Some notes on the multi-level Gevrey solutions of singularly perturbed linear partial differential equations

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Abstract

This paper is a slightly modified, abridged version of the work [7]. It deals with some questions made to the authors during the conference Analytic, Algebraic and Geometric Aspects of Differential Equations, held in Bedlewo (Poland) during the second week of september, 2015.

We study analytic and formal solutions related to a singularly perturbed partial differential equation and relate them by means of a multi-level Gevrey order asymptotic behaviour, with respect to the perturbation parameter.

Key words: Linear partial differential equations, singular perturbation, formal power series, Borel-Laplace transform, Borel summability, Gevrey asymptotic expansion.

2000 MSC: 35C10, 35C20

1 Introduction

The main aim of the present work is to give answer to certain questions and fruitful mathematical discussions held with some participants of the conference Analytic, Algebraic and Geometric Aspects of Differential Equations (AAGADE), held in Bedlewo (Poland) during the second week of september, 2015, where we presented the work [7]. For the sake of completeness and clarity, we provide an sketch of the results in that work.

The main purpose in [7] is to study a family of singularly perturbed linear partial differential equations of the form

(1)
$$(\epsilon^{r_2}(t^{k+1}\partial_t)^{s_2}+a_2)(\epsilon^{r_1}(t^{k+1}\partial_t)^{s_1}+a_1)\partial_z^S X(t,z,\epsilon)=\sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}X)(t,z,\epsilon),$$

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for given initial conditions

$$(\partial_z^j X)(t,0,\epsilon) = \phi_j(t,\epsilon), \quad 0 \le j \le S-1, \tag{2}$$

where r_1 stands for a nonnegative integer (i. e. it belongs to $\mathbb{N} = \{0,1,...\}$), and r_2, s_1, s_2 are positive integers. We also fix $a_1, a_2 \in \mathbb{C}^\star$. $\mathcal{S}$ consists of a finite subset of elements $(s,\kappa_0,\kappa_1) \in \mathbb{N}^3$. We assume that $S > \kappa_0$ for every $(s,\kappa_0,\kappa_1) \in \mathcal{S}$, and also that $b_{s,\kappa_0,\kappa_1}(z,\epsilon)$ belongs to the space of holomorphic functions in a neighborhood of the origin in $\mathbb{C}^2$, $\mathcal{O}\{z,\epsilon\}$.

The initial data consist of holomorphic functions defined in a product of finite sectors with vertex at the origin.

The case for complex perturbation parameter ϵ has also been studied when solving partial differential equations; in particular, when dealing with solutions belonging to spaces of analytic functions for singularly perturbed partial differential equations which exhibit several singularities of different nature. On this direction, one can cite the work by M. Canalis-Durand, J. Mozo-Fernández and R. Schäfke [2], S. Kamimoto [4], the second author [8, 9], and the first and the second author and J. Sanz [5]. In this last work, the appearance of both, irregular and fuchsian singularities in the problem causes that the Gevrey type concerning the asymptotic representation of the formal solution varies with respect to a problem in which only one type of such singularities appears.

The asymptotic behavior of the solution in the problem (1), (2) distinguish both singularly perturbed irregular operators located at the head of the main equation, in the sense that different Gevrey orders would appear relating asymptotically the analytic and the formal solution in the perturbation parameter ϵ. The main purpose of this work is to exhibit this interesting behaviour of the asymptotics related. For this reason, we do not consider an equation (1) in which nonlinear terms have been taken into consideration. In our opinion, the relevant asymptotic phenomenon coming from the problem would not change, but computations would become tedious and unclear.

We construct actual holomorphic solutions $X(t,z,\epsilon)$ of (1), (2) which are represented by the formal solution

$$\hat{X}(t,z,\epsilon) = \sum_{\beta \ge 0} H_\beta(t,z)\frac{\epsilon^\beta}{\beta!} \in \mathbb{E}[[\epsilon]], \tag{3}$$

where $\mathbb{E}$ is an adecquate complex Banach space. The solution is holomorphic in a domain of the form $\mathcal{T} \times \mathcal{U} \times \mathcal{E}$, where $\mathcal{T}$ and $\mathcal{E}$ are sectors of finite radius and vertex at the origin, and $\mathcal{U}$ is a neighborhood of the origin. In the asymptotic representation several Gevrey orders will appear.

In these notes, we also present some improvements wih respect to the restrictions made on the coefficients appearing in the equation, and the geometry in which the problem rests. Moreover, we provide some details on the appearance of a higher number of operators appearing at the head of the equation and the asymptotic dependence on this data.

2 Summary of the strategy followed and main results

In this section, we present the main results in [7] giving only some detail on the crucial points for this notes. We refer to [7] for the complete details.

Let $S \ge 1$ be an integer. We also consider a nonnegative integer r_1 and positive integers r_2, s_1, s_2, k. Let $r := \frac{r_2}{s_2 k}$. We fix $a_1, a_2 \in \mathbb{C}^\star$ and a finite subset $\mathcal{S}$ of $\mathbb{N}^3$. For every $(s,\kappa_0,\kappa_1) \in \mathcal{S}$, let $b_{\kappa_0\kappa_1}(z,\epsilon)$ be a holomorphic and bounded function in a product of discs centered at the origin.

The problem (1) is studied for ϵ in each of the elements in a good covering in $\mathbb{C}^\star$.

Definition 1 *Let $(\mathcal{E}_i)_{0\leq i\leq \nu-1}$ be a finite family of open sectors such that $\mathcal{E}_i$ has its vertex at the origin and common finite radius $r_{\mathcal{E}_i} := r_{\mathcal{E}} > 0$ for every $0 \leq i \leq \nu-1$. We say this family conforms a good covering in $\mathbb{C}^\star$ if $\mathcal{E}_i \cap \mathcal{E}_{i+1} \neq \emptyset$ for $0 \leq i \leq \nu-1$ (we put $\mathcal{E}_\nu := \mathcal{E}_0$) and $\cup_{0\leq i\leq \nu-1}\mathcal{E}_i = \mathcal{U}\setminus\{0\}$ for some neighborhood of the origin $\mathcal{U}$.*

Definition 2 *Let $(\mathcal{E}_i)_{0\leq i\leq \nu-1}$ be a good covering in $\mathbb{C}^\star$. For every $0\leq i\leq \nu-1$, we assume*

$$\mathcal{E}_i = \{\epsilon \in \mathbb{C}^\star : |\epsilon| < r_{\mathcal{E}}, \theta_{1,\mathcal{E}_i} < \arg(\epsilon) < \theta_{2,\mathcal{E}_i}\},$$

for some $r_{\mathcal{E}} > 0$ and $0 \leq \theta_{1,\mathcal{E}_i} < \theta_{2,\mathcal{E}_i} < 2\pi$. We write $d_{\mathcal{E}_i}$ for the bisecting direction of $\mathcal{E}_i$, $(\theta_{1,\mathcal{E}_i}+\theta_{2,\mathcal{E}_i})/2$. Let $\mathcal{T}$ be an open sector with vertex at 0 and finite radius, say $r_{\mathcal{T}} > 0$. We also fix a family of open sectors

$$S_{d_i,\theta,r_{\mathcal{E}}^r r_{\mathcal{T}}} = \left\{t \in \mathbb{C}^\star : |t| \leq r_{\mathcal{E}}^r r_{\mathcal{T}}, |d_i - \arg(t)| < \frac{\theta}{2}\right\},$$

with $d_i \in [0,2\pi)$ for $0\leq i\leq \nu-1$, and $\theta > \pi/k$, under the following properties:

1. ***Assumption (A)**: one has $\arg(d_i) \neq \frac{\pi(2j+1)+\arg(a_2)}{ks_2}$, for every $j = 0,...,ks_2-1$.*
2. ***Assumption (B)**: one has $s_1r_2 - s_2r_1 > s_2 > 0$ and $|\arg(d_i) - d_{\mathcal{E}_i,j}| > \delta_{2i}$, for $j = 0,...,ks_1-1$, where $\delta_{2i} := \frac{s_1r_2-s_2r_1}{2ks_1s_2}(\theta_{2,\mathcal{E}_i}-\theta_{1,\mathcal{E}_i})$, and $d_{\mathcal{E}_i,j} = \frac{1}{ks_1}(\pi(2j+1)+\arg(a_1)+\frac{s_1r_2-s_2r_1}{s_2}\left(\frac{\theta_{1,\mathcal{E}_i}+\theta_{2,\mathcal{E}_i}}{2}\right))$.*
3. *for every $0\leq i\leq \nu-1$, $t\in\mathcal{T}$ and $\epsilon\in\mathcal{E}_i$, one has $\epsilon^r t \in S_{d_i,\theta,r_{\mathcal{E}}^r r_{\mathcal{T}}}$.*

Under the previous settings, we say the family $\{(S_{d_i,\theta,r_{\mathcal{E}}^r r_{\mathcal{T}}})_{0\leq i\leq \nu-1},\mathcal{T}\}$ is associated to the good covering $(\mathcal{E}_i)_{0\leq i\leq \nu-1}$.

Assumption (A) in the previous definition is concerned with the existence of $d_i \in [0,2\pi)$ such that the argument of every root of the polynomial $\tau \mapsto (k\tau^k)^{s_2} + a_2$ has positive distance to d_i, for every $0\leq i\leq \nu-1$.

The first part in Assumption (B) is motivated by the next

Assumption (C):

$$\theta_{2,\mathcal{E}_i} - \theta_{1,\mathcal{E}_i} < \frac{2\pi s_2}{s_1r_2 - s_2r_1},$$

which guarantees the existence of possible choices for directions $d_i \in [0,2\pi)$ compatible with Assumption (B), in the sense that

$$d_i \notin \frac{1}{ks_1}\left[\pi(2j+1) + \arg(a_1) + \frac{s_1r_2 - s_2r_1}{s_2}\arg(\epsilon)\right],$$

for every $\tau \in S_{d_i,\theta,r_{\mathcal{E}}^r r_{\mathcal{T}}}$, $j = 0,\dots,ks_1-1$, $\epsilon\in\mathcal{E}_i$ and $0\leq i\leq \nu-1$.

The second part in Assumption (B) is related to the existence of $d_i\in[0,2\pi)$ such that the argument of every root of the polynomial $\tau \mapsto \epsilon^{r_1-s_1rk}(k\tau^k)^{s_1} + a_1$ has positive distance to d_i, for every $0\leq i\leq \nu-1$, independently of $\epsilon\in\mathcal{E}_i$.

We also make the further assumption that for every $(s,\kappa_0,\kappa_1)\in\mathcal{S}$, one has $S > \kappa_0 \geq 1$, $S > \kappa_1$, and there exists an integer $\delta_{\kappa_0} \geq k$ such that $s = \kappa_0(k+1)+\delta_{\kappa_0}$, and that $S > \left\lfloor b\left(\frac{\delta_{\kappa_0}}{k}+\kappa_0\right)\right\rfloor + 1$, for some $b > 1$.

This last assumption allows to write the operators $T^s\partial_T^{\kappa_0}$ in such a way that the initial problem is transformed into an auxiliary equation via a slightly modified formal Borel transform (see [10] for the source of this idea and [6] for the properties held by this transformation).

Let $(\mathcal{E}_i)_{0\le i\le \nu-1}$ be a good covering, and let $\{(S_{d_i,\theta,r_{\mathcal{E}}^r r_{\mathcal{T}}})_{0\le i\le \nu-1},\mathcal{T}\}$ be a family associated to that good covering. For every $0\le i\le \nu-1$, we study the Cauchy problem

(4)
$$(\epsilon^{r_2}(t^{k+1}\partial_t)^{s_2}+a_2)(\epsilon^{r_1}(t^{k+1}\partial_t)^{s_1}+a_1)\partial_z^S X_i(t,z,\epsilon)=\sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}X_i)(t,z,\epsilon),$$

for given initial conditions

$$(\partial_z^j X)(t,0,\epsilon)=\phi_{i,j}(t,\epsilon),\quad 0\le j\le S-1, \tag{5}$$

where the functions $\phi_{i,j}$ are constructed as follows: for every $0\le i\le\nu-1$ and all $0\le j\le S-1$, let $W_{i,j}(\tau,\epsilon)\in\mathcal{O}((S_{d_i}\cup D)\times\mathcal{E}_i)$, for some neighborhood of the origin D, and $S_{d_i}=\{t\in\mathbb{C}^\star : |d_i-\arg(t)|<\theta/2\}$. Moreover, we make the assumption that

$$|W_{i,j}(\tau,\epsilon)|\le M_0\left|\frac{\tau}{\epsilon^r}\right|\frac{1}{1+\left|\frac{\tau}{\epsilon^r}\right|^{2k}}\exp\left(\sigma\left|\frac{\tau}{\epsilon^r}\right|^k\right),\qquad (\tau,\epsilon)\in(S_{d_i}\cup D)\times\mathcal{E}_i, \tag{6}$$

for some $M_0,\sigma>0$. Also, we assume $W_{i,j}\equiv W_{i+1,j}$ in the domain $(S_{d_i}\cup D)\times(\mathcal{E}_i\cap\mathcal{E}_{i+1})$, for all $0\le i\le\nu-1$ and every $0\le j\le S-1$. Let $L_{d_i}=[0,\infty)e^{\sqrt{-1}d_i}$. For every $0\le i\le\nu-1$ and all $0\le j\le S-1$, we define

$$\phi_{i,j}(t,\epsilon)k\int_{L_{d_i}}W_{i,j}(u,\epsilon)e^{-(\frac{u}{\epsilon^r t})^k}\frac{du}{u},$$

for $(t,\epsilon)\in\mathcal{T}\times\mathcal{E}_i$. $\phi_{i,j}$ turns out to be a holomorphic function in $\mathcal{T}\times\mathcal{E}_i$.

Under these settings, one is able to construct the solution of (4) with initial conditions (5). We have $X_i(t,z,\epsilon)\in\mathcal{O}(\mathcal{T}\times D'\times\mathcal{E}_i)$, for some neighborhood of the origin D' in the form

$$X_i(t,z,\epsilon)=\sum_{\beta\ge0}X_{i,\beta}(t,\epsilon)\frac{z^\beta}{\beta!}, \tag{7}$$

where

$$X_{i,\beta}(t,\epsilon)=k\int_{L_{d_i}}W_{i,\beta}(u,\epsilon)e^{-(\frac{u}{\epsilon^r t})^k}\frac{du}{u}. \tag{8}$$

The elements $(W_{i,\beta}(\tau,\epsilon))_{\beta\ge0}$ are constructed by a recurrence relation provided that the formal power series $W_i(\tau,z,\epsilon)=\sum_{\beta\ge0}W_{\beta,i}(\tau,\epsilon)\frac{z^\beta}{\beta!}$ is a formal solution of

$$\begin{aligned}((k\tau^k)^{s_2}+a_2)(\epsilon^{r_1-s_1rk}(k\tau^k)^{s_1}+a_1)\partial_z^S W_i(\tau,z,\epsilon)&\\ =\sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}}b_{\kappa_0\kappa_1}(z,\epsilon)\epsilon^{-r(s-\kappa_0)}\Bigg[\frac{\tau^k}{\Gamma\left(\frac{\delta_{\kappa_0}}{k}\right)}\int_0^{\tau^k}(\tau^k-s)^{\frac{\delta_{\kappa_0}}{k}-1}(ks)^{\kappa_0}\partial_z^{\kappa_1}W_i(s^{1/k},z,\epsilon)\frac{ds}{s}&\\ +\sum_{1\le p\le\kappa_0-1}A_{\kappa_0,p}\frac{\tau^k}{\Gamma\left(\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}\right)}\int_0^{\tau^k}(\tau^k-s)^{\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}-1}(ks)^p\partial_z^{\kappa_1}W_i(s^{1/k},z,\epsilon)\frac{ds}{s}\Bigg],&\end{aligned} \tag{9}$$

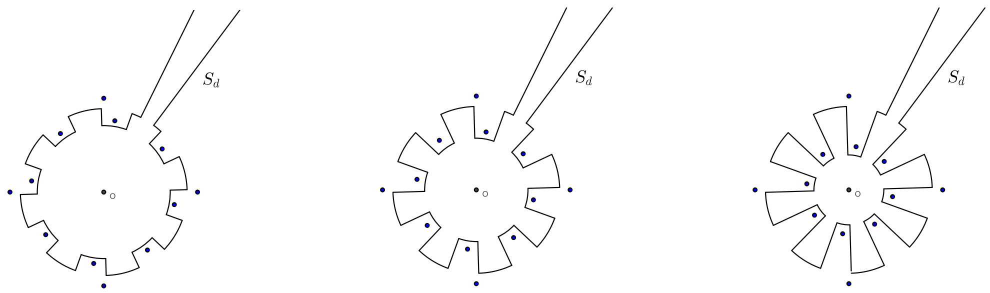

Figure 1: Roots of the polynomials at the head of (9) and domain $\Omega(\epsilon)$.

for given initial data

$$(\partial_z^j W_i)(\tau,0,\epsilon) = W_{i,j}(\tau,\epsilon), \quad 0\le j\le S-1. \tag{10}$$

Here, $A_{\kappa_0,p}\in\mathbb{C}$. The previous equation is the result of applying formal Borel transform to both sides in (4), bearing in mind its properties (see Proposition 3, [7]), and bearing in mind Assumption (C) in order to rewrite the right-hand side of the main equation.

One can observe from equation (9) that a small denominator phenomenon appears when calculating the coefficients $W_{i,\beta}(\tau,\epsilon)$. The domain of definition depends on ϵ and has to avoid the roots of the two polynomials at the head of the equation. This implies the domain of definition of the funtion $\tau\mapsto W_{i,\beta}(\tau,\epsilon)$ depends on $\epsilon\in\mathcal{E}_i$. Indeed, it is defined for $\tau\in\Omega(\epsilon)=S_{d_i}\cup(D\setminus\Omega_1(\epsilon))$, where $\Omega_1(\epsilon)$ turns out to be a finite collection of sets of the form $\{\tau\in\mathbb{C}:|\tau|>\rho(|\epsilon|),|\arg(\tau)-d_i|<\delta_2\}$, where $x\in(0,r_{\mathcal{E}})\mapsto\rho(x)$ is a monotone increasing function with $\rho(x)\to 0$ when $x\to 0$ (see Figure 1).

A fixed point technique allow us to conclude the existence of $M,Z_0>0$ such that

$$|W_{i,\beta}(\tau,\epsilon)|\le MZ_0^\beta\beta!\left|\frac{\tau}{\epsilon^r}\right|\frac{1}{1+\left|\frac{\tau}{\epsilon^r}\right|^{2k}}\exp\left(\sigma r_b(\beta)\left|\frac{\tau}{\epsilon^r}\right|^k\right), \qquad (\tau,\epsilon)\in(S_{d_i}\cup D)\times\mathcal{E}_i, \tag{11}$$

with $r_b(\beta)=\sum_{n=0}^{\beta}1/(n+1)^b$. The estimates in (11) yield (8) is well-defined for $(t,\epsilon)\in\mathcal{T}\times\mathcal{E}_i$.

Theorem 1 *Under the assumptions made, there exist $K,M>0$ (not depending on ϵ), such that*

$$\sup_{t\in\mathcal{T},z\in D'}|X_{i+1}(t,z,\epsilon)-X_i(t,z,\epsilon)|\le K\exp\left(-\frac{M}{|\epsilon|^{\hat{r}_i}}\right),$$

for every $\epsilon\in\mathcal{E}_i\cap\mathcal{E}_{i+1}$, and some positive real number $\hat{r}_i$ which depends on i.

Proof This result corresponds to Theorem 2 in [7]. We give some detail at certain steps of the proof. There are three different situations when estimating the difference of two solutions defined in consecutive elements in the good covering.

1. If there are no singular directions $\frac{\pi(2j+1)+\arg(a_2)}{ks_2}$ for $j=0,...,ks_2-1$ (we will refer to such directions as singular directions of first kind) nor $\tilde{d}$ with $|\tilde{d}_i-\arg(d_{\mathcal{E}_i,j})|\le\delta_{2i}$ for $j=0,...,ks_1$ (we will say these are singular directions of second kind) in between d_i and d_{i+1}, then one can deform the path $d_{\gamma_{i+1}}-d_{\gamma_i}$ to a point by means of Cauchy theorem so that the difference $X_{i+1}-X_i$ is null. In this case, one can reformulate the problem by considering a new good covering combining $\mathcal{E}_i$ and $\mathcal{E}_{i+1}$ in a unique sector.

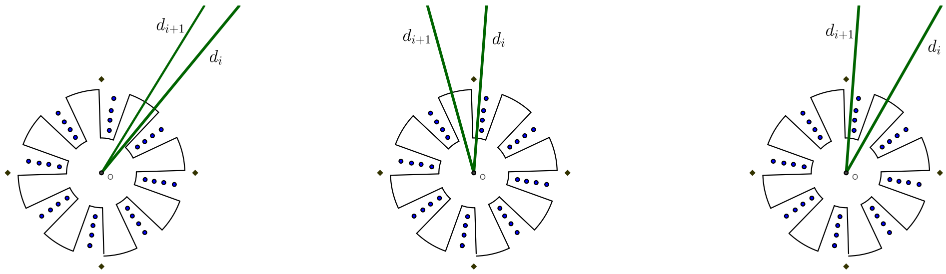

Figure 2: First case (left), second case (center) and third case (right) to be considered in Theorem 1.

2. If there exists at least a singular direction of first kind but no singular directions of second kind in between d_i and d_{i+1}, then the movable singularities depending on ϵ do not affect the geometry of the problem, whereas the path can only be deformed taking into account those singularities which do not depend on ϵ. In this case $\hat{r}_i := r_2/s_2$.

3. If there is at least a singular direction of second kind in between d_i and d_{i+1}, then the movable singularities depend on ϵ, and tend to zero. As a consequence, this affects the geometry of the problem, and the path deformation has to be made accordingly. In this case, $\hat{r}_i := r_1/s_1$.

Regarding the situation in which only singular directions of first kind appear, one can deform the integration path for the integrals along direction d_i and d_{i+1} in (8).

For every $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$ and $t \in \mathcal{T}$ one has

$$X_{i+1,\beta}(t,\epsilon) - X_{i,\beta}(t,\epsilon) = +k \int_{L_{\rho_0/2,d_{i+1}}} W_{i+1,\beta}(u,\epsilon) e^{-(\frac{u}{t\epsilon^r})^k} \frac{du}{u}$$
$$- k \int_{L_{\rho_0/2,d_i}} W_{i,\beta}(u,\epsilon) e^{-(\frac{u}{t\epsilon^r})^k} \frac{du}{u} + \int_{C(\rho_0/2,d_i,d_{i+1})} W_{i,i+1,\beta}(u,\epsilon) e^{-(\frac{u}{t\epsilon^r})^k} \frac{du}{u}.$$

Here, $\rho_0 > 0$ such that $\rho_0 \in D'$, $L_{\rho_0/2,d_{i+1}} := [\frac{\rho_0}{2}, +\infty) e^{\sqrt{-1}d_{i+1}}$, $L_{\rho_0/2,d_i} := [\frac{\rho_0}{2}, +\infty) e^{\sqrt{-1}d_i}$ and $C(\rho_0/2, d_i, d_{i+1})$ is an arc of circle with radius $\rho_0/2$ connecting $\rho_0/2e^{\sqrt{-1}d_{i+1}}$ and $\rho_0/2e^{\sqrt{-1}d_i}$ with a well chosen orientation. Moreover, $W_{i,i+1,\beta}$ denotes the function $W_{i,\beta}$ in an open domain which contains the closed path $(L_{d_{i+1}} \setminus L_{\rho_0/2,d_{i+1}}) - C(\rho_0/2, d_i, d_{i+1}) - (L_{d_i} \setminus L_{\rho_0/2,d_i})$, in which $W_{i,\beta}$ and $W_{i+1,\beta}$ coincide. This is a consequence of the construction of the initial data in our problem.

In the third situation, an analogous argument can be followed. One has to substitute ρ_0 by the function $\epsilon \mapsto \rho(|\epsilon|)$.

The result follows from here after usual estimates.

□

The classical definition of Gevrey asymptotics on functions with values in complex Banach space are considered to describe the asymptotic behaviour relating the analytic and the formal solution of the mein problem under study.

Definition 3 *Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a complex Banach space and $\mathcal{E}$ be an open and bounded sector with vertex at 0. We also consider a positive real number α.*

We say that a function $f:\mathcal{E}\to\mathbb{E}$, holomorphic on $\mathcal{E}$, admits a formal power series $\hat{f}(\epsilon)=\sum_{k\geq 0}a_k\epsilon^k\in\mathbb{E}[[\epsilon]]$ as its $\alpha-$Gevrey asymptotic expansion if, for any closed proper subsector $\mathcal{W}\subseteq\mathcal{E}$ with vertex at the origin, there exist $C,M>0$ such that

$$\left\|f(\epsilon)-\sum_{k=0}^{N-1}a_k\epsilon^k\right\|_{\mathbb{E}}\leq CM^N N!^{\alpha}|\epsilon|^N,$$

for every $N\geq 1$, and all $\epsilon\in\mathcal{W}$.

For the existence of a formal power series in ϵ and the asymptotic relation to the analytic solutions, we make use of a novel version of Ramis-Sibuya theorem in two levels and Theorem 1, in order to conclude with the main result in [7]. For a classical reference on this result, we provide [3] as a reference.

Theorem 2 *Under the previous assumptions, there exists a formal power series*

$$\hat{X}(t,z,\epsilon)=\sum_{\beta\geq 0}H_{\beta}(t,z)\frac{\epsilon^{\beta}}{\beta!}\in\mathbb{E}[[\epsilon]], \tag{12}$$

where $\mathbb{E}$ stands for the Banach space of holomorphic and bounded functions on the set $\mathcal{T}\times D'$ equipped with the supremum norm, which formally solves the equation

$$(\epsilon^{r_2}(t^{k+1}\partial_t)^{s_2}+a_2)(\epsilon^{r_1}(t^{k+1}\partial_t)^{s_1}+a_1)\partial_z^S\hat{X}(t,z,\epsilon)=\sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}}b_{\kappa_0\kappa_1}(z,\epsilon)t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}\hat{X})(t,z,\epsilon). \tag{13}$$

Moreover, $\hat{X}$ can be written in the form

$$\hat{X}(t,z,\epsilon)=a(t,z,\epsilon)+\hat{X}^1(t,z,\epsilon)+\hat{X}^2(t,z,\epsilon), \tag{14}$$

where $a(t,z,\epsilon)\in\mathbb{E}\{\epsilon\}$ is a convergent series on some neighborhood of $\epsilon=0$ and $\hat{X}^1(t,z,\epsilon)$, $\hat{X}^2(t,z,\epsilon)$ are elements in $\mathbb{E}[[\epsilon]]$. Moreover, for every $0\leq i\leq\nu-1$, the $\mathbb{E}$-valued function $\epsilon\mapsto X_i(t,z,\epsilon)$ constructed in (7) is of the form

$$X_i(t,z,\epsilon)=a(t,z,\epsilon)+X_i^1(t,z,\epsilon)+X_i^2(t,z,\epsilon), \tag{15}$$

where $\epsilon\mapsto X_i^j(t,z,\epsilon)$ is a $\mathbb{E}$-valued function which admits $\hat{X}_i^j(t,z,\epsilon)$ as its $\hat{r}_j$-Gevrey asymptotic expansion on $\mathcal{E}_i$, for $j=1,2$.

Corollary 1 *Observe that $r_1/s_1<r_2/s_2$. If one assumes the existence of $i_0\in\{0,\dots,\nu-1\}$ such that $\mathcal{E}_{i_0}$ has opening larger than $\pi s_2/r_2$, such that every index in the set $I_{\delta_1,i,\delta_2}=\{i_0-\delta_1,...,i_0,...,i_0+\delta_2\}$ satisfies 2. in the proof of Theorem 1, for some $\delta_1,\delta_2\geq 0$ and also*

$$\mathcal{E}_{i_0}\subseteq S_{\pi s_1/r_1}\subseteq\cup_{h\in I_{\delta_1,i,\delta_2}}\mathcal{E}_h,$$

where $S_{\pi s_1/r_1}$ stands for a sector with vertex at 0 and opening larger than $\pi s_1/r_1$, then the decomposition in (14) and (15) is unique. In terms of [1], $\hat{X}(t,z,\epsilon)$, as a formal power series in ϵ, with coefficients in $\mathbb{E}$ is $(r_2/s_2,r_1,s_1)-$summable on $\mathcal{E}_{i_0}$, and its $(r_2/s_2,r_1,s_1)-$sum is the function $X_{i_0}(t,z,\epsilon)$ on $\mathcal{E}_{i_0}$.

A practical situation has been considered in [7].

3 Some additional comments and further work

We focus our attention on Assumption (B), which is considered for geometric reasons, as we pointed out before. We now provide an alternative approach to avoid the assumption $s_1r_2 - s_2r_1 > s_2$, following different strategies.

Case 1: Assumption (B.1) $s_1r_2 - s_2r_1 < -s_1$.

Under Assumption (B.1), one can interchange the roles of the operators involved at the head of the main equation in (4), namely $\epsilon^{r_2}(t^{k+1}\partial_t)^{s_2}+a_2$ and $\epsilon^{r_1}(t^{k+1}\partial_t)^{s_1}+a_1$. We consider $r := \frac{r_1}{ks_1}$ and put $T := \epsilon^r t$. After this change of variable, one rewrites the equation obtained by means of the idea in [10], as before. The operators $T^s\partial_t^{\kappa_0}$ can be rewritten so that the properties of formal Borel transform applied at both sides of the transformed equation lead to an auxiliary problem within the Borel plane. We omit all the details here because they follow analogous arguments as in the former construction. After this procedure, one gets the next problem, instead of (9):

$$(\epsilon^{r_2-s_2rk}(k\tau^k)^{s_2}+a_2)((k\tau^k)^{s_1}+a_1)\partial_z^S W_i(\tau,z,\epsilon) \tag{16}$$
$$= \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)\epsilon^{-r(s-\kappa_0)}\left[\frac{\tau^k}{\Gamma\left(\frac{\delta_{\kappa_0}}{k}\right)}\int_0^{\tau^k}(\tau^k-s)^{\frac{\delta_{\kappa_0}}{k}-1}(ks)^{\kappa_0}\partial_z^{\kappa_1}W_i(s^{1/k},z,\epsilon)\frac{ds}{s}\right.$$
$$\left.+\sum_{1\le p\le\kappa_0-1}A_{\kappa_0,p}\frac{\tau^k}{\Gamma\left(\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}\right)}\int_0^{\tau^k}(\tau^k-s)^{\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}-1}(ks)^p\partial_z^{\kappa_1}W_i(s^{1/k},z,\epsilon)\frac{ds}{s}\right].$$

Regarding Assumption (B.1), parallel results to Lemma1 and Lemma 2 in [7] can be proved. More precisely, Lemma 2 in [7] reads as follows:

Lemma 1 *Let $0 \le i \le \nu-1$ and $\epsilon \in \mathcal{E}_i$. Under Assumption (B.1), there exists a constant $C_2 > 0$, not depending on ϵ, such that*

$$\left|\frac{1}{\epsilon^{r_2-s_2rk}(k\tau^k)^{s_2}+a_2}\right| \le C_2,$$

for every $\tau \in \Omega(\epsilon)$.

Indeed, this lemma holds under the less restrictive condition $s_1r_2 - s_2r_1 < 0$. By means of a fixed point argument (analogous to that in Section 3 in [7]) we guarantee a formal solution of (16) under initial conditions (10) in the form $W_i(\tau,z,\epsilon) = \sum_{\beta\ge0} W_{\beta,i}(\tau,\epsilon)\frac{z^\beta}{\beta!}$, ans such that (11) holds.

In Theorem 1, the situations to handle differ. Indeed, the singularities of first kind and of second kind interchange their roles: singular directions $\frac{\pi(2j+1)+\arg(a_1)}{ks_1}$ for $j = 0,\dots,ks_1-1$ become fixed singular directions not depending on $\epsilon \in \mathcal{E}_i$ for any fixed i, i.e. of first kind; whilst directions $\tilde{d}_i \in [0,2\pi)$ with $|\tilde{d}_i - \arg(d_{\mathcal{E}_i,j})| \le \delta_{2i}$ for $j = 0,\dots,ks_2$ turn into movable singular directions with respect to $\epsilon \in \mathcal{E}_i$. If there exist a singular direction of first kind but no singular directions of second kind in between d_i and d_{i+1}, we define $\hat{r}_i := r_1/s_1$. If there is at least a singular direction of second kind in between d_i and d_{i+1}, then we put $\hat{r}_i := r_2/s_2$.

Then, Theorem 2 holds under Assumption (B.1) with the same enunciate.

Case 2: Assumption (B.2) $s_1r_2 - s_2r_1 = 0$.

It is worth mentioning this particular case because under Assumption (B.2), the geometry of the problem changes. There is no longer a distinction between singularities depending on the perurbation parameter and fixed singularities, only remaining the fixed ones. Indeed, $r := \frac{r_2}{s_2 k} = \frac{r_1}{s_1 k}$. The same procedure leads to the auxiliary equation

$$((k\tau^k)^{s_2} + a_2)((k\tau^k)^{s_1} + a_1)\partial_z^S W_i(\tau, z, \epsilon) \tag{17}$$

$$= \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)\epsilon^{-r(s-\kappa_0)} \left[\frac{\tau^k}{\Gamma\left(\frac{\delta_{\kappa_0}}{k}\right)} \int_0^{\tau^k} (\tau^k - s)^{\frac{\delta_{\kappa_0}}{k}-1} (ks)^{\kappa_0} \partial_z^{\kappa_1} W_i(s^{1/k}, z, \epsilon) \frac{ds}{s} \right.$$

$$\left. + \sum_{1\le p\le \kappa_0-1} A_{\kappa_0,p} \frac{\tau^k}{\Gamma\left(\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}\right)} \int_0^{\tau^k} (\tau^k - s)^{\frac{\delta_{\kappa_0}+k(\kappa_0-p)}{k}-1} (ks)^p \partial_z^{\kappa_1} W_i(s^{1/k}, z, \epsilon) \frac{ds}{s} \right],$$

which can be solved by a fixed point theorem, leading to a unique Gevrey order appearing in the asymptotic representation of the solution of (4). More precisely, one has

Theorem 3 *There exists a formal power series $\hat{X}(t,z,\epsilon)$ in the form of (12) which formally solves (13). Moreover, for every $0 \le i \le \nu - 1$, the $\mathbb{E}$-valued function $\epsilon \mapsto X_i(t,z,\epsilon)$ constructed in (7) admits $\hat{X}(t,z,\epsilon)$ as its r_1/s_1-Gevrey asymptotic expansion on $\mathcal{E}_i$.*

Corollary 1 is reduced to the existence of an index $0 \le i_0 \le \nu - 1$ such that the opening of the sector $\mathcal{E}_{i_0}$ is larger than $\pi s_1/r_1$. In this case, $\hat{X}$, as a formal power series in ϵ with coefficients in $\mathbb{E}$ is r_1/s_1-summable in $\mathcal{E}_{i_0}$ by Watson's lemma.

Case 3: Assumption (B.3) $0 < s_1 r_2 - s_2 r_1 < s_2$.

As it has been pointed out, the condition $s_2 < s_1 r_2 - s_2 r_1$ in Assumption (B) is of geometric nature. It is imposed to guarantee the existence of rays from the origin which do not cross the movable singularities appearing at the head of the equation. One may substitute the good covering by any other consisting of sectors with small enough openings.

Case 4: Assumption (B.4) $-s_1 < s_1 r_2 - s_2 r_1 < 0$. Can be studied in the same way as Case 3.

Regarding the geometry of the problem involved, one can consider a more general problem under study, which can be solved analogously. A first approach could be to study the equation

$$(\epsilon^{r_2}(t^{k+1}\partial_t)^{s_2} + a_2)^{m_2}(\epsilon^{r_1}(t^{k+1}\partial_t)^{s_1} + a_1)^{m_1}\partial_z^S X(t,z,\epsilon) = \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon) t^s (\partial_t^{\kappa_0}\partial_z^{\kappa_1} X)(t,z,\epsilon),$$

for any positive integers m_1, m_2.

This more general consideration does not change the configuration of the problem. Indeed, one can follow the same arguments to arrive at the auxiliary equation (9) in which the head of the equation has been substituted by

$$((k\tau^k)^{s_2} + a_2)^{m_2}(\epsilon^{r_1 - s_1 r k}(k\tau^k)^{s_1} + a_1)^{m_1}\partial_z^S W_i(\tau, z, \epsilon).$$

It is straight to check that no additional assumptions have to be added, because the roots of the polynomials $((k\tau^k)^{s_2} + a_2)^{m_2}$ coincide for any positive integer m_2. Also, the same holds for the polynomial $(\epsilon^{r_1 - s_1 r k}(k\tau^k)^{s_1} + a_1)^{m_1}$ for any positive integer m_1. The direction d_i at positive distance to the roots of both polynomials can be chosen independently of m_1 nor m_2. The problem can be solved following the same arguments as in [7]. The main result can be rewritten word by word.

A more general approach to this one could be to consider more than two singularly perturbed terms at the head of the equation. More precisely, one may consider the equation

$$(\epsilon^{r_h}(t^{k+1}\partial_t)^{s_h}+a_h)^{m_h}(\epsilon^{r_{h-1}}(t^{k+1}\partial_t)^{s_{h-1}}+a_{h-1})^{m_1}\dots(\epsilon^{r_1}(t^{k+1}\partial_t)^{s_1}+a_1)\partial_z^S X_i(t,z,\epsilon)$$
$$=\sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{\kappa_0\kappa_1}(z,\epsilon)t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}X_i)(t,z,\epsilon),$$

for some integer $h\ge 2$, $a_j\in\mathbb{C}^\star$, and where r_j stands for a nonnegative integer whilst s_j,m_j are positive integers for every $j=1,\dots,h$. Under this situation, one chooses the indices $\{h_1,\dots,h_\ell\}\subseteq\{1,\dots,h\}$ such that r_{h_μ}/s_{h_μ} coincide for every $\mu=1,\dots,\ell$ and $r_{h_\mu}/s_{h_\mu}>r_p/s_p$ for every $p\in\{1,\dots,h\}\setminus\{h_1,\dots,h_\ell\}$. We write $\overline{r}/\overline{s}:=r_{h_\mu}/s_{h_\mu}$ for any $\mu\in\{1,\dots,\ell\}$.

An analogous procedure can be followed in this situation. We do not enter into details for the sake of clarity, but it is worth mentioning that, under an appropriate geometry for the problem, several Gevrey orders appear in the asymptotic study of the equation. More precisely, the analytic solution can be split in several terms, in the shape of (15) and the formal solution can be written in the form of a sum of the same number of terms as the formal one. One of the terms in the analytic solution admits the corresponding one in the formal solution, as its Gevrey asymptotic expansion of order $\overline{r}/\overline{s}$ in each of the domains of definition of the perturbation parameter. The asymptotic expansions have to be considered as in Theorem 2, with coefficients of the formal power series, and functions with values in the Banach space $\mathbb{E}$. This term corresponds to the fixed singularity appearing in the auxiliary equation, in the Borel plane. The roots to be avoided are all the roots of the polynomials $(k\tau^k)^{s_{h_\mu}}+a_\mu=0$, for $\mu=1,\dots,\ell$.

Regarding the remaining terms at the head of the equation, corresponding to $(\epsilon^{r_p}(t^{k+1}\partial_t)^{s_p}+a_p)^{m_p}$, for $p\in\{1,\dots,h\}\setminus\{h_1,\dots,h_\ell\}$, one observes the phenomenon of movable singularties described in Theorem 2 at each term. The geometry becomes more complicated and one has to choose the direction d_i so that it avoids all singularities.

More precisely, Assumption (A) and Assumption (B) are substituted by the following ones.

Assumption (A): For every $0\le i\le\nu-1$ and $\mu\in\{1,\dots,\ell\}$ one has $\arg(d_i)\neq\frac{\pi(2j+1)+\arg(a_{h_\mu})}{ks_{h_\mu}}$ for every $j=0,\dots,ks_{h_\mu}-1$.

Assumption (B): For every $p\in\{1,\dots,h\}\setminus\{h_1,\dots,h_\ell\}$ and $\mu=1,\dots,\ell$, one has $s_pr_{h_\mu}-s_{h_\mu}r_p>s_p>0$ and $|\arg(d_i)-d_{\mathcal{E}_{i,j,p,\mu}}|>\delta_{2,i,p,\mu}$ for $j=0,\dots,ks_p-1$, where $\delta_{2,i,p,\mu}:=\frac{s_pr_{h_\mu}-s_{h_\mu}r_p}{2ks_{h_\mu}s_p}(\theta_{2,\mathcal{E}_i}-\theta_{1,\mathcal{E}_i})$, and $d_{\mathcal{E}_{i,j,p,\mu}}=\frac{1}{ks_p}(\pi(2j+1)+\arg(a_p)+\frac{s_pr_{h_\mu}-s_{h_\mu}r_p}{s_{h_\mu}}\left(\frac{\theta_{1,\mathcal{E}_i}+\theta_{2,\mathcal{E}_i}}{2}\right))$.

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