SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME NON FREDHOLM OPERATORS WITH DRIFT

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Abstract: We study solvability of certain linear nonhomogeneous elliptic equations and prove that under reasonable technical conditions the convergence in L^2 of their right sides yields the existence and the convergence in H^2 of the solutions. The problems involve second order differential operators with or without Fredholm property, on the whole real line or on a finite interval with periodic boundary conditions. We show that the drift term involved in these equations provides the regularization for the solutions of our problems.

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1. Introduction

Let us consider the problem

$$-\Delta u + V(x)u - au = f, \tag{1.1}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and $V(x)$ is a function decaying to 0 at infinity. If $a \geq 0$, then the essential spectrum of the operator $A : E \to F$, which corresponds to the left-hand side of equation (1.1) contains the origin. Consequently, such operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimensions of its kernel and the codimension of its image are not finite. In this article we will study some properties of such operators. Note that elliptic equations involving non-Fredholm operators were treated extensively in recent years (see [11], [12], [15], [16], [17], [19],

[20], [21], also [3]) along with their potential applications to the theory of reaction-diffusion equations (see [7], [8]). In the particular case where $a = 0$ the operator A satisfies the Fredholm property in some properly chosen weighted spaces [1], [2], [5], [6], [3]. However, the case with $a \neq 0$ is essentially different and the method developed in these articles cannot be applied.

One of the important issues about equations with non-Fredholm operators concerns their solvability. We will study it in the following setting. Let f_n be a sequence of functions in the image of the operator A, such that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Denote by u_n a sequence of functions from $H^2(\mathbb{R}^d)$ such that

$$Au_n = f_n, \ n \in \mathbb{N}.$$

Since the operator A fails to satisfy the Fredholm property, the sequence u_n may not be convergent. Let us call a sequence u_n such that $Au_n \to f$ a solution in the sense of sequences of equation $Au = f$ (see [10]). If this sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this equation. Solution in the sense of sequences is equivalent in this sense to the usual solution. However, in the case of non-Fredholm operators this convergence may not hold or it can occur in some weaker sense. In such case, solution in the sense of sequences may not imply the existence of the usual solution. In this work we will find sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, the conditions on sequences f_n under which the corresponding sequences u_n are strongly convergent.

In the first part of the work we consider the equation with the drift term

$$-\frac{d^2u}{dx^2} - b\frac{du}{dx} - au = f(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $a \geq 0$ and $b \in \mathbb{R}, \ b \neq 0$ are constants and the right side is square integrable. The equation with drift in the context of the Darcy's law describing the fluid motion in the porous medium was treated in [19]. The drift term arises when studying the emergence and propagation of patterns arising in the theory of speciation (see [13]). Nonlinear propagation phenomena for the reaction-diffusion type equations involving the drift term was studied in [4]. The operator involved in the left side of (1.2)

$$L_{a,\ b} := -\frac{d^2}{dx^2} - b\frac{d}{dx} - a : \quad H^2(\mathbb{R}) \to L^2(\mathbb{R}) \tag{1.3}$$

is non-selfadjoint. By means of the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ipx} dx, \quad p \in \mathbb{R} \tag{1.4}$$

it can be easily verified that the essential spectrum of the operator $L_{a,\ b}$ is given by

$$\lambda_{a,\ b}(p) := p^2 - a - ibp, \quad p \in \mathbb{R}.$$

Evidently, when $a > 0$ the operator $L_{a,\ b}$ is Fredholm, because its essential spectrum does not contain the origin. But when $a = 0$ the operator $L_{0,\ b}$ is non Fredholm since its essential spectrum contains the origin.

Note that in the absence of the drift term we are dealing with the self-adjoint operator

$$-\frac{d^2}{dx^2} - a : \quad H^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad a > 0,$$

which fails to satisfy the Fredholm property (see [14], [21]). Let us write down the corresponding sequence of iterated equations with $m \in \mathbb{N}$ as

$$-\frac{d^2 u_m}{dx^2} - b\frac{du_m}{dx} - au_m = f_m(x), \quad x \in \mathbb{R}, \tag{1.5}$$

where the right sides converge to the right side of (1.2) in $L^2(\mathbb{R})$ as $m \to \infty$. The inner product of two functions

$$(f(x), g(x))_{L^2(\mathbb{R})} := \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx, \tag{1.6}$$

with a slight abuse of notations when these functions are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R})$ and $g(x) \in L^\infty(\mathbb{R})$, then clearly the integral considered above makes sense, like for example in the case of functions involved in the orthogonality relations (1.8) and (1.9) of Theorems 1 and 2 below. For our problem on the finite interval $I := [0, 2\pi]$ with periodic boundary conditions, we will use inner product analogously to (1.6), replacing the real line with I. In the article we will consider the space $H^2(\mathbb{R})$ equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \left\|\frac{d^2u}{dx^2}\right\|^2_{L^2(\mathbb{R})}. \tag{1.7}$$

Our first main proposition is as follows.

Theorem 1. *Let* $f(x) : \mathbb{R} \to \mathbb{R}$ *and* $f(x) \in L^2(\mathbb{R})$.

a) When $a > 0$ *equation (1.2) admits a unique solution* $u(x) \in H^2(\mathbb{R})$.

b) When $a = 0$ *let in addition* $xf(x) \in L^1(\mathbb{R})$. *Then problem (1.2) possesses a unique solution* $u(x) \in H^2(\mathbb{R})$ *if and only if the orthogonality condition*

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{1.8}$$

holds.

Note that the expression in the left side of (1.8) is well defined via the elementary argument analogous to the proof of Fact 1 of [17]. Then we turn our attention to proving the solvability in the sense of sequences for our equation on the real line.

Theorem 2. *Let $m \in \mathbb{N}$, $f_m(x) : \mathbb{R} \to \mathbb{R}$ and $f_m(x) \in L^2(\mathbb{R})$. Moreover, $f_m(x) \to f(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$.*

a) When $a > 0$ equations (1.2) and (1.5) have unique solutions $u(x) \in H^2(\mathbb{R})$ and $u_m(x) \in H^2(\mathbb{R})$ respectively, such that $u_m(x) \to u(x)$ in $H^2(\mathbb{R})$ as $m \to \infty$.

b) When $a = 0$ let in addition $xf_m(x) \in L^1(\mathbb{R})$, such that $xf_m(x) \to xf(x)$ in $L^1(\mathbb{R})$ as $m \to \infty$. Furthermore,

$$(f_m(x), 1)_{L^2(\mathbb{R})} = 0, \quad m \in \mathbb{N} \tag{1.9}$$

holds. Then problems (1.2) and (1.5) admit unique solutions $u(x) \in H^2(\mathbb{R})$ and $u_m(x) \in H^2(\mathbb{R})$ respectively, such that $u_m(x) \to u(x)$ in $H^2(\mathbb{R})$ as $m \to \infty$.

Note that in the parts a) of Theorems 1 and 2 above the orthogonality conditions are not required, as distinct from the situation without a drift term discussed in the part a) of Lemma 5 of [21] and in the part a) of Theorem 1.1 of [14]. In the part b) of Theorems 1 and 2 of the present article only a single orthogonality condition is required, as distinct from the situation when $b = 0$ studied in the part b) of Lemma 5 of [21] and in the part a) of Theorem 1.1 of [14], where there solvability was based on the two orthogonality relations. These facts show that the introduction of the drift term provides the regularization for the solutions of our problems.

In the second part of the work we study the analogous equation on the finite interval with periodic boundary conditions, i.e. $I := [0,\ 2\pi]$, namely

$$-\frac{d^2u}{dx^2} - b\frac{du}{dx} - au = f(x), \quad x \in I \tag{1.10}$$

where $a \geq 0$ and $b \in \mathbb{R},\ b \neq 0$ are constants and the right side is bounded and periodic. Clearly,

$$\|f\|_{L^1(I)} \leq 2\pi\|f\|_{L^\infty(I)} < \infty, \quad \|f\|_{L^2(I)} \leq \sqrt{2\pi}\|f\|_{L^\infty(I)} < \infty, \tag{1.11}$$

such that $f(x) \in L^1(I) \cap L^2(I)$ as well. We will use the Fourier transform

$$f_n := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)e^{-inx}dx, \quad n \in \mathbb{Z}, \tag{1.12}$$

such that

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{inx}}{\sqrt{2\pi}}.$$

Evidently, the non-selfadjoint operator involved in the left side of (1.10)

$$\mathcal{L}_{a,\ b} := -\frac{d^2}{dx^2} - b\frac{d}{dx} - a : \quad H^2(I) \to L^2(I) \tag{1.13}$$

is Fredholm. By applying (1.12), it can be easily verified that the spectrum of $\mathcal{L}_{a,\ b}$ is given by

$$\lambda_{a,\ b}(n) := n^2 - a - ibn, \quad n \in \mathbb{Z}$$

and the corresponding eigenfunctions are the Fourier harmonics $\frac{e^{inx}}{\sqrt{2\pi}},\ n \in \mathbb{Z}$. Obviously, the eigenvalues of the operator $\mathcal{L}_{a,\ b}$ are simple, as distinct from the situation without the drift term, when the eigenvalues corresponding to $n \neq 0$ are doubly degenerate (see [18]). The appropriate functional space here $H^2(I)$ is

$$\{u(x) : I \to \mathbb{R} \mid u(x), u''(x) \in L^2(I), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi)\}.$$

For the technical purposes, we will use the following auxiliary constrained subspace

$$H_0^2(I) = \{u(x) \in H^2(I) \mid (u(x), 1)_{L^2(I)} = 0\}, \tag{1.14}$$

which is a Hilbert spaces as well (see e.g. Chapter 2.1 of [9]). When $a > 0$, the kernel of the operator $\mathcal{L}_{a,\ b}$ is empty. For $a = 0$, we consider

$$\mathcal{L}_{0,\ b} : \quad H_0^2(I) \to L^2(I).$$

Such operator has the empty kernel as well. We write down the corresponding sequence of iterated equations with $m \in \mathbb{N}$ as

$$-\frac{d^2u_m}{dx^2} - b\frac{du_m}{dx} - au_m = f_m(x), \quad x \in I, \tag{1.15}$$

where the right sides are bounded, periodic and converge to the right side of (1.10) in $L^\infty(I)$ as $m \to \infty$. Theorems 3 and 4 below are given to show the formal similarity of the results on the finite interval with periodic boundary conditions to the ones established for the whole real line case in Theorems 1 and 2 above.

Theorem 3. *Let $f(x) : I \to \mathbb{R}$, such that $f(0) = f(2\pi)$ and $f(x) \in L^\infty(I)$.*

a) When $a > 0$ equation (1.10) has a unique solution $u(x) \in H^2(I)$.

b) When $a = 0$ problem (1.10) admits a unique solution $u(x) \in H_0^2(I)$ if and only if the orthogonality condition

$$(f(x), 1)_{L^2(I)} = 0 \tag{1.16}$$

holds.

Our final main proposition deals with the solvability in the sense of sequences for our equation on the finite interval I.

Theorem 4. *Let $m \in \mathbb{N},\ f_m(x) : I \to \mathbb{R}$, such that $f_m(0) = f_m(2\pi)$. Moreover, $f_m(x) \in L^\infty(I)$ and $f_m(x) \to f(x)$ in $L^\infty(I)$ as $m \to \infty$.*

a) When $a > 0$ equations (1.10) and (1.15) have unique solutions $u(x) \in H^2(I)$ and $u_m(x) \in H^2(I)$ respectively, such that $u_m(x) \to u(x)$ in $H^2(I)$ as $m \to \infty$.

b) When $a = 0$ let

$$(f_m(x), 1)_{L^2(I)} = 0, \quad m \in \mathbb{N}. \tag{1.17}$$

Then (1.10) and (1.15) admit unique solutions $u(x) \in H_0^2(I)$ and $u_m(x) \in H_0^2(I)$ respectively, such that $u_m(x) \to u(x)$ in $H_0^2(I)$ as $m \to \infty$.

Note that in the parts a) of Theorems 3 and 4 above the orthogonality relations are not required, as distinct from the situation when $a = n_0^2,\ n_0 \in \mathbb{N}$ considered in part II) of Theorem 2 of [18], see also part II) of Theorem 2.2 of [22], where the two orthogonality conditions were involved.

2. The whole real line case

Proof of Theorem 1. First of all, let us show that it would be sufficient to solve our problem in $L^2(\mathbb{R})$. Indeed, if $u(x)$ is a square integrable solution of (1.2), directly from this equation under our assumptions we obtain that

$$-\frac{d^2u}{dx^2} - b\frac{du}{dx} \in L^2(\mathbb{R})$$

as well. By using the standard Fourier transform (1.4), we arrive at $(p^2 - ibp)\widehat{u}(p) \in L^2(\mathbb{R})$. Hence, $\int_{-\infty}^{\infty} p^4|\widehat{u}(p)|^2 dp < \infty$, such that $\frac{d^2u}{dx^2} \in L^2(\mathbb{R})$. Therefore, $u(x) \in H^2(\mathbb{R})$ as well.

To show the uniqueness of solutions of (1.2), we suppose that $u_1(x),\ u_2(x) \in L^2(\mathbb{R})$ satisfy this equation. Then their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R})$ is a solution of the homogeneous problem

$$\frac{d^2w}{dx^2} - b\frac{dw}{dx} - aw = 0.$$

Since the operator $L_{a,\,b}$, which is defined in (1.3) does not possess any nontrivial square integrable zero modes on the real line, the function $w(x)$ vanishes on $\mathbb{R}$.

We apply the standard Fourier transform to both sides of equation (1.2). This yields

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{p^2 - a - ibp}. \tag{2.18}$$

Therefore,

$$\|u\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} \frac{|\widehat{f}(p)|^2}{(p^2 - a)^2 + b^2p^2} dp. \tag{2.19}$$

Let us first consider the case a) of the theorem. By means of (2.19), we obtain

$$\|u\|^2_{L^2(\mathbb{R})} \leq \frac{1}{C}\|f\|^2_{L^2(\mathbb{R})} < \infty$$

due to the one of our assumptions. Here and throughout the article C will denote a finite, positive constant.

Then we turn our attention to the situation when $a = 0$. From (2.18), we easily express

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{ib(p-ib)} - \frac{\widehat{f}(p)}{ibp}. \tag{2.20}$$

The first term in the right side of (2.20) is square integrable, since

$$\int_{-\infty}^{\infty} \frac{|\widehat{f}(p)|^2}{b^2(p^2+b^2)}dp \leq \frac{1}{b^4}\|f\|^2_{L^2(\mathbb{R})} < \infty$$

as assumed. The second term in the right side of (2.20) can be written as

$$\frac{i\widehat{f}(p)}{bp}\chi_{\{|p|\leq 1\}} + \frac{i\widehat{f}(p)}{bp}\chi_{\{|p|>1\}}. \tag{2.21}$$

Here and further down χ_A will denote the characteristic function of a set $A \subseteq \mathbb{R}$. Clearly, the second term in (2.21) can be estimated from above in the absolute value by $\frac{|\widehat{f}(p)|}{|b|} \in L^2(\mathbb{R})$ since $f(x)$ is square integrable according to our assumption. Let us express

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^p \frac{d\widehat{f}(s)}{ds}ds.$$

Hence, the first term in (2.21) can be written as

$$\frac{i\widehat{f}(0)}{bp}\chi_{\{|p|\leq 1\}} + \frac{i\int_0^p \frac{d\widehat{f}(s)}{ds}ds}{bp}\chi_{\{|p|\leq 1\}}. \tag{2.22}$$

Using definition (1.4), we easily estimate

$$\left|\frac{d\widehat{f}(p)}{dp}\right| \leq \frac{1}{\sqrt{2\pi}}\|xf(x)\|_{L^1(\mathbb{R})}.$$

Thus, the second term in (2.22) can be bounded from above in the absolute value by

$$\frac{1}{\sqrt{2\pi}|b|}\|xf(x)\|_{L^1(\mathbb{R})}\chi_{\{|p|\leq 1\}} \in L^2(\mathbb{R}).$$

Obviously, the first term in (2.22) is square integrable if and only if $\widehat{f}(0)$ vanishes, which is equivalent to orthogonality relation (1.8). ■

Then we proceed to establishing the solvability in the sense of sequences for our problem on the real line.

Proof of Theorem 2. First of all, we suppose that equations (1.2) and (1.5) admit unique solutions $u(x) \in H^2(\mathbb{R})$ and $u_m(x) \in H^2(\mathbb{R}),\ m \in \mathbb{N}$ respectively, such that $u_m(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$. This will imply that $u_m(x)$ also converges to $u(x)$ in $H^2(\mathbb{R})$ as $m \to \infty$. Indeed, from (1.2) and (1.5) we easily obtain

$$\left\| -\frac{d^2(u_m-u)}{dx^2} - b\frac{d(u_m-u)}{dx}\right\|_{L^2(\mathbb{R})} \leq \|f_m - f\|_{L^2(\mathbb{R})} + a\|u_m - u\|_{L^2(\mathbb{R})}.$$

The right side of the inequality above tends to zero as $m \to \infty$ due to our assumptions. By applying the standard Fourier transform (1.4), we arrive at

$$\int_{-\infty}^{\infty} p^4 |\widehat{u}_m(p) - \widehat{u}(p)|^2 dp \to 0, \quad m \to \infty,$$

such that $\dfrac{d^2u_m}{dx^2} \to \dfrac{d^2u}{dx^2}$ in $L^2(\mathbb{R})$ as $m \to \infty$. Therefore, $u_m(x) \to u(x)$ in $H^2(\mathbb{R})$ as $m \to \infty$ as well.

By applying the standard Fourier transform (1.4) to both sides of (1.5), we arrive at

$$\widehat{u}_m(p) = \frac{\widehat{f}_m(p)}{p^2 - a - ibp}, \quad m \in \mathbb{N} \tag{2.23}$$

Let us first consider the case a) of the theorem. By means of the part a) of Theorem 1, equations (1.2) and (1.5) admit unique solutions $u(x) \in H^2(\mathbb{R})$ and $u_m(x) \in H^2(\mathbb{R}),\ m \in \mathbb{N}$ respectively. By virtue of (2.23) along with (2.18), we derive

$$\|u_m - u\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \frac{|\widehat{f}_m(p) - \widehat{f}(p)|^2}{(p^2-a)^2 + b^2p^2} dp.$$

Hence,

$$\|u_m - u\|_{L^2(\mathbb{R})} \leq \frac{1}{C}\|f_m - f\|_{L^2(\mathbb{R})} \to 0, \quad m \to \infty,$$

which implies that in the case of $a > 0$ we have $u_m(x) \to u(x)$ in $H^2(\mathbb{R})$ as $m \to \infty$ as discussed above.

Let us conclude the proof of the theorem by considering the situation when the parameter $a = 0$. By means of the result of the part a) of Lemma 3.3 of [14], under our assumptions we obtain that

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{2.24}$$

holds. Then by virtue of the part b) of Theorem 1, problems (1.2) and (1.5) have unique solutions $u(x) \in H^2(\mathbb{R})$ and $u_m(x) \in H^2(\mathbb{R}),\ m \in \mathbb{N}$ respectively when a

vanishes. Formulas (2.23) and (2.18) give us

$$\widehat{u}_m(p) - \widehat{u}(p) = \frac{\widehat{f}_m(p) - \widehat{f}(p)}{ib(p - ib)} - \frac{\widehat{f}_m(p) - \widehat{f}(p)}{ibp},$$

which yields

$$\|u_m - u\|_{L^2(\mathbb{R})} \leq \frac{1}{|b|}\left\|\frac{\widehat{f}_m(p) - \widehat{f}(p)}{p - ib}\right\|_{L^2(\mathbb{R})} + \frac{1}{|b|}\left\|\frac{\widehat{f}_m(p) - \widehat{f}(p)}{p}\right\|_{L^2(\mathbb{R})}. \tag{2.25}$$

Clearly, the norm involved in the first term in the right side of (2.25) can be bounded from above by

$$\frac{1}{|b|}\|f_m - f\|_{L^2(\mathbb{R})} \to 0, \quad m \to \infty$$

as assumed. Let us express

$$\frac{\widehat{f}_m(p) - \widehat{f}(p)}{p} = \frac{\widehat{f}_m(p) - \widehat{f}(p)}{p}\chi_{\{|p|\leq 1\}} + \frac{\widehat{f}_m(p) - \widehat{f}(p)}{p}\chi_{\{|p|>1\}}.$$

Hence

$$\left\|\frac{\widehat{f}_m(p) - \widehat{f}(p)}{p}\right\|_{L^2(\mathbb{R})} \leq$$

$$\leq \left\|\frac{\widehat{f}_m(p) - \widehat{f}(p)}{p}\chi_{\{|p|\leq 1\}}\right\|_{L^2(\mathbb{R})} + \left\|\frac{\widehat{f}_m(p) - \widehat{f}(p)}{p}\chi_{\{|p|>1\}}\right\|_{L^2(\mathbb{R})}. \tag{2.26}$$

Evidently, the second term in the right side of (2.26) can be estimated from above by

$$\|f_m - f\|_{L^2(\mathbb{R})} \to 0, \quad m \to \infty$$

due to the one of our assumptions. Orthogonality conditions (2.24) and (1.9) give us

$$\widehat{f}(0) = 0, \quad \widehat{f}_m(0) = 0, \quad m \in \mathbb{N},$$

such that

$$\widehat{f}(p) = \int_0^p \frac{d\widehat{f}(s)}{ds}ds, \quad \widehat{f}_m(p) = \int_0^p \frac{d\widehat{f}_m(s)}{ds}ds, \quad m \in \mathbb{N}. \tag{2.27}$$

Hence, it remains to estimate the norm of the term

$$\frac{\int_0^p [\frac{d\widehat{f}_m(s)}{ds} - \frac{d\widehat{f}(s)}{ds}]ds}{p}\chi_{\{|p|\leq 1\}}.$$

Using the definition of the standard Fourier transform (1.4), we easily derive

$$\Big|\frac{d\widehat{f}_m(p)}{dp} - \frac{d\widehat{f}(p)}{dp}\Big| \le \frac{1}{\sqrt{2\pi}}\|xf_m(x) - xf(x)\|_{L^1(\mathbb{R})}.$$

Thus,

$$\left\|\frac{\widehat{f}_m(p) - \widehat{f}(p)}{p}\chi_{\{|p|\le 1\}}\right\|_{L^2(\mathbb{R})} \le \frac{1}{\sqrt{\pi}}\|xf_m(x) - xf(x)\|_{L^1(\mathbb{R})} \to 0, \quad m \to \infty$$

as assumed. Therefore, $u_m(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$, which implies that $u_m(x) \to u(x)$ in $H^2(\mathbb{R})$ as $m \to \infty$ as discussed above. ■

3. The problem on the finite interval

Proof of Theorem 3. First of all, we prove that it would be sufficient to solve our problem in $L^2(I)$. Indeed, if $u(x)$ is a square integrable solution of (1.10), periodic on I along with its first derivative, directly from our equation under the given conditions we arrive at

$$-\frac{d^2u}{dx^2} - b\frac{du}{dx} \in L^2(I).$$

Using (1.12), we obtain $(n^2 - ibn)u_n \in l^2$. Thus, $\sum_{n=-\infty}^{\infty} n^4|u_n|^2 < \infty$, such that $\frac{d^2u}{dx^2} \in L^2(I)$. This implies that $u(x) \in H^2(I)$ as well.

To prove the uniqueness of solutions of (1.10), we treat the case of $a > 0$. When a vanishes, we are able to use the similar argument in the constrained subspace $H_0^2(I)$. Let us suppose that $u_1(x),\ u_2(x) \in H^2(I)$ solve (1.10). Then their difference $w(x) := u_1(x) - u_2(x) \in H^2(I)$ satisfies the homogeneous equation

$$\frac{d^2w}{dx^2} - b\frac{dw}{dx} - aw = 0.$$

Since the operator $\mathcal{L}_{a,\,b} : H^2(I) \to L^2(I)$ defined in (1.13) does not have any nontrivial $H^2(I)$ zero modes, the function $w(x)$ vanishes on I.

Let us apply the Fourier transform (1.12) to both sides of problem (1.10). This yields

$$u_n = \frac{f_n}{n^2 - a - ibn}, \quad n \in \mathbb{Z}. \tag{3.28}$$

First we consider the case a) of our theorem. By virtue of (3.28), we arrive at

$$\|u\|^2_{L^2(I)} \le \frac{1}{C}\|f\|^2_{L^2(I)} < \infty$$

via the one of our assumptions (see (1.11)). Let us conclude the proof of the theorem by treating the situation when $a = 0$. From (3.28), we easily derive

$$u_n = \frac{f_n}{ib(n - ib)} - \frac{f_n}{ibn}, \quad n \in \mathbb{Z}. \tag{3.29}$$

The first term in the right side of (3.29) belongs to l^2, since

$$\sum_{n=-\infty}^{\infty} \frac{|f_n|^2}{b^2(n^2 + b^2)} \leq \frac{1}{b^4}\|f\|_{L^2(I)}^2 < \infty$$

as discussed above. The second term in the right side of (3.29) belongs to l^2 if and only if $f_0 = 0$ and the square of its l^2 norm can be easily bounded from above by $\frac{1}{b^4}\|f\|_{L^2(I)}^2 < \infty$, which is equivalent to orthogonality condition (1.16). ∎

Let us conclude the article with proving the solvability in the sense of sequences for our problem on the interval I with periodic boundary conditions.

Proof of Theorem 4. Evidently,

$$|f(0) - f(2\pi)| \leq |f(0) - f_m(0)| + |f_m(2\pi) - f(2\pi)| \leq 2\|f_m - f\|_{L^\infty(I)} \to 0$$

as $m \to \infty$. Hence, $f(0) = f(2\pi)$. By means of (1.11) under our assumptions, we have $f_m(x) \in L^1(I) \cap L^2(I), \quad m \in \mathbb{N}$. Using (1.11), we arrive at

$$\|f_m(x) - f(x)\|_{L^1(I)} \leq 2\pi\|f_m(x) - f(x)\|_{L^\infty(I)} \to 0, \quad m \to \infty \tag{3.30}$$

as assumed, such that $f_m(x) \to f(x)$ in $L^1(I)$ as $m \to \infty$. Similarly, via (1.11) we obtain

$$\|f_m(x) - f(x)\|_{L^2(I)} \leq \sqrt{2\pi}\|f_m(x) - f(x)\|_{L^\infty(I)} \to 0, \quad m \to \infty, \tag{3.31}$$

such that $f_m(x) \to f(x)$ in $L^2(I)$ as $m \to \infty$ as well. By applying the Fourier transform (1.12) to both sides of (1.15), we derive

$$u_{m,n} = \frac{f_{m,n}}{n^2 - a - ibn}, \quad m \in \mathbb{N}, \quad n \in \mathbb{Z}. \tag{3.32}$$

First we consider the case a) of our theorem. By virtue of the part a) of Theorem 3, problems (1.10) and (1.15) possess unique solutions $u(x) \in H^2(I)$ and $u_m(x) \in H^2(I),\ m \in \mathbb{N}$ respectively. (3.32) along with (3.28) yield

$$\|u_m - u\|_{L^2(I)}^2 = \sum_{n=-\infty}^{\infty} \frac{|f_{m,n} - f_n|^2}{(n^2 - a)^2 + b^2n^2} \leq \frac{1}{C}\|f_m - f\|_{L^2(I)}^2 \to 0, \quad m \to \infty$$

via (3.31). Thus, $u_m(x) \to u(x)$ in $L^2(I)$ as $m \to \infty$. Let us show that $u_m(x)$ converges to $u(x)$ in $H^2(I)$ as $m \to \infty$. Indeed, (1.10) and (1.15) give us

$$\left\| -\frac{d^2(u_m - u)}{dx^2} - b\frac{d(u_m - u)}{dx} \right\|_{L^2(I)} \leq \|f_m - f\|_{L^2(I)} + a\|u_m - u\|_{L^2(I)}.$$

The right side of this inequality converges to zero as $m \to \infty$ due to (3.31). Using the Fourier transform (1.12), we derive

$$\sum_{n=-\infty}^{\infty} n^4 |u_{m,n} - u_n|^2 \to 0, \quad m \to \infty.$$

Hence, $\dfrac{d^2 u_m}{dx^2} \to \dfrac{d^2 u}{dx^2}$ in $L^2(I)$ as $m \to \infty$, such that $u_m(x) \to u(x)$ in $H^2(I)$ as $m \to \infty$ as well.

We conclude the article by dealing with the situation when the parameter a vanishes. By means of (1.17) along with (3.30), we obtain

$$|(f(x), 1)_{L^2(I)}| = |(f(x) - f_m(x), 1)_{L^2(I)}| \leq \|f_m - f\|_{L^1(I)} \to 0, \quad m \to \infty,$$

such that

$$(f(x), 1)_{L^2(I)} = 0 \tag{3.33}$$

holds. By virtue of the part b) of Theorem 3 above equations (1.10) and (1.15) admit unique solutions $u(x) \in H_0^2(I)$ and $u_m(x) \in H_0^2(I), \quad m \in \mathbb{N}$ respectively when $a = 0$. Formulas (3.28) and (3.32) yield

$$u_{m,n} - u_n = \frac{f_{m,n} - f_n}{ib(n - ib)} - \frac{f_{m,n} - f_n}{ibn}, \quad m \in \mathbb{N}, \quad n \in \mathbb{Z}. \tag{3.34}$$

By means of orthogonality relations (3.33) and (1.17), we have

$$f_0 = 0, \quad f_{m,0} = 0, \quad m \in \mathbb{N}.$$

Obviously, the l^2 norm of the first term in the right side of (3.34) can be bounded from above by

$$\frac{1}{b^2}\|f_m - f\|_{L^2(I)} \to 0, \quad m \to \infty$$

via (3.31). The l^2 norm of the second term in the right side of (3.34) can be estimated from above by

$$\sqrt{\sum_{n=-\infty,\, n\neq 0}^{\infty} \frac{|f_{m,n} - f_n|^2}{b^2 n^2}} \leq \frac{1}{|b|}\|f_m - f\|_{L^2(I)} \to 0, \quad m \to \infty$$

as above. Therefore, $u_m(x) \to u(x)$ in $L^2(I)$ as $m \to \infty$, which implies that $u_m(x) \to u(x)$ in $H_0^2(I)$ as $m \to \infty$ as well via the argument analogous to the one we had in the proof of the part a) of the theorem. ■

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