explained in Section 4.3. Recall that an exponential polynomial is a sum of finitely many terms, each of which is a product of a rational function and an exponential; *cf.* the review paper [Lan31] and the monographs [BC63, BG95]. In order to obtain the asymptotics (1.3) we employ a classical result on the distribution of zeros of exponential polynomials, recalled in Section 2 for the convenience of the reader.

A configuration of points X for which $W_X = V_X$ is said to be of *Weyl-type*. We show that for any $N \in \mathbb{N}$ there exist Weyl-type configurations consisting of N points. For two and three points ($N \leq 3$), in fact, any configuration is of Weyl-type, as shown in Section 5.1. On the other hand, we present in Section 5.2 an example of a non-Weyl configuration for $N = 4$, for which strict inequality $W_X < V_X$ holds. We expect that such configurations can also be constructed for any $N > 4$. One can trace an analogy with non-Weyl quantum graphs studied in [DEL10, DP11]. Non-uniqueness of the permutation at which the maximum in (1.2) is attained, is a necessary condition for a configuration of points X to be non-Weyl. Exact geometric characterization of non-Weyl-type point configurations remains an open question. Besides that a physical interpretation of this mathematical observation still needs to be clarified.

It is worth pointing out that $\mathcal{N}_{\alpha,X}(R)$ is *asymptotically linear* similarly as the counting function for resonances of the one-dimensional Schrödinger operator $-\frac{\mathsf{d}^2}{\mathsf{d}x^2} + V$ with a potential $V \in C_0^\infty(\mathbb{R};\mathbb{R})$; see [Zwo87]. The exact asymptotics of the counting function for resonances of the three-dimensional Schrödinger operator $-\Delta + V$ with a potential $V \in C_0^\infty(\mathbb{R}^3;\mathbb{R})$ is known only in some special cases, but for "generic" potentials this counting function behaves as $\sim R^3$, thus being *not asymptotically linear*; see [CH08] for details.

2. **Exponential polynomials**

In this section we introduce exponential polynomials and recall a classical result on the asymptotic distribution of their zeros. This result was first obtained by Pólya [Pol20] and later improved by many authors, including Schwengeler [Sch25] and Moreno [Mor73]. We refer the reader to the review [Lan31] by Langer and to the monographs [BC63, BG95].

Definition 2.1. *An exponential polynomial $F\colon \mathbb{C} \to \mathbb{C}$ is a function of the form*

$$F(z) = \sum_{m=1}^{M} z^{\nu_m} A_m(z) \mathrm{e}^{\mathrm{i}z\sigma_m}\,, \tag{2.1}$$

where $\nu_m \in \mathbb{R}$, $m = 1, 2, \ldots, M$, $A_m(z)$ *are rational functions in* z *not vanishing identically, and the constants* $\sigma_m \in \mathbb{R}$ *are ordered increasingly* ($\sigma_{\min} := \sigma_1 < \sigma_2 < \cdots < \sigma_M =: \sigma_{\max}$).

For example, for the exponential polynomial

$$F(z) = z\frac{z+\mathsf{i}}{z-\mathsf{i}}\mathrm{e}^{\mathrm{i}z} + z^2\frac{z^2+\mathsf{i}}{z^2+1}\mathrm{e}^{2\mathrm{i}z}$$

we have $M = 2$, $\nu_1 = 1$, $\nu_2 = 2$, $\sigma_1 = 1$, $\sigma_2 = 2$, $A_1(z) = \frac{z+\mathsf{i}}{z-\mathsf{i}}$, $A_2(z) = \frac{z^2+\mathsf{i}}{z^2+1}$.

The zero set of an exponential polynomial F is defined by

$$\mathcal{Z}_F := \{z \in \mathbb{C} \colon F(z) = 0\}\,. \tag{2.2}$$

For any $z \in \mathcal{Z}_F$ we define its multiplicity $\mathsf{m}_F(z) \in \mathbb{N}$ as the algebraic multiplicity of the root z of the function (2.1). Moreover, we introduce the counting function for an exponential polynomial F by

$$\mathcal{N}_F(R) = \sum_{z \in \mathcal{Z}_F \cap \mathcal{D}_R} \mathsf{m}_F(z)\,,$$

where $\mathcal{D}_R := \{z \in \mathbb{C} \colon |z| < R\}$ is the disc in the complex plane centered at the origin and having the radius $R > 0$. Thus, the value $\mathcal{N}_F(R)$ equals the number of zeros of F counted with multiplicities and lying inside $\mathcal{D}_R$. Now we have all the tools at our disposal to state the result on the asymptotics of $\mathcal{N}_F(R)$, proven in [Lan31, Thm. 6], see also [DEL10, Thm. 3.1].

Theorem 2.2. *Let* F *be an exponential polynomial as in* (2.1) *such that*

$$\lim_{z\to\infty} A_m(z) = a_m \in \mathbb{C} \setminus \{0\}\,, \qquad \forall\, m = 1, 2, \ldots, M\,.$$

Then the counting function for F *asymptotically behaves as*

$$\mathcal{N}_F(R) = \frac{\sigma_{\max} - \sigma_{\min}}{\pi} R + \mathcal{O}(1)\,, \qquad R \to \infty\,.$$

3. Rigorous definition of $\mathsf{H}_{\alpha,X}$

The Schrödinger operator $\mathsf{H}_{\alpha,X}$ associated with the formal differential expression (1.1) can be rigorously defined as a self-adjoint extension in $L^2(\mathbb{R}^3)$ of the closed, densely defined, symmetric operator

$$\mathsf{S}_X u := -\Delta u\,, \qquad \operatorname{dom} \mathsf{S}_X := \{u \in H^2(\mathbb{R}^3) \colon u|_X = 0\}\,, \tag{3.1}$$

where the vector $u|_X = (u(x_1), u(x_2), \ldots, u(x_N))^\top \in \mathbb{C}^N$ is well-defined by the Sobolev embedding theorem [McL00, Thm. 3.26]. The self-adjoint extensions of S_X with $N = 1$ have been first analyzed in the seminal paper [BF61]. For $N > 1$ the symmetric operator S_X possesses a rich family of self-adjoint extensions, not all of which correspond to point interactions. The self-adjoint extensions of S_X corresponding to point interactions are investigated in detail in the monographs [AGHH05, AK99], see also the references therein. Several alternative ways for parameterizing of all the self-adjoint extensions of S_X can be found in a more recent literature; see *e.g.* [GMZ12, Pos08, Tet90]. Below we follow the strategy of [GMZ12] and use some of notations therein. According to [GMZ12, Prop. 4.1], the adjoint of S_X can be characterized as follows

$$\mathrm{dom}\,\mathsf{S}_X^* = \left\{ u = u_0 + \sum_{n=1}^N \left(\xi_{0n} \frac{\mathrm{e}^{-r_n}}{r_n} + \xi_{1n}\mathrm{e}^{-r_n} \right) : u_0 \in \mathrm{dom}\,\mathsf{S}_X, \xi_0, \xi_1 \in \mathbb{C}^N \right\},$$

$$\mathsf{S}_X^* u = -\Delta u_0 - \sum_{n=1}^N \left(\xi_{0n} \frac{\mathrm{e}^{-r_n}}{r_n} + \xi_{1n} \left(\mathrm{e}^{-r_n} - \frac{2\mathrm{e}^{-r_n}}{r_n} \right) \right),$$

where $r_n\colon \mathbb{R}^3 \to \mathbb{R}_+$, $r_n(x) := |x - x_n|$ for all $n = 1, 2, \dots, N$ and $\xi_0 = \{\xi_{0n}\}_{n=1}^N$, $\xi_1 = \{\xi_{1n}\}_{n=1}^N$. Next, we introduce the mappings $\Gamma_0, \Gamma_1 \colon \mathrm{dom}\,\mathsf{S}_X^* \to \mathbb{C}^N$ by

$$\Gamma_0 u := 4\pi\xi_0 \qquad \text{and} \qquad \Gamma_1 u := \left\{ \lim_{x\to x_n} \left(u(x) - \frac{\xi_{0n}}{r_n} \right) \right\}_{n=1}^N . \tag{3.2}$$

Eventually, the operator $\mathsf{H}_{\alpha,X}$ is defined as the restriction of S_X^*

$$\mathsf{H}_{\alpha,X} u := \mathsf{S}_X^* u\,, \qquad \mathrm{dom}\,\mathsf{H}_{\alpha,X} := \{ u \in \mathrm{dom}\,\mathsf{S}_X^* \colon \Gamma_1 u = \alpha\Gamma_0 u \}\,, \tag{3.3}$$

cf. [GMZ12, Rem. 4.3]. Finally, by [GMZ12, Prop. 4.2], the operator $\mathsf{H}_{\alpha,X}$ is self-adjoint in $L^2(\mathbb{R}^3)$. Note also that the operator $\mathsf{H}_{\alpha,X}$ is the same as the one considered in [AGHH05, Chap. II.1]. We remark that the usual self-adjoint free Laplacian in $L^2(\mathbb{R}^3)$ formally corresponds to the case $\alpha = \infty$.

4. Resonances of $\mathsf{H}_{\alpha,X}$

The main aim of this section is to prove asymptotics of resonances given in (1.3). Apart from that we provide a condition on resonances through the pseudo-orbit expansion, which is of independent interest.

4.1. A condition on resonances for $\mathsf{H}_{\alpha,X}$.

First, we recall the definition of resonances for $\mathsf{H}_{\alpha,X}$ borrowed from [AGHH05, Sec. II.1.1]. This definition provides at the same time a way to find them. To this aim we introduce the function

$$F_{\alpha,X}(\kappa) := \det \left[\left\{ \left(\alpha - \frac{\mathrm{i}\kappa}{4\pi} \right) \delta_{nn'} - \widetilde{G}_\kappa(x_n - x_{n'}) \right\}_{n,n'=1}^{N,N} \right], \tag{4.1}$$

where $\delta_{nn'}$ is the Kronecker symbol and $\widetilde{G}_\kappa(\cdot)$ is given by

$$\widetilde{G}_\kappa(x) := \begin{cases} 0\,, & x = 0\,, \\ \frac{\mathrm{e}^{\mathrm{i}\kappa|x|}}{4\pi|x|}\,, & x \neq 0\,. \end{cases}$$

We say that $\kappa_0 \in \mathbb{C}$ is a resonance of $\mathsf{H}_{\alpha,X}$ if

$$F_{\alpha,X}(\kappa_0) = 0\,, \tag{4.2}$$

holds. The multiplicity of the resonance κ_0 equals the multiplicity of the zero of $F_{\alpha,X}(\cdot)$ at $\kappa = \kappa_0$. In our convention true resonances and negative eigenvalues of $\mathsf{H}_{\alpha,X}$ correspond to $\operatorname{Im}\kappa_0 < 0$ and $\operatorname{Im}\kappa_0 > 0$, respectively. According to [AGHH05, Thm. II.1.1.4] the number of negative eigenvalues of $\mathsf{H}_{\alpha,X}$ is finite and in the end it does not contribute to the asymptotics of the counting function for resonances of $\mathsf{H}_{\alpha,X}$.

It is not difficult to see using standard formula for the determinant of a matrix that $F_{\alpha,X}$ is an exponential polynomial as in Definition 2.1 with the coefficients dependent on α and on the set X.

4.2. **Asymptotics of the number of resonances.** Recall the definition of the counting function for resonances of $\mathsf{H}_{\alpha,X}$.

Definition 4.1. *We define the* counting function $\mathcal{N}_{\alpha,X}(R)$ *as the number of resonances of* $\mathsf{H}_{\alpha,X}$ *with multiplicities lying inside the disc* $\mathcal{D}_R$*.*

Now, we have all the tools to provide a proof for the asymptotics of resonances (1.3) stated in the introduction.

Theorem 4.2. *The counting function for resonances of* $\mathsf{H}_{\alpha,X}$ *asymptotically behaves as*

$$\mathcal{N}_{\alpha,X}(R) = \frac{W_X}{\pi}R + \mathcal{O}(1)\,, \qquad R \to +\infty\,, \tag{4.3}$$

with $W_X \in [0, V_X]$*, where* V_X *is the size of* X *defined in* (1.2)*. In addition,* W_X *is independent of* α*.*

Proof. The argument relies on the resonance condition (4.2). Note that the element of the matrix under the determinant in (4.1) located in the n-th row and the n'-th column is a product of a polynomial in κ and the exponential $\exp(\mathrm{i}\kappa\ell_{nn'})$ with $\ell_{nn'} = |x_n - x_{n'}|$. Hence, expanding $F_{\alpha,X}$ by means of a standard formula for the determinant, we get that each single term in $F_{\alpha,X}$ is a product of a polynomial in κ and the exponential $\exp(\mathrm{i}\kappa\sum_{n=1}^N \ell_{n\pi(n)})$, where $\pi \in \Pi_N$ is a permutation of the set $\{1, 2, \ldots, N\}$.

The term with the lowest multiple of $\mathrm{i}\kappa$ in the exponential is $\left(\alpha - \frac{\mathrm{i}\kappa}{4\pi}\right)^N$, *i.e.* there is no exponential at all and hence $\sigma_{\min} = 0$. The largest possible multiple of $\mathrm{i}\kappa$ in the exponentials of $F_{\alpha,X}$ is V_X. Hence, we get $\sigma_{\max} \le V_X$. The equality $\sigma_{\max} = V_X$ is not always satisfied. If the coefficient by $\exp(\mathrm{i}\kappa V_X)$ vanishes, we have strict inequality $\sigma_{\max} < V_X$. Finally, Theorem 2.2 yields

$$\mathcal{N}_{\alpha,X}(R) = \mathcal{N}_{F_{\alpha,X}}(R) = \frac{W_X}{\pi}R + \mathcal{O}(1)\,, \qquad R \to \infty\,,$$

with some $W_X \in [0, V_X]$.

The term with the largest multiple of $\mathrm{i}\kappa$ in the exponent can be represented as a product $P\left(\alpha - \frac{\mathrm{i}\kappa}{4\pi}\right)\exp(\mathrm{i}\kappa\sigma_{\max})$, where P is a polynomial with real coefficients of degree $< N$. For simple algebraic reasons, if this term does not identically vanish as a function of κ for some $\alpha = \alpha_0 \in \mathbb{R}$, then it does not identically vanish in the same sense for all $\alpha \in \mathbb{R}$. Hence, we obtain by Theorem 2.2 that W_X is independent of α. ☐

Remark 4.1. The proof of Theorem 4.2 gives slightly more, namely the case $W_X < V_X$ can occur only if the maximum in the definition (1.2) of the size V_X of X is attained at more than one permutation, as otherwise cancellation of the principal term in the exponential polynomial $F_{\alpha,X}$ can not occur.

4.3. **Pseudo-orbit expansion for the resonance condition.** The resonance condition (4.2) can be alternatively expressed by contributions of the irreducible pseudo-orbits similarly as for quantum graphs [BHJ12, Lip15, Lip16]. This expression is just yet another way how to write the determinant. However, in some cases one can easier find the terms of the determinant by studying pseudo-orbits on the corresponding directed graph and, eventually, verify their cancellations.

Consider a complete metric graph G having N vertices identified with the respective points in the set X and connected by $\frac{N(N-1)}{2}$ edges of lengths $\ell_{nn'} = |x_n - x_{n'}|$. To this graph we associate its oriented G' counterpart, which is obtained from G by replacing each edge e of G (e is the edge between the points with indices n and n') by two oriented bonds b, $\widehat{b}$ of lengths $|b| = |\widehat{b}| = \ell_{nn'}$. The orientation of the bonds is opposite; b goes from x_n to $x_{n'}$, whereas $\widehat{b}$ goes from $x_{n'}$ to x_n.

Definition 4.3. *With the graph G' we associate the following concepts.*

(a) *A* periodic orbit γ *in the graph G' is a closed path, which begins and ends at the same vertex, we label it by the oriented bonds, which it subsequently visits $\gamma = (b_1, b_2, \dots, b_n)$.*

(b) *A* pseudo-orbit $\widetilde{\gamma}$ *is a collection of periodic orbits $\widetilde{\gamma} = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$. The number of periodic orbits contained in the pseudo-orbit $\widetilde{\gamma}$ will be denoted by $|\widetilde{\gamma}|_{\mathrm{o}} \in \mathbb{N}_0$.*

(c) *An* irreducible pseudo-orbit $\bar{\gamma}$ *is a pseudo-orbit which does not contain any bond more than once. Furthermore, we define*

$$B_{\bar{\gamma}}(\kappa) = \prod_{b_j \in \bar{\gamma}} \left(-\frac{\mathrm{e}^{\mathrm{i}\kappa|b_j|}}{4\pi|b_j|} \right).$$

For $|\bar{\gamma}|_{\mathrm{o}} = 0$ we set $B_{\bar{\gamma}} := 1$. We denote by $\bar{\mathcal{O}}_m$ the set of all irreducible pseudo-orbits in G' containing exactly $m \in \mathbb{N}_0$ bonds.

Note that any permutation $\pi \in \Pi_N$ can be represented as a product of disjoint cycles [Bon04, Sec. 3.1]

$$\pi = (v_1, v_2, \dots, v_{n_1})\,(v_{n_1+1}, \dots, v_{n_1+n_2}) \cdots (v_{n_1+\dots+n_{m(\pi)-1}+1}, \dots, v_{n_1+\dots+n_{m(\pi)}})\,,$$

where $m(\pi)$ is the number of them, $n_j = n_j(\pi)$ is the length of the j^{th}-cycle, and $n(\pi)$ is the number of cycles in π of length one. In this notation, each parenthesis denotes one cycle and *e.g.* for a cycle $(v_1, v_2, \dots, v_{n_1})$ it holds that $\pi(v_1) = v_2$, $\pi(v_2) = v_3$, $\dots$, $\pi(v_{n_1}) = v_1$. Permutations Π_N are in one-to-one correspondence with irreducible pseudo-orbits in Definition 4.3 through the decomposition into cycles; *cf.* [BHJ12, Sec. 3]. Namely, an irreducible pseudo-orbit $\overline{\gamma} = \overline{\gamma}(\pi)$ consists of periodic orbits, each of which is a cycle of π in its decomposition, satisfying $n_j(\pi) > 1$.

With these definitions in hands, we can state the following proposition, whose proof is inspired by the proof of [BHJ12, Thm. 1].

Proposition 4.2. *The resonance condition $F_{\alpha,X}(\kappa) = 0$ in (4.2) can be alternatively written as*

$$\begin{aligned}\sum_{\pi\in\Pi_N} \operatorname{sign}\pi \prod_{n=1}^{N} \left(\left(\alpha - \frac{\mathsf{i}\kappa}{4\pi}\right) \delta_{n\pi(n)} - \widetilde{G}_\kappa(x_n - x_{\pi(n)}) \right) \\ = (-1)^N \sum_{n=0}^{N} \sum_{\bar\gamma\in\bar{\mathcal{O}}_n} (-1)^{|\bar\gamma|_\circ} B_{\bar\gamma}(\kappa) \left(\frac{\mathsf{i}\kappa}{4\pi} - \alpha\right)^{N-n} = 0\,.\end{aligned} \tag{4.4}$$

Proof. Expanding the determinant in the definition of $F_{\alpha,X}$ we get

$$F_{\alpha,X}(\kappa) = \sum_{\pi\in\Pi_N} \operatorname{sign}\pi \prod_{n=1}^{N} \left(\left(\alpha - \frac{\mathsf{i}\kappa}{4\pi}\right) \delta_{n\pi(n)} - \widetilde{G}_\kappa(x_n - x_{\pi(n)}) \right)\,. \tag{4.5}$$

According to [BC09, Sec 4.1], we have $\operatorname{sign}\pi = (-1)^{N+m(\pi)}$. Substituting this formula for $\operatorname{sign}\pi$ into (4.5), making use of the correspondence between irreducible periodic orbits and permutations, the formula $m(\pi) = n(\pi) + |\bar\gamma(\pi)|_\circ$, and performing some simple rearrangements, we find

$$\begin{aligned}F_{\alpha,X}(\kappa) &= \sum_{n=0}^{N} \sum_{\substack{\pi\in\Pi_N \\ n(\pi)=N-n}} \operatorname{sign}\pi \prod_{s=1}^{N} \left(\left(\alpha - \frac{\mathsf{i}\kappa}{4\pi}\right) \delta_{s\pi(s)} - \widetilde{G}_\kappa(x_s - x_{\pi(s)}) \right) \\ &= \sum_{n=0}^{N} \sum_{\substack{\pi\in\Pi_N \\ n(\pi)=N-n}} (-1)^{N+n(\pi)} (-1)^{|\bar\gamma(\pi)|_\circ} B_{\overline{\gamma}(\pi)}(\kappa) \left(\alpha - \frac{\mathsf{i}\kappa}{4\pi}\right)^{N-n} \\ &= (-1)^N \sum_{n=0}^{N} \sum_{\overline{\gamma}\in\mathcal{O}_n} (-1)^{|\bar\gamma|_\circ} B_{\overline{\gamma}}(\kappa) \left(\frac{\mathsf{i}\kappa}{4\pi} - \alpha\right)^{N-n}\,.\end{aligned}$$

□

5. Point configurations of Weyl- and non-Weyl-types

Recall that a configuration of points is said to be of Weyl-type if $W_X = V_X$ and of non-Weyl-type if $W_X < V_X$. In this section we provide examples for both types of point configurations and discuss related questions. For the sake of convenience, for a configuration of points $X = \{x_n\}_{n=1}^N$ and a permutation $\pi \in \Pi_N$ we define

$$V_X(\pi) := \sum_{n=1}^{N} |x_n - x_{\pi(n)}| \,.$$

5.1. **Weyl-type configurations.** First, we show that for low number of points non-Weyl configurations do not exist.

Proposition 5.1. *For $N = 2, 3$, $W_X = V_X$ holds for any $X = \{x_n\}_{n=1}^N$.*

Proof. For $N = 2$, we have $V_X = 2\ell_{12}$. From (4.1) and (4.2) we obtain the resonance condition

$$\left(\frac{\mathrm{i}\kappa}{4\pi} - \alpha\right)^2 - \frac{\mathrm{e}^{2\mathrm{i}\kappa\ell_{12}}}{(4\pi\ell_{12})^2} = 0\,.$$

Obviously, the coefficient at $\mathrm{e}^{\mathrm{i}\kappa V_X}$ does not identically vanish and the claim follows from Theorem 2.2.

Let $N = 3$. Without loss of generality we assume that $\ell_{12} \geq \ell_{23} \geq \ell_{13}$. By triangle inequality we have $\ell_{12}+\ell_{23}+\ell_{13} \geq 2\ell_{12}$. The equality is attained only if all three points belong to a straight line. Hence, we have $V_X = \ell_{12}+\ell_{23}+\ell_{13}$, which is attained at the cyclic shift, having the decomposition $\pi = (1, 2, 3)$. From (4.2) we obtain the resonance condition

$$\left(\frac{\mathrm{i}\kappa}{4\pi} - \alpha\right)^3 - \left(\frac{\mathrm{i}\kappa}{4\pi} - \alpha\right) f(\kappa) + g(\kappa) = 0\,, \quad \text{where}$$

$$f(\kappa) := \frac{1}{(4\pi)^2}\left(\frac{\mathrm{e}^{2\mathrm{i}\kappa\ell_{12}}}{(\ell_{12})^2} + \frac{\mathrm{e}^{2\mathrm{i}\kappa\ell_{23}}}{(\ell_{23})^2} + \frac{\mathrm{e}^{2\mathrm{i}\kappa\ell_{13}}}{(\ell_{13})^2}\right)\,, \quad g(\kappa) := \frac{2\mathrm{e}^{\mathrm{i}\kappa(\ell_{12}+\ell_{23}+\ell_{13})}}{(4\pi)^3\ell_{12}\ell_{23}\ell_{13}}\,.$$

For simple algebraic reasons, in both cases $\ell_{12} + \ell_{23} + \ell_{13} > 2\ell_{12}$ and $\ell_{12} + \ell_{23} + \ell_{13} = 2\ell_{12}$ the coefficient at $\mathrm{e}^{\mathrm{i}\kappa V_X}$ does not vanish identically and the claim also follows from Theorem 2.2. ☐

Next, we show that Weyl-type configurations are not something specific for low number of points and they can be constructed for any number of them.

Theorem 5.1. *For any $N \geq 2$ there exist a configuration of points $X = \{x_n\}_{n=1}^N$ such that $W_X = V_X$.*

Proof. We provide two different constructions for the cases of even and odd number of points in the set X.

For $N = 2m$, $m \in \mathbb{N}$, we choose the configuration $X = \{x_n\}_{n=1}^{2m}$ as follows. First, we fix arbitrary distinct point $x_1, x_2, \ldots, x_m$ on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, so that none of them is diametrically opposite to the other. Second, we select the point $x_{m+k} \in \mathbb{S}^2$, $k = 1, \ldots, m$ to be diametrically opposite to x_k. For simple geometric reasons, we have $V_X = 4m$ and this maximum is attained at the unique permutation π having the following decomposition into cycles

$\pi = (1, m+1)(2, m+2)\dots(m, 2m)$. In view of Remark 4.1, we conclude that $W_X = V_X$.

For $N = 2m+1$, $m \in \mathbb{N}$, we choose the configuration $X = \{x_n\}_{n=1}^{2m+1}$, as follows. First, we distribute the points $\{x_n\}_{n=1}^{2m}$ on $\mathbb{S}^2$ as in the case of even N. Second, we put the point x_{2m+1} into the center of $\mathbb{S}^2$. If a permutation $\pi \in \Pi_{2m+1}$ does not contain the cycle $(2m+1)$, then we have $V_X(\pi) \le 4m$ and the case of equality occurs only for the permutations

$$\begin{aligned}
\pi_1 &= (1, m+1)(2, m+2)\dots(m-1, 2m-1)(m, 2m, 2m+1),\\
\pi_2 &= (1, m+1)(2, m+2)\dots(m-1, 2m-1)(m, 2m+1, 2m),\\
\pi_3 &= (1, m+1)(2, m+2)\dots(m-2, 2m-2)(m-1, 2m-1, 2m+1)(m, 2m),\\
\pi_4 &= (1, m+1)(2, m+2)\dots(m-2, 2m-2)(m-1, 2m+1, 2m-1)(m, 2m),\\
&\dots\dots\\
\pi_{2m-1} &= (2, m+2)\dots(m-1, 2m-1)(m, 2m)(1, m+1, 2m+1),\\
\pi_{2m} &= (2, m+2)...(m-1, 2m-1)(m, 2m)(1, 2m+1, m+1).
\end{aligned}$$

If a permutation $\pi \in \Pi_{2m+1}$ contains the cycle $(2m+1)$, then we again have $V_X(\pi) \le 4m$ and the case of equality happens for the unique permutation

$$\pi_{2m+1} = (1, m+1)(2, m+2)...(m, 2m)(2m+1)\,.$$

Hence, we obtain that $V_X = 4m$. Moreover, the exponential polynomial $F_{\alpha,X}$ in (4.1) can be written as

$$F_{\alpha,X}(\kappa) = (-1)^m \frac{4m + 4\pi\alpha - \mathrm{i}\kappa}{2^{2m}(4\pi)^{2m+1}} \mathrm{e}^{\mathrm{i}(4m)\kappa} + g_0(\kappa) + \sum_{l=1}^{L} g_l(\kappa)\mathrm{e}^{\mathrm{i}\sigma_l\kappa}\,,$$

where $\sigma_l \in (0, 4m)$ and g_0, g_l are polynomials, $l = 1, 2, \dots, L$. Finally, by Theorem 2.2 we get $W_X = V_X = 4m$. □

5.2. **An example of a non-Weyl-type configuration.** Eventually, we provide an example of a configuration of points $X = \{x_n\}_{n=1}^4$ for which $W_X < V_X$ in Theorem 4.2, since there will be a significant cancellation of some terms.

For $a, b, c > 0$, we consider a configuration of points $X = \{x_n\}_{n=1}^4$, where

$$x_1 = (0,0,0)^\top\,, \quad x_2 = (a, -b, 0)^\top\,, \quad x_3 = (a, b, 0)^\top\,, \quad x_4 = (c, 0, 0)^\top\,;$$

see Figure 5.1. Notice that

$$\ell_{12} = \sqrt{a^2+b^2}, \quad \ell_{23} = 2b, \quad \ell_{34} = \sqrt{(a-c)^2+b^2}, \quad \ell_{14} = c. \tag{5.1}$$

Let us assume that b and c are sufficiently small in comparison to a, being more precise $2b + c < \sqrt{a^2+b^2} + \sqrt{(a-c)^2+b^2}$. Let us first write down the general resonance condition (4.2) for four points.

$$\begin{aligned}
c_0^4 - c_0^2(c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 + c_6^2) + 2c_0(c_1c_2c_4 + c_1c_3c_5 + c_2c_3c_6 + c_4c_5c_6)\\
+ c_1^2c_6^2 + c_2^2c_5^2 + c_3^2c_4^2 - 2(c_1c_2c_5c_6 + c_1c_3c_4c_6 + c_2c_3c_4c_5) = 0\,,
\end{aligned}$$

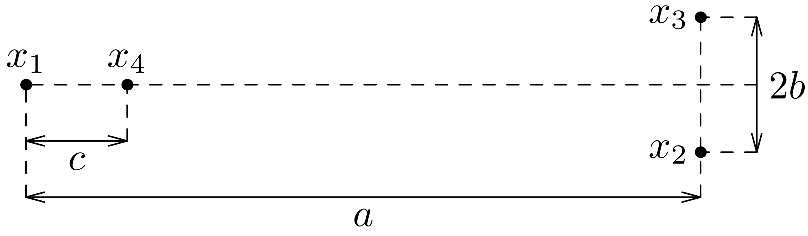

FIGURE 5.1. Discrete set $X = \{x_n\}_{n=1}^4$ related to example in Section 5.2.

where

$$c_0 = \alpha - \frac{\mathrm{i}\kappa}{4\pi}, \quad c_1 = -\frac{\mathrm{e}^{\mathrm{i}\kappa\ell_{12}}}{4\pi\ell_{12}}, \quad c_2 = -\frac{\mathrm{e}^{\mathrm{i}\kappa\ell_{13}}}{4\pi\ell_{13}}, \quad c_3 = -\frac{\mathrm{e}^{\mathrm{i}\kappa\ell_{14}}}{4\pi\ell_{14}},$$
$$c_4 = -\frac{\mathrm{e}^{\mathrm{i}\kappa\ell_{23}}}{4\pi\ell_{23}}, \quad c_5 = -\frac{\mathrm{e}^{\mathrm{i}\kappa\ell_{24}}}{4\pi\ell_{24}}, \quad c_6 = -\frac{\mathrm{e}^{\mathrm{i}\kappa\ell_{34}}}{4\pi\ell_{34}}.$$

In our special case we have

$$\ell_{12} = \ell_{13}, \quad \ell_{34} = \ell_{24} \qquad \text{and} \qquad \ell_{12} + \ell_{34} > \ell_{14} + \ell_{23}. \tag{5.2}$$

Moreover, using (5.1) we get

$$\begin{aligned} \ell_{12} + \ell_{23} + \ell_{34} + \ell_{14} &= 2b + c + \sqrt{a^2+b^2} + \sqrt{(a-c)^2+b^2} \\ &< 2\sqrt{a^2+b^2} + 2\sqrt{(a-c)^2+b^2} \\ &= \ell_{12} + \ell_{34} + \ell_{13} + \ell_{24}. \end{aligned} \tag{5.3}$$

The elements of the group Π_4 can be decomposed into disjoint cycles as

$$\begin{array}{lll} \pi_1 = (1)(2)(3)(4), & \pi_9 = (1,2,3)(4), & \pi_{17} = (1,3)(2,4), \\ \pi_2 = (3,4)(1)(2), & \pi_{10} = (1,2,3,4), & \pi_{18} = (1,3,2,4), \\ \pi_3 = (2,3)(1)(4), & \pi_{11} = (1,2,4,3), & \pi_{19} = (1,4,3,2), \\ \pi_4 = (2,3,4)(1), & \pi_{12} = (1,2,4)(3), & \pi_{20} = (1,4,2)(3), \\ \pi_5 = (2,4,3)(1), & \pi_{13} = (1,3,2)(4), & \pi_{21} = (1,4,3)(2), \\ \pi_6 = (2,4)(1)(3), & \pi_{14} = (1,3,4,2), & \pi_{22} = (1,4)(2)(3), \\ \pi_7 = (1,2)(3)(4), & \pi_{15} = (1,3)(2)(4), & \pi_{23} = (1,4,2,3), \\ \pi_8 = (1,2)(3,4), & \pi_{16} = (1,3,4)(2), & \pi_{24} = (1,4)(2,3). \end{array}$$

Using the above decompositions of permutations and (5.2), (5.3) we find

$$\begin{aligned} &V_X(\pi_8) = V_X(\pi_{11}) = V_X(\pi_{14}) = V_X(\pi_{17}) \\ &\qquad > V_X(\pi_{10}) = V_X(\pi_{18}) = V_X(\pi_{19}) = V_X(\pi_{23}) > \cdots > V_X(\pi_1) = 0. \end{aligned}$$

Hence, $V_X = V_X(\pi_8) = V_X(\pi_{11}) = V_X(\pi_{14}) = V_X(\pi_{17})$ and in view of (5.2) the leading term corresponding to $\exp(\mathrm{i}\kappa V_X)$ in the resonance condition (4.2) cancels

$$\frac{\mathrm{e}^{2\mathrm{i}\kappa(\ell_{12}+\ell_{34})}}{(4\pi)^4\ell_{12}^2\ell_{34}^2} + \frac{\mathrm{e}^{2\mathrm{i}\kappa(\ell_{13}+\ell_{24})}}{(4\pi)^4\ell_{13}^2\ell_{24}^2} - \frac{2\mathrm{e}^{\mathrm{i}\kappa(\ell_{12}+\ell_{34}+\ell_{13}+\ell_{24})}}{(4\pi)^4\ell_{12}\ell_{34}\ell_{13}\ell_{24}} = 0.$$

However, the succeeding term in the condition (4.2) corresponding to the exponent $\exp(\mathrm{i}\kappa V_X(\pi_{10}))$ does not cancel

$$-\frac{2}{(4\pi)^4}\left(\frac{\mathrm{e}^{\mathrm{i}\kappa(\ell_{12}+\ell_{23}+\ell_{34}+\ell_{14})}}{\ell_{12}\ell_{23}\ell_{34}\ell_{14}}+\frac{\mathrm{e}^{\mathrm{i}\kappa(\ell_{13}+\ell_{23}+\ell_{24}+\ell_{14})}}{\ell_{13}\ell_{23}\ell_{24}\ell_{14}}\right)\neq 0\,.$$

Finally, we end up with

$$W_X = V_X(\pi_{10}) = V_X(\pi_{18}) = V_X(\pi_{19}) = V_X(\pi_{23}) < V_X.$$

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