Existence of solutions for some non-Fredholm integro-differential equations with the bi-Laplacian

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Abstract. We prove the existence of solutions for some semilinear elliptic equations in the appropriate H^4 spaces using the fixed point technique where the elliptic equation contains fourth order differential operators with and without Fredholm property, generalizing the results of the preceding work [22].

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1 Introduction

We recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. Consequently, the problem $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. Such properties of Fredholm operators are extensively used in many methods of linear and nonlinear analysis.

Elliptic equations in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, proper ellipticity and Lopatinskii conditions are fulfilled (see e.g. [1], [10], [15]). This is the main result of the theory of linear elliptic problems. In the case of unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For example, Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ fails to satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

Linear elliptic equations in unbounded domains satisfy the Fredholm property if and only if, in addition to the conditions stated above, limiting operators are invertible (see [17]). In some trivial cases, limiting operators can be explicitly constructed. For instance, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have limits at infinity,

$$a_{\pm} = \lim_{x \to \pm\infty} a(x), \quad b_{\pm} = \lim_{x \to \pm\infty} b(x), \quad c_{\pm} = \lim_{x \to \pm\infty} c(x),$$

the limiting operators are:

$$L_{\pm}u = a_{\pm}u'' + b_{\pm}u' + c_{\pm}u.$$

Since the coefficients are constant, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L - \lambda$ fails to satisfy the Fredholm property, can be explicitly found by means of the Fourier transform:

$$\lambda_{\pm}(\xi) = -a_{\pm}\xi^2 + b_{\pm}i\xi + c_{\pm}, \quad \xi \in \mathbb{R}.$$

Invertibility of limiting operators is equivalent to the condition that the essential spectrum does not contain the origin.

In the case of general elliptic equations, the same assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or if the limiting operators are invertible. However, these conditions may not be explicitly written.

In the case of non-Fredholm operators the usual solvability conditions may not be applicable and solvability conditions are, in general, not known. There are some classes of operators for which solvability conditions are obtained. Let us illustrate them with the following example. Consider the problem

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, where a is a positive constant. The operator L coincides with its limiting operators. The homogeneous equation admits a nonzero bounded solution. Hence the Fredholm property is not satisfied. However, since the operator has constant coefficients, we can apply the Fourier transform and find the solution explicitly. Solvability conditions can be formulated as follows. If $f \in L^2(\mathbb{R}^d)$ and $xf \in L^1(\mathbb{R}^d)$, then there exists a solution of this equation in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}} \right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see [24]). Here and further down S^d_r stands for the sphere in $\mathbb{R}^d$ of radius r centered at the origin. Hence, though the operator fails to satisfy the Fredholm property, solvability conditions are formulated in a similar way. However, this similarity is only formal since the range of the operator is not closed.

In the case of the operator with a potential,

$$Lu \equiv \Delta u + a(x)u = f,$$

Fourier transform is not directly applicable. Nevertheless, solvability conditions in $\mathbb{R}^3$ can be obtained by a rather sophisticated application of the theory of self-adjoint operators (see [21]). As before, solvability conditions are formulated in terms of orthogonality to solutions of the homogeneous adjoint equation. There are several other examples of linear elliptic operators without Fredholm property for which solvability conditions can be obtained (see [17]-[24]). The bi-Laplacian is relevant to the studies of the solvability conditions for a linearized Cahn-Hilliard equation (see e.g. [18]). The boundedness of the gradient of a solution for the biharmonic equation was established in [11]. The behavior near the boundary of solutions to the Dirichlet problem for the biharmonic operator was studied in [12] and [13]. The Dirichlet problem in Lipschitz domains for higher order elliptic systems with rough coefficients was considered in [14].

Solvability relations play a significant role in the analysis of nonlinear elliptic equations. In the case of operators without Fredholm property, in spite of some progress in understanding of linear problems, there exist only few examples where nonlinear non-Fredholm operators are analyzed (see [5]-[7]). In the present work we treat another class of nonlinear problems, for which the Fredholm property may not be satisfied:

$$-\Delta^2 u + a^2 u + \int_\Omega G(x-y)F(u(y),y)dy = 0, \quad a > 0, \tag{1.2}$$

generalizing the results derived in [22] for the equation analogous to (1.2) but containing the standard Laplace operator. Here Ω is a domain in $\mathbb{R}^d$, $d = 1, 2, 3$, the more physically interesting dimensions. Some applications of this model are presented in Appendix 2. The linear part of the corresponding operator here is similar to problem (1.1) above, it only contains the negative bi-Laplacian. We will use the explicit form of solvability relations and will study the existence of solutions of the nonlinear problem.

2 Formulation of the results

The nonlinear part of problem (1.2) will satisfy the following regularity conditions.

Assumption 1. *Function $F(u,x) : \mathbb{R} \times \Omega \to \mathbb{R}$ is such that*

$$|F(u,x)| \le k|u| + h(x) \quad for \quad u \in \mathbb{R}, \ x \in \Omega \tag{2.1}$$

with a constant $k > 0$ and $h(x) : \Omega \to \mathbb{R}^+$, $h(x) \in L^2(\Omega)$. Furthermore, it is a Lipschitz continuous function, such that

$$|F(u_1,x) - F(u_2,x)| \le l|u_1 - u_2| \quad for \quad any \quad u_{1,2} \in \mathbb{R}, \quad x \in \Omega \tag{2.2}$$

with a constant $l > 0$.

We introduce the auxiliary problem

$$\Delta^2 u - a^2 u = \int_{\Omega} G(x-y)F(v(y),y)dy, \quad a>0. \tag{2.3}$$

Let us denote

$$(f_1(x), f_2(x))_{L^2(\Omega)} := \int_{\Omega} f_1(x)\bar{f}_2(x)dx, \tag{2.4}$$

with a slight abuse of notations when these functions are not square integrable, like for example those used in the orthogonality relations (6.4) and (6.5) of Lemma A1 of Appendix 1. Indeed, if $f_1(x) \in L^1(\Omega)$ and $f_2(x) \in L^\infty(\Omega)$, the integral in the right side of (2.4) makes sense. In the first part of the work we treat the case of $\Omega = \mathbb{R}^d,\ 1 \le d \le 3$, such that the appropriate Sobolev space is equipped with the norm

$$\|u\|^2_{H^4(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|\Delta^2 u\|^2_{L^2(\mathbb{R}^d)}.$$

The main issue for the equation above is that the operator $\Delta^2 - a^2 : H^4(\mathbb{R}^d) \to L^2(\mathbb{R}^d),\ a>0$ fails to satisfy the Fredholm property, which is the obstacle to solve problem (2.3). The similar situations but in linear problems, both self- adjoint and non self-adjoint involving non Fredholm second or fourth order differential operators or even systems of equations with non Fredholm operators have been studied extensively in recent years (see [16], [17], [18], [19], [20], [21], [22], [23], [24]). However, we manage to establish that problem (2.3) in such case defines a map $T_a : H^4(\mathbb{R}^d) \to H^4(\mathbb{R}^d),\ a>0$, which is a strict contraction under the given technical conditions.

Theorem 1. *Let $\Omega = \mathbb{R}^d,\ d \le 3,\ G(x) : \mathbb{R}^d \to \mathbb{R},\ G(x) \in L^1(\mathbb{R}^d),\ xG(x) \in L^1(\mathbb{R}^d)$, the constant $a > 0$, $N_{a,\ d}$ is defined in (6.3) and Assumption 1 holds.*

Let orthogonality relations (6.4) hold for $d = 1$ and (6.5) when $d = 2, 3$ and $\sqrt{2}(2\pi)^{\frac{d}{2}} N_{a,\ d}\, l < 1$. Then the map $T_a v = u$ on $H^4(\mathbb{R}^d)$ defined by equation (2.3) has a unique fixed point v_a, which is the only solution of problem (1.2) in $H^4(\mathbb{R}^d)$.

The fixed point $v_a,\ a > 0$ is nontrivial provided the intersection of supports of the Fourier transforms of functions $supp\widehat{F(0,x)} \cap supp\widehat{G}$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$.

In the second part of the article we consider the analogous problem on the finite interval with periodic boundary conditions, i.e. $\Omega = I := [0,\ 2\pi]$ and the appropriate function space is

$$H^4(I) = \{u(x) : I \to \mathbb{R} \mid u(x), u''''(x) \in L^2(I),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),$$
$$u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi)\}.$$

We introduce the following auxiliary constrained subspace

$$H^4_0(I) := \{u \in H^4(I) \mid \Big(u(x), \frac{e^{\pm i n_0 x}}{\sqrt{2\pi}}\Big)_{L^2(I)} = 0\},\ n_0 \in \mathbb{N}, \tag{2.5}$$

which is a Hilbert spaces as well (see e.g. Chapter 2.1 of [9]). Let us establish that equation (2.3) in this situation defines a map $\tau_a,\ a>0$ on the above mentioned spaces which will be a strict contraction under the given assumptions.

Theorem 2. *Let $\Omega = I,\ G(x) : I \to \mathbb{R},\ G(x) \in L^1(I),\ G(0) = G(2\pi),\ F(u,0) = F(u,2\pi)$ for $u \in \mathbb{R}$ and Assumption 1 holds.*

I) If $a > 0$ and $a \neq n_0^2,\ n_0 \in \mathbb{N}$, we assume that $2\sqrt{\pi}\mathcal{N}_a l < 1$. Then the map $\tau_a v = u$ on $H^4(I)$ defined by equation (2.3) has a unique fixed point v_a, the only solution of problem (1.2) in $H^4(I)$.

II) If $a = n_0^2,\ n_0 \in \mathbb{N}$, we assume that orthogonality conditions (6.11) hold and $2\sqrt{\pi}\mathcal{N}_{n_0^2} l < 1$. Then the map $\tau_{n_0^2} v = u$ on $H_0^4(I)$ defined by equation (2.3) possesses a unique fixed point $v_{n_0^2}$, the only solution of equation (1.2) in $H_0^4(I)$.

In both cases I) and II) the fixed point $v_a,\ a > 0$ is nontrivial provided the Fourier coefficients $G_n F(0,x)_n \neq 0$ for some $n \in \mathbb{Z}$.

Remark. *We use the constrained subspace $H_0^4(I)$ in case II), such that the operator*

$$\frac{d^4}{dx^4} - n_0^4 : H_0^4(I) \to L^2(I), \quad n_0 \in \mathbb{N}$$

which possesses the Fredholm property, has the empty kernel.

Let us conclude the article with the studies of our problem on the product of spaces, where one is the finite interval with periodic boundary conditions as before and another is the whole space of dimension not exceeding two. Hence, in our notations $\Omega = I \times \mathbb{R}^d = [0,2\pi] \times \mathbb{R}^d,\ d = 1,2$ and $x = (x_1, x_\perp)$ with $x_1 \in I$ and $x_\perp \in \mathbb{R}^d$. The appropriate Sobolev space for our problem is $H^4(\Omega)$ defined as

$$\Big\{ u(x) : \Omega \to \mathbb{R} \mid u(x),\ \Delta^2 u(x) \in L^2(\Omega),\ u(0,x_\perp) = u(2\pi, x_\perp),$$

$$\frac{\partial u}{\partial x_1}(0,x_\perp) = \frac{\partial u}{\partial x_1}(2\pi,x_\perp),\ \frac{\partial^2 u}{\partial x_1^2}(0,x_\perp) = \frac{\partial^2 u}{\partial x_1^2}(2\pi,x_\perp),\ \frac{\partial^3 u}{\partial x_1^3}(0,x_\perp) = \frac{\partial^3 u}{\partial x_1^3}(2\pi,x_\perp) \Big\},$$

where $x_\perp \in \mathbb{R}^d$ *a.e.* Analogously to the whole space case considered in Theorem 1 above, the operator $\Delta^2 - a^2 : H^4(\Omega) \to L^2(\Omega), \quad a > 0$ is non Fredholm. We prove that equation (2.3) in this context defines a map $t_a : H^4(\Omega) \to H^4(\Omega),\ a > 0$, a strict contraction under the given technical conditions.

Theorem 3. *Let $\Omega = I \times \mathbb{R}^d,\ d = 1,2, \quad G(x) : \Omega \to \mathbb{R}, \quad G(x) \in L^1(\Omega), \quad G(0,x_\perp) = G(2\pi,x_\perp), \quad F(u,0,x_\perp) = F(u,2\pi,x_\perp)$ for $x_\perp \in \mathbb{R}^d$ a.e. and $u \in \mathbb{R}$ and Assumption 1 holds.*

I) When $n_0^2 < a < (n_0+1)^2, \quad n_0 \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ let $x_\perp G(x) \in L^1(\Omega)$, condition (6.23) holds if dimension $d = 1$ and (6.24) if $d = 2$ and $\sqrt{2}(2\pi)^{\frac{d+1}{2}} M_a l < 1$. Then the map $t_a v = u$

on $H^4(\Omega)$ defined by equation (2.3) has a unique fixed point v_a, the only solution of problem (1.2) in $H^4(\Omega)$.

II) When $a = n_0^2,\ n_0 \in \mathbb{N}$ let $x_\perp^2 G(x) \in L^1(\Omega)$, conditions (6.20), (6.22) hold if dimension $d = 1$ and conditions (6.21), (6.22) hold if $d = 2$ and $\sqrt{2}(2\pi)^{\frac{d+1}{2}} M_{n_0^2} l < 1$. Then the map $t_{n_0^2} v = u$ on $H^4(\Omega)$ defined by equation (2.3) possesses a unique fixed point $v_{n_0^2}$, the only solution of equation (1.2) in $H^4(\Omega)$.

In both cases I) and II) the fixed point $v_a,\ a > 0$ is nontrivial provided that for some $n \in \mathbb{Z}$ the intersection of supports of the Fourier images of functions $supp\widehat{F(0,x)}_n \cap supp\widehat{G}_n$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$.

Remark. *Note that the maps discussed above act on real valued functions by means of the assumptions on $F(u,x)$ and $G(x)$ involved in the nonlocal term of equation (2.3).*

3 The Whole Space Case

Proof of Theorem 1. First we suppose that in the case of $\Omega = \mathbb{R}^d$ for some $v \in H^4(\mathbb{R}^d)$ there exist two solutions $u_{1,2} \in H^4(\mathbb{R}^d)$ of equation (2.3). Then their difference $w(x) := u_1(x) - u_2(x) \in H^4(\mathbb{R}^d)$ will be a solution to the homogeneous equation

$$\Delta^2 w = a^2 w.$$

Since the bi-Laplace operator acting in the whole space does not possess any nontrivial square integrable eigenfunctions, $w(x)$ vanishes a.e. in $\mathbb{R}^d$.

We choose arbitrarily $v(x) \in H^4(\mathbb{R}^d)$. Let us apply the standard Fourier transform (6.1) to both sides of (2.3). This yields

$$\widehat{u}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G}(p)\widehat{f}(p)}{p^4 - a^2}, \tag{3.1}$$

where $\widehat{f}(p)$ stands for the Fourier image of $F(v(x),x)$. Evidently, we have the estimates from above

$$|\widehat{u}(p)| \le (2\pi)^{\frac{d}{2}} N_{a,\ d} |\widehat{f}(p)| \quad and \quad |p^4 \widehat{u}(p)| \le (2\pi)^{\frac{d}{2}} N_{a,\ d} |\widehat{f}(p)|,$$

where $N_{a,\ d} < \infty$ by virtue of Lemma A1 of Appendix 1 under orthogonality conditions (6.4) in dimension $d = 1$ and (6.5) for $d = 2, 3$ respectively. This allows us to obtain the bound from above for the norm

$$\|u\|^2_{H^4(\mathbb{R}^d)} = \|\widehat{u}(p)\|^2_{L^2(\mathbb{R}^d)} + \|p^4 \widehat{u}(p)\|^2_{L^2(\mathbb{R}^d)} \le 2(2\pi)^d N^2_{a,\ d} \|F(v(x),x)\|^2_{L^2(\mathbb{R}^d)},$$

which is finite due to (2.1) of Assumption 1. Thus, for any $v(x) \in H^4(\mathbb{R}^d)$ there is a unique solution $u(x) \in H^4(\mathbb{R}^d)$ of equation (2.3) with its Fourier image given by (3.1) and the map

$T_a : H^4(\mathbb{R}^d) \to H^4(\mathbb{R}^d)$ is well defined. This enables us to choose arbitrarily $v_{1,2}(x) \in H^4(\mathbb{R}^d)$ such that their images $u_{1,2} = T_a v_{1,2} \in H^4(\mathbb{R}^d)$. Hence

$$\Delta^2 u_1 - a^2 u_1 = \int_{\mathbb{R}^d} G(x-y)F(v_1(y), y)dy,$$

$$\Delta^2 u_2 - a^2 u_2 = \int_{\mathbb{R}^d} G(x-y)F(v_2(y), y)dy.$$

Let us apply the standard Fourier transform (6.1) to both sides of these equations. This yields

$$\widehat{u}_1(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G}(p)\widehat{f}_1(p)}{p^4 - a^2}, \quad \widehat{u}_2(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G}(p)\widehat{f}_2(p)}{p^4 - a^2},$$

where $\widehat{f}_{1,2}(p)$ denote the Fourier images of $F(v_{1,2}(x), x)$. We estimate

$$|\widehat{u}_1(p) - \widehat{u}_2(p)| \le (2\pi)^{\frac{d}{2}} N_{a,\ d} |\widehat{f}_1(p) - \widehat{f}_2(p)|, \quad |p^4 \widehat{u}_1(p) - p^4 \widehat{u}_2(p)| \le (2\pi)^{\frac{d}{2}} N_{a,\ d} |\widehat{f}_1(p) - \widehat{f}_2(p)|.$$

Therefore, for the appropriate norms of functions we arrive at

$$\|u_1 - u_2\|^2_{H^4(\mathbb{R}^d)} \le 2(2\pi)^d N^2_{a,\ d} \|F(v_1(x), x) - F(v_2(x), x)\|^2_{L^2(\mathbb{R}^d)}.$$

Clearly, $v_{1,2}(x) \in H^4(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d),\ d \le 3$ by virtue of the Sobolev embedding. By means of condition (2.2), we easily obtain

$$\|T_a v_1 - T_a v_2\|_{H^4(\mathbb{R}^d)} \le \sqrt{2}(2\pi)^{\frac{d}{2}} N_{a,\ d} l \|v_1 - v_2\|_{H^4(\mathbb{R}^d)},$$

where the constant in the right side of this bound is less than one due to the assumption of the theorem. Therefore, by virtue of the Fixed Point Theorem, there exists a unique function $v_a \in H^4(\mathbb{R}^d)$ with the property $T_a v_a = v_a$, which is the only solution of problem (1.2) in $H^4(\mathbb{R}^d)$. Suppose $v_a(x)$ vanishes a.e. in $\mathbb{R}^d$. This will contradict to our assumption that the Fourier transforms of $G(x)$ and $F(0, x)$ do not vanish on a set of nonzero Lebesgue measure in $\mathbb{R}^d$. ■

4 The Problem on the Finite Interval

Proof of Theorem 2. We present the proof of our theorem in case I) and when $a = n_0^2,\ n_0 \in \mathbb{N}$ the ideas will be similar, using the constrained subspace (2.5) instead of $H^4(I)$. Let us first suppose that for a certain $v \in H^4(I)$ there exist two solutions $u_{1,2} \in H^4(I)$ of equation (2.3) when $\Omega = I$. Then function $w(x) := u_1(x) - u_2(x) \in H^4(I)$ will satisfy the equation

$$w'''' = a^2 w.$$

Since $a > 0,\ a \ne n_0^2,\ n_0 \in \mathbb{N},\ a^2$ is not an eigenvalue of the operator $\dfrac{d^4}{dx^4}$ on $L^2(I)$ with periodic boundary conditions. Hence, $w(x)$ vanishes a.e. in I.

Let us suppose $v(x) \in H^4(I)$ is arbitrary. We apply the Fourier transform (6.8) to equation (2.3) treated on the interval I and arrive at

$$u_n = \sqrt{2\pi}\frac{G_n f_n}{n^4 - a^2}, \quad n \in \mathbb{Z} \tag{4.1}$$

with $f_n := F(v(x), x)_n$. Evidently, for the transform of the fourth derivative we obtain

$$(u'''')_n = \sqrt{2\pi}\frac{n^4 G_n f_n}{n^4 - a^2}, \quad n \in \mathbb{Z}.$$

This allows us to estimate

$$\|u\|^2_{H^4(I)} = \sum_{n=-\infty}^{\infty} |u_n|^2 + \sum_{n=-\infty}^{\infty} |n^4 u_n|^2 \le 4\pi \mathcal{N}_a^2 \|F(v(x), x)\|^2_{L^2(I)} < \infty$$

via (2.1) of Assumption 1 and Lemma A2 of Appendix 1. Thus, for an arbitrary $v(x) \in H^4(I)$ there is a unique $u(x) \in H^4(I)$ satisfying equation (2.3) with its Fourier transform given by (4.1) and the map $\tau_a : H^4(I) \to H^4(I)$ in case I) is well defined.

We consider any $v_{1,2} \in H^4(I)$, such that their images under the map discussed above $u_{1,2} = \tau_a v_{1,2} \in H^4(I)$ and obtain easily the estimate from above

$$\|u_1 - u_2\|^2_{H^4(I)} = \sum_{n=-\infty}^{\infty} |u_{1n} - u_{2n}|^2 + \sum_{n=-\infty}^{\infty} |n^4(u_{1n} - u_{2n})|^2 \le$$

$$\le 4\pi \mathcal{N}_a^2 \|F(v_1(x), x) - F(v_2(x), x)\|^2_{L^2(I)}.$$

Evidently, $v_{1,2}(x) \in H^4(I) \subset L^\infty(I)$ by means of the Sobolev embedding. By virtue of (2.2), we easily arrive at

$$\|\tau_a v_1 - \tau_a v_2\|_{H^4(I)} \le 2\sqrt{\pi}\mathcal{N}_a l \|v_1 - v_2\|_{H^4(I)}.$$

The constant in the right side of this inequality is less than one as assumed. Therefore, the Fixed Point Theorem yields the existence and uniqueness of a function $v_a \in H^4(I)$ satisfying $\tau_a v_a = v_a$, which is the only solution of equation (1.2) in $H^4(I)$. Suppose $v_a(x) = 0$ a.e. in I. Then we obtain the contradiction to our assumption that $G_n F(0, x)_n \ne 0$ for some $n \in \mathbb{Z}$. Note that in the case of $a > 0, \ a \ne n_0^2, \ n_0 \in \mathbb{N}$ the argument does not rely on any orthogonality relations. ■

5 The Problem on the Product of Spaces

Proof of Theorem 3. Let us first suppose that there exists $v(x) \in H^4(\Omega)$ which generates $u_{1,2}(x) \in H^4(\Omega)$ satisfying equation (2.3). Then their difference $w(x) := u_1(x) - u_2(x) \in H^4(\Omega)$ will be a solution of

$$\Delta^2 w = a^2 w$$

in our domain Ω. We apply the partial Fourier transform (6.16) to both sides of this equation and easily obtain

$$(n^2 - \Delta_\perp)^2 w_n(x_\perp) = a^2 w_n(x_\perp), \quad n \in \mathbb{Z}.$$

Here $\Delta_\perp$ is the transversal Laplace operator acting on $x_\perp$. Obviously,

$$\|w\|^2_{L^2(\Omega)} = \sum_{n=-\infty}^{\infty} \|w_n(x_\perp)\|^2_{L^2(\mathbb{R}^d)}.$$

Hence, $w_n(x_\perp) \in L^2(\mathbb{R}^d),\ n \in \mathbb{Z}$. Since each operator $(n^2 - \Delta_\perp)^2,\ n \in \mathbb{Z}$ does not possess any nontrivial square integrable eigenfunctions belonging to $L^2(\mathbb{R}^d),\ w(x)$ vanishes a.e. in Ω.

We choose arbitrarily $v(x) \in H^4(\Omega)$ and apply the Fourier transform (6.14) to both sides of equation (2.3). This yields

$$\widehat{u}_n(p) = (2\pi)^{\frac{d+1}{2}} \frac{\widehat{G}_n(p)\widehat{f}_n(p)}{(p^2+n^2)^2 - a^2}, \quad n \in \mathbb{Z}, \quad p \in \mathbb{R}^d, \quad d = 1, 2, \tag{5.1}$$

where $\widehat{f}_n(p)$ denotes the Fourier image of $F(v(x), x)$. Evidently,

$$|\widehat{u}_n(p)| \leq (2\pi)^{\frac{d+1}{2}} M_a |\widehat{f}_n(p)| \quad and \quad |(p^2+n^2)^2 \widehat{u}_n(p)| \leq (2\pi)^{\frac{d+1}{2}} M_a |\widehat{f}_n(p)|.$$

Note that $M_a < \infty$ by virtue of Lemma A3 of Appendix 1 for $a = n_0^2,\ n_0 \in \mathbb{N}$ and of Lemma A4 when $n_0^2 < a < (n_0+1)^2,\ n_0 \in \mathbb{Z}^+$ under the appropriate orthogonality relations stated there. Therefore,

$$\|u\|^2_{H^4(\Omega)} = \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |\widehat{u}_n(p)|^2 dp + \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |(p^2+n^2)^2 \widehat{u}_n(p)|^2 dp \leq$$

$$\leq 2(2\pi)^{d+1} {M_a}^2 \|F(v(x), x)\|^2_{L^2(\Omega)} < \infty$$

by virtue of (2.1) of Assumption 1, such that for any $v(x) \in H^4(\Omega)$ there exists a unique $u(x) \in H^4(\Omega)$ satisfying equation (2.3) with its Fourier image given by (5.1) and the map $t_a : H^4(\Omega) \to H^4(\Omega)$ is well defined.

Then we choose arbitrarily $v_{1,2}(x) \in H^4(\Omega)$ such that their images under the map are $u_{1,2} = t_a v_{1,2} \in H^4(\Omega)$ and derive

$$\|u_1 - u_2\|^2_{H^4(\Omega)} = \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |\widehat{u_1}_n(p) - \widehat{u_2}_n(p)|^2 dp + \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |(p^2+n^2)^2 (\widehat{u_1}_n(p) - \widehat{u_2}_n(p))|^2 dp \leq$$

$$\leq 2(2\pi)^{d+1} {M_a}^2 \|F(v_1(x), x) - F(v_2(x), x)\|^2_{L^2(\Omega)}.$$

Obviously, $v_{1,2}(x) \in H^4(\Omega) \subset L^\infty(\Omega)$ due to the Sobolev embedding theorem. By means of (2.2) we easily obtain the upper bound

$$\|t_a v_1 - t_a v_2\|_{H^2(\Omega)} \leq \sqrt{2}(2\pi)^{\frac{d+1}{2}} M_a l \|v_1 - v_2\|_{H^4(\Omega)}$$

with the constant in the right side of it less than one as assumed. Hence, the Fixed Point Theorem implies the existence and uniqueness of a function $v_a \in H^4(\Omega)$ which satisfies $t_a v_a = v_a$ and is the only solution of equation (1.2) in $H^4(\Omega)$. Suppose $v_a(x)$ vanishes a.e. in Ω. This gives us the contradiction to the assumption that there exists $n \in \mathbb{Z}$ for which $supp\widehat{G}_n \cap supp\widehat{F(0,x)}_n$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$. ■

6 Appendix 1

Let $G(x)$ be a function, $G(x) : \mathbb{R}^d \to \mathbb{R}, \quad d \leq 3$ for which we designate its standard Fourier transform using the hat symbol as

$$\widehat{G}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} G(x) e^{-ipx} dx, \ p \in \mathbb{R}^d, \tag{6.1}$$

such that

$$\|\widehat{G}(p)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \|G\|_{L^1(\mathbb{R}^d)} \tag{6.2}$$

and $G(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \widehat{G}(q) e^{iqx} dq, \ x \in \mathbb{R}^d$. We define the auxiliary quantities

$$N_{a,\ d} := max\Big\{ \Big\| \frac{\widehat{G}(p)}{p^4 - a^2} \Big\|_{L^\infty(\mathbb{R}^d)}, \quad \Big\| \frac{p^4 \widehat{G}(p)}{p^4 - a^2} \Big\|_{L^\infty(\mathbb{R}^d)} \Big\} \tag{6.3}$$

with $a > 0$. In our notations p^4 denotes $|p|^4$ for $p \in \mathbb{R}^d, \ d \leq 3$. We have the following technical statement.

Lemma A1. *Let $G(x) \in L^1(\mathbb{R}^d), \ d \leq 3, \ xG(x) \in L^1(\mathbb{R}^d)$ and $a > 0$.*
a) If $d = 1$ then $N_{a,\ 1} < \infty$ if and only if

$$\Big(G(x), \frac{e^{\pm i\sqrt{a}x}}{\sqrt{2\pi}} \Big)_{L^2(\mathbb{R})} = 0. \tag{6.4}$$

b) If $d = 2, 3$ then $N_{a,\ d} < \infty$ if and only if

$$\Big(G(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}} \Big)_{L^2(\mathbb{R}^d)} = 0 \quad for \quad p \in S^d_{\sqrt{a}} \quad a.e. \tag{6.5}$$

Proof. Evidently, we have

$$\frac{p^4 \widehat{G}(p)}{p^4 - a^2} = \widehat{G}(p) + a^2 \frac{\widehat{G}(p)}{p^4 - a^2}. \tag{6.6}$$

The first term in the right side of (6.6) is bounded under the given conditions by means of estimate (6.2). Then if we establish that $\frac{\widehat{G}(p)}{p^4-a^2} \in L^\infty(\mathbb{R}^d)$, it will imply that $\frac{p^4\widehat{G}(p)}{p^4-a^2} \in L^\infty(\mathbb{R}^d)$ as well. Obviously,

$$\frac{\widehat{G}(p)}{p^4-a^2} = \frac{1}{2a}\frac{\widehat{G}(p)}{p^2-a} - \frac{1}{2a}\frac{\widehat{G}(p)}{p^2+a}. \tag{6.7}$$

Under our assumptions, the second term in the right side of (6.7) belongs to $L^\infty(\mathbb{R}^d)$, since it can be easily bounded from above in the absolute value by $\frac{1}{(2\pi)^{\frac{d}{2}}2a^2}\|G(x)\|_{L^1(\mathbb{R}^d)}$ by means of (6.2). Let us recall the part a) of Lemma A1 and the part a) of Lemma A2 of [22]. Hence, we have $\frac{\widehat{G}(p)}{p^2-a} \in L^\infty(\mathbb{R}^d)$ if and only if orthogonality conditions (6.4) and (6.5) hold on the real line and for $d = 2, 3$ respectively. ■

Let the function $G(x) : I \to \mathbb{R}, \quad G(0) = G(2\pi)$ and its Fourier transform on the finite interval is given by

$$G_n := \int_0^{2\pi} G(x)\frac{e^{-inx}}{\sqrt{2\pi}}dx, \quad n \in \mathbb{Z} \tag{6.8}$$

and $G(x) = \sum_{n=-\infty}^{\infty} G_n \frac{e^{inx}}{\sqrt{2\pi}}$. Evidently, we have the estimate

$$\|G_n\|_{l^\infty} \leq \frac{1}{\sqrt{2\pi}}\|G\|_{L^1(I)}. \tag{6.9}$$

Similarly to the whole space case we introduce

$$\mathcal{N}_a := max\left\{\left\|\frac{G_n}{n^4-a^2}\right\|_{l^\infty}, \quad \left\|\frac{n^4G_n}{n^4-a^2}\right\|_{l^\infty}\right\} \tag{6.10}$$

with $a > 0$. We have the following trivial proposition.

Lemma A2. *Let $G(x) \in L^1(I)$ and $G(0) = G(2\pi)$.*
a) If $a > 0$ and $a \neq n_0^2,\ n_0 \in \mathbb{N}$ then $\mathcal{N}_a < \infty$.
b) If $a = n_0^2,\ n_0 \in \mathbb{N}$ then $\mathcal{N}_a < \infty$ if and only if

$$\left(G(x), \frac{e^{\pm in_0x}}{\sqrt{2\pi}}\right)_{L^2(I)} = 0. \tag{6.11}$$

Proof. Obviously, we have

$$\frac{n^4G_n}{n^4-a^2} = G_n + a^2\frac{G_n}{n^4-a^2}. \tag{6.12}$$

The first term in the right side of (6.12) is bounded under the stated conditions due to inequality (6.9). Then if we prove that $\frac{G_n}{n^4 - a^2} \in l^\infty$, it will give us that $\frac{n^4 G_n}{n^4 - a^2} \in l^\infty$ as well. Evidently,

$$\frac{G_n}{n^4 - a^2} = \frac{1}{2a}\frac{G_n}{n^2 - a} - \frac{1}{2a}\frac{G_n}{n^2 + a}. \tag{6.13}$$

Under the given conditions, the second term in the right side of (6.13) belongs to l^∞, since it can be easily estimated from above in the absolute value by $\frac{1}{\sqrt{2\pi}2a^2}\|G(x)\|_{L^1(I)} < \infty$ by virtue of (6.9).

We recall the results of Lemma A3 of [22]. Hence, in the case a) of the present lemma, the first term in the right side of (6.13) is contained in l^∞ as well, such that $\mathcal{N}_a < \infty$.

If $a = n_0^2$ with some $n_0 \in \mathbb{N}$, orthogonality relations (6.11) are necessary and sufficient for $\mathcal{N}_a < \infty$. ∎

Let $G(x)$ be a function on the product of spaces treated in Theorem 3, $G(x) : \Omega = I \times \mathbb{R}^d \to \mathbb{R}$, $d = 1, 2$, $G(0, x_\perp) = G(2\pi, x_\perp)$ for $x_\perp \in \mathbb{R}^d$ a.e. and its Fourier transform on the product of spaces is given by

$$\widehat{G}_n(p) := \frac{1}{(2\pi)^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} dx_\perp e^{-ipx_\perp} \int_0^{2\pi} G(x_1, x_\perp) e^{-inx_1} dx_1, \quad p \in \mathbb{R}^d, \ n \in \mathbb{Z}. \tag{6.14}$$

The norm

$$\|\widehat{G}_n(p)\|_{L^\infty_{n,p}} := \sup_{\{p \in \mathbb{R}^d, \ n \in \mathbb{Z}\}} |\widehat{G}_n(p)| \leq \frac{1}{(2\pi)^{\frac{d+1}{2}}} \|G\|_{L^1(\Omega)} \tag{6.15}$$

and $G(x) = \frac{1}{(2\pi)^{\frac{d+1}{2}}} \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} \widehat{G}_n(p) e^{ipx_\perp} e^{inx_1} dp$. It is also useful to consider the Fourier transform only in the first variable, such that

$$G_n(x_\perp) := \int_0^{2\pi} G(x_1, x_\perp) \frac{e^{-inx_1}}{\sqrt{2\pi}} dx_1, \quad n \in \mathbb{Z}. \tag{6.16}$$

We define $\xi_n^a(p) := \frac{\widehat{G}_n(p)}{(p^2 + n^2)^2 - a^2}$ and introduce

$$M_a := max\{\|\xi_n^a(p)\|_{L^\infty_{n,p}}, \quad \|(p^2 + n^2)^2 \xi_n^a(p)\|_{L^\infty_{n,p}}\} \tag{6.17}$$

with $a > 0$. In our notations the momentum vector $p \in \mathbb{R}^d$. Clearly, we have

$$(p^2 + n^2)^2 \xi_n^a(p) = \widehat{G}_n(p) + a^2 \xi_n^a(p). \tag{6.18}$$

The first term in the right side of (6.18) is bounded under the conditions of Lemmas A3 and A4 below via inequality (6.15). Then if we show that $\xi_n^a(p) \in L^\infty_{n,p}$, it will imply that $(p^2 + n^2)^2 \xi_n^a(p) \in L^\infty_{n,p}$ as well. Apparently,

$$\xi_n^a(p) = \frac{\widehat{G}_n(p)}{2a(p^2 + n^2 - a)} - \frac{\widehat{G}_n(p)}{2a(p^2 + n^2 + a)}. \tag{6.19}$$

Under the conditions of Lemmas A3 and A4 below, the second term in the right side of (6.19) is contained in $L^{\infty}_{n,p}$, because it can be easily bounded from above in the absolute value by $\frac{1}{(2\pi)^{\frac{d+1}{2}}2a^2}\|G(x)\|_{L^1(\Omega)} < \infty$ via (6.15). Let us first consider the case when the constant a is a square of a natural number.

Lemma A3. *Let $G(x) \in L^1(\Omega)$, $x_{\perp}^2 G(x) \in L^1(\Omega)$ and $G(0, x_{\perp}) = G(2\pi, x_{\perp})$ for $x_{\perp} \in \mathbb{R}^d$ a.e., $d = 1, 2$ and $a = n_0^2$, $n_0 \in \mathbb{N}$. Then $M_a < \infty$ if and only if*

$$\left(G(x_1, x_{\perp}), \frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{\pm i\sqrt{n_0^2-n^2}x_{\perp}}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad |n| \le n_0 - 1, \quad d = 1, \tag{6.20}$$

$$\left(G(x_1, x_{\perp}), \frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{ipx_{\perp}}}{2\pi}\right)_{L^2(\Omega)} = 0, \quad p \in S^2_{\sqrt{n_0^2-n^2}} \quad a.e., \quad |n| \le n_0 - 1, \quad d = 2, \tag{6.21}$$

$$\left(G(x_1, x_{\perp}), \frac{e^{\pm in_0x_1}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad \left(G(x_1, x_{\perp}), \frac{e^{\pm in_0x_1}}{\sqrt{2\pi}}x_{\perp,\ k}\right)_{L^2(\Omega)} = 0, \quad 1 \le k \le d. \tag{6.22}$$

Proof. Let us recall the result of Lemma A5 of [22]. Under the stated conditions, the first term in the right side of (6.19) belongs to $L^{\infty}_{n,p}$ if and only if orthogonality relations (6.20) for $d = 1$, (6.21) for $d = 2$ and (6.22) for $d = 1, 2$ hold. Since the boundedness of $\frac{\widehat{G}_n(p)}{2a(p^2 + n^2 - n_0^2)}$ is equivalent to $\xi_n^{n_0^2}(p) \in L^{\infty}_{n,p}$, which follows from formula (6.19), the result of the lemma is obvious. ■

Finally, we treat the case when the parameter a is located on an open interval between the squares of two consecutive nonnegative integers.

Lemma A4. *Let $G(x) \in L^1(\Omega)$, $x_{\perp} G(x) \in L^1(\Omega)$ and $G(0, x_{\perp}) = G(2\pi, x_{\perp})$ for $x_{\perp} \in \mathbb{R}^d$ a.e., $d = 1, 2$ and $n_0^2 < a < (n_0 + 1)^2$, $n_0 \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. Then $M_a < \infty$ if and only if*

$$\left(G(x_1, x_{\perp}), \frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{\pm i\sqrt{a-n^2}x_{\perp}}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad |n| \le n_0, \quad d = 1, \tag{6.23}$$

$$\left(G(x_1, x_{\perp}), \frac{e^{inx_1}}{\sqrt{2\pi}}\frac{e^{ipx_{\perp}}}{2\pi}\right)_{L^2(\Omega)} = 0, \quad p \in S^2_{\sqrt{a-n^2}} \ a.e., \quad |n| \le n_0, \quad d = 2. \tag{6.24}$$

Proof. We recall the result of Lemma A6 of [22]. Under the given conditions, the first term in the right side of (6.19) is contained in $L^{\infty}_{n,p}$ if and only if orthogonality relations (6.23) for $d = 1$, (6.24) for $d = 2$ hold. The boundedness of $\frac{\widehat{G}_n(p)}{2a(p^2 + n^2 - a)}$ is equivalent to $\xi_n^a(p) \in L^{\infty}_{n,p}$, which stems from (6.19). Therefore, the result of the lemma is evident. ■

7 Appendix 2

We will discuss here the derivation and some applications of equation (1.2). Let us begin with the system of two second-order equations

$$\frac{\partial A}{\partial t} = D_1 \Delta A + W_1, \tag{7.1}$$

$$\frac{\partial B}{\partial t} = D_2 \Delta B + W_2, \tag{7.2}$$

where A and B denotes the concentrations of some entities, the first terms in the right-hand sides of equations (7.1), (7.2) describe their random motion, W_1 and W_2 are the source terms. The concentrations A and B are considered here in a general framework without discussing their specific origin. In population dynamics they can be related to biological species, in neuroscience to neuron activation and inhibition, in chemistry to concentrations of some substances. We suppose that A acts on B, and B act on A in such a way that W_1 is a function of B and W_2 is a function of A. Furthermore, we set

$$W_1 = k_1 B, \qquad W_2 = k_2 \int_\Omega G(x-y) F(A(y,t)) dy + k_3 A.$$

The first dependence is conventional, and it does not require further comments. Nonlocal dependence of the source term W_2 can arise in various applications. In population dynamics it describes intra-specific competition and nonlocal consumption of resources (see e.g. [2], [3], [8]). In neuroscience it corresponds to the nonlocal interaction of neurons (see [4] the references therein). If $G(x)$ is a δ-function, then W_2 is reduced to a conventional local dependence. Hence, the integral term corresponds to a more general and interesting situation.

In the stationary case, we can use equation (7.1) in order to express B through the Laplacian of A and substitute into equation (7.2). Thus, we obtain equation (1.2) up to notation.

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