

KAM ESTIMATES FOR THE DISSIPATIVE STANDARD MAP

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ABSTRACT. From the beginning of KAM theory, it was realized that its applicability to realistic problems depended on developing quantitative estimates on the sizes of the perturbations allowed. In this paper we present results on the existence of quasi-periodic solutions for conformally symplectic systems in non-perturbative regimes. We recall that, for conformally symplectic systems, finding the solution requires also to find a *drift parameter*. We present a proof on the existence of solutions for values of the parameters which agree with more than three figures with the numerically conjectured optimal values.

The first step of the strategy is to establish a very explicit quantitative theorem in an a-posteriori format. We recall that in numerical analysis, an a-posteriori theorem assumes the existence of an approximate solution, which satisfies an invariance equation up to an error which is small enough with respect to explicit condition numbers, and then concludes the existence of a solution. In the case of conformally symplectic systems, an a-posteriori theorem was proved in [12]. Our first task is to make all the constants fully explicit.

We emphasize that our result allows to conclude the existence of the true solution by verifying mainly that the approximate solution satisfies the equation up to a small error and that some condition numbers are finite. The method used to produce the approximate solution does not need to be examined.

The second step in the strategy is to produce numerically very accurate solutions in a concrete problem. We have implemented the algorithm indicated in [12] in a model problem, widely considered in the literature; we constructed numerically very accurate solutions of the invariance equations (discretizations with 2^{18} Fourier coefficients, each one computed with 100 digits of precision). From the point of view of rigorous mathematics, we note that the first step is a fully rigorous theorem, the second step is a high precision calculation which produces an impact for the theorem in the first part.

The third and final step is to present a numerical verification of the hypotheses of the theorem stated in the first part on the numerical solutions presented in the second part. Using these estimates we would conclude the existence of tori for certain values of the drift parameter. The perturbation parameters we can consider coincide with more than 3 significant figures with the values conjectured as optimal by numerical experiments.

The verification of the estimates presented here is not completely rigorous since we do not control the round-off error. Nevertheless, running with different precision shows very little difference in the results. Given the high precision of the calculation and the simplicity of the estimates, this does not seem to affect the results. A full verification should be done implementing interval arithmetic.

We make available the approximate solutions, the highly efficient algorithms to generate them (incorporating high precision based on the MPFR library) and the routines used to verify the applicability of the theorem.

1. INTRODUCTION

The goal of this paper is to develop a methodology to compute efficiently and to verify rigorously the existence of quasi-periodic solutions in concrete systems (compare with [23, 17, 16, 18, 19, 20, 21, 22, 32, 33, 29, 59, 58, 60]).

The celebrated KAM theory, started in [38, 2, 44], solved the outstanding problem of establishing the persistence of quasi-periodic orbits under small perturbations. An important motivation was represented by problems in celestial mechanics ([3]). By now, KAM theory has developed into a very useful paradigm. Surveys of KAM theory and its applications are: [3, 46, 46, 45, 5, 61, 28, 19, 31].

At the beginning of the theory, the quantitative requirements for applicability led to unrealistic smallness estimates. In a well known calculation ([35]), M. Hénon made a preliminary study of the parameters required to apply to the three-body problem ([2]) and obtained that the small parameter (representing the Jupiter–Sun mass–ratio) should be smaller than 10^{-48} , whereas of course, the real value for Jupiter is about 10^{-3} . Discouraged by this result, the often quoted conclusion of [35] was that¹

*“Ainsi, ces théorèmes, bien que d’un très grand intérêt théorique, ne semblent pas pouvoir **en leur état actuel** être appliqués á des problèmes pratiques”.*

Even if the statement of [35] is perfectly correct as stated, removing the words we have set in bold (as it is often done), one obtains a statement invalid 50 years after the original statement.

It is also true that the first attempts to study the problem numerically were disappointing. The persistence of quasi-periodic solutions indeed depends on rather higher regularity of the perturbation (the smoothness requirements of some versions of KAM theory are optimal, [36, 24, 48]) and attempts based on low regularity discretizations such as finite elements were discouraging ([6]). Furthermore, unless one is careful, one can be misled by spurious solutions. It is also true that many of the original proofs were based on transformation theory, which is difficult to implement numerically (one needs to deal with functions of a high number of variables and impose that they satisfy geometric constraints). More successful studies such as [34, 25, 4, 50, 51, 41] were based on indirect methods for very specific systems and were challenged because of being indirect.

By the late 70’s it was folklore belief that the estimates of KAM theory were essentially optimal (the optimal steps for one step were known to good approximation, and one could hope that, by some Baire category argument, one could find systems that saturate the bounds to all steps).

By now, the situation has changed drastically. There are general bounds based on different schemes ([7, 62, 27, 36]), which lead to substantially better bounds. In practical

¹“It does not seem that these theorems, though having a great theoretical interest, can be applied, **in their present state**, to practical problems” [35].

applications, one is interested in concrete systems, not on generic ones and there has been also progress in obtaining estimates in some specific systems. Later on, many more situations leading to better bounds were found, see [17, 16, 19, 20].

More related to the present paper, in recent times there has been a rapid development in proofs of KAM theorems in the “*a-posteriori*” format common in numerical analysis. We recall that an a-posteriori theorem in numerical analysis is a theorem of the following format.

Theorem Format 1. *Let $\mathcal{X}_0 \subset \mathcal{X}_1$ be Banach spaces and $\mathcal{U} \subset \mathcal{X}_0$ an open set. Consider the map*

$$\mathcal{F} : \mathcal{U} \subset \mathcal{X}_0 \rightarrow \mathcal{X}_0 ;$$

assume that there are functionals $m_1, \dots, m_n : \mathcal{U} \rightarrow \mathbb{R}^+$ for some $x_0 \in \mathcal{X}_0$, such that:

- (1) $\|\mathcal{F}(x_0)\|_{\mathcal{X}_0} < \varepsilon$ for some $\varepsilon \in \mathbb{R}$;
- (2) $m_1(x_0) \leq M_1, \dots, m_n(x_0) \leq M_n$ for some condition numbers M_1, \dots, M_n ;
- (3) $\varepsilon \leq \varepsilon^*(M_1, \dots, M_n)$, where ε^* is an explicit function of the condition numbers.

Then, there exists an $x^ \in \mathcal{X}_1$ such that $\mathcal{F}(x^*) = 0$ and $\|x_0 - x^*\|_{\mathcal{X}_1} \leq C_{M_1, \dots, M_n} \varepsilon$ for some positive constant C_{M_1, \dots, M_n} .*

Of course, to obtain the statement of a theorem in the Format 1, one has to specify all the ingredients, $\mathcal{X}_0, \mathcal{X}_1, \mathcal{F}, m_1, \dots, m_n$, the function ε^* and provide a proof; Theorems of this form are very common e.g. in finite elements theory ([49]) or in linear algebra.

As it turns out, one can formulate several KAM theorems in this format. One needs to choose an appropriate functional \mathcal{F} whose zeros imply the existence of quasi-periodic solutions (in such applications x is an embedding that belongs to a suitable space of functions, see 2.1.1.)

Notice that in contrast with other more customary versions of KAM theory, this formulation does not involve that we are considering a system close to integrable and it does not require any global assumption on the map, but only some functionals evaluated in the approximate solution. In the problems considered in this paper, the condition numbers are just averages of algebraic expressions involving derivatives of the embedding x_0 and do not include any global assumption in the maps such as the twist assumption.

Of course, KAM theory (and a fortiori KAM theory in an a-posteriori format) usually makes assumptions on geometric properties of the dynamical system. Roughly, the geometric properties are used to eliminate adding parameters to the system.

There are different geometric properties that lead to a KAM theory (see [47, 8] for a discussion of the classical contexts – general, symplectic, volume preserving, reversible – formulated in a format which is not a-posteriori). Other more modern contexts are presymplectic [1], or closer to the goals of this paper, conformally symplectic [12].

Notice that an a-posteriori theorem allows to validate the existence of an approximate solution, independently of how it has been obtained. For example, one can take as an approximate solution a numerically computed one (typically this will be a trigonometric polynomial whose coefficients are chosen among the numbers representable in a computer). If one can perform a finite (but too large for pencil-paper) number of operations taking care of the rounding off, one can obtain estimates on ε and M_1, \dots, M_n .

As it turns out, there exist computer science techniques (interval analysis [42, 43, 37]) which allow one to perform these rigorous bounds mechanically. The coupling of an a-posteriori theorem with interval arithmetic has led to many *computer assisted proofs* of mathematically relevant problems that are reduced to the existence of a fixed point². A particularly emblematic computer assisted proof based on an a-posteriori theorem and interval arithmetic is [39], but there are many other proofs based on a-posteriori theorems for fixed points³.

Therefore, a way to prove the existence of a quasi-periodic solution has different stages, each of them requiring a different methodology.

- A) For a fixed geometric context, prove an a posteriori KAM theorem.
- B) Make sure that the conditions of the a-posteriori theorem in part A are made explicit and computable.
- C) Produce approximate solutions.
- D) Verify the conditions given in B) on the approximate solutions produced in C).

This strategy for two dimensional symplectic mappings was implemented in [52] and in [32]. The paper [52] also considered upper and lower bounds of Siegel radius and proved they would converge to the right value if given enough computer resources. The paper [32] gives a very innovative implementation of a-posteriori KAM estimates by proposing an efficient computer-assisted method. The technique is successfully applied to the standard

²We note, however, that, besides computer assisted proofs based on fixed point theorems, there are other computer assisted proofs which do not involve fixed points theorems, but which are based on other arguments (exclusion of matches, algebraic operations, etc.).

³The proof of [39] used only a Banach contraction argument and indeed most of the computer assisted proofs rely on a contraction mapping argument. In our case, we need to rely on more sophisticated Nash-Moser arguments.

map, obtaining estimates in agreement of 99.9% with the numerical threshold. The paper has also considered applications to the non-twist standard map and to the Froeschlé map.

We note that, in principle, the above methodology can continue in parameters and establish the results even arbitrarily close to the values of the parameters where the result is no longer true. Of course, in practice, one is limited by the computer resources available (e.g. computer memory or time). We will show that for some emblematic problems, even modest resources (a common today's desktop) can produce results quite close to optimal.

The parts A), B), C) and D) above require different methodologies and are, in principle, independent. In practice, there are some relations (e.g. the choice of spaces in the mathematical proofs is related to the numerical methods used). This is why we decided to present the results in a single paper rather than separate it in logically independent units.

Part A) requires the traditional methods of mathematics, but the goals should be an efficient and explicit formulation that makes efficient the other parts of the strategy. Notably, the functional equations should involve functions of as little variables as possible – the difficulty of dealing with functions grows very fast with the number of variables. This is known as the *curse of dimensionality*. Moreover, the norms should be easy to evaluate. The spaces one is dealing with should be easily parameterizable, preferably by linear combinations of functions – for example, parametrizing symplectic transformations requires using generating functions to impose the very nonlinear constraint of preserving the symplectic form.

Part B) is in principle straightforward, but a high quality implementation requires taking advantage of the cancellations and organize the estimates very efficiently. Also, some non-constructive arguments need to be replaced by constructive arguments.

Part C) is very traditional in numerical analysis and can be accomplished in many ways, for example discretizing the invariance equation, but we stress that there are some interactions with the other parts.

Many of the more modern proofs in Part A) are based in describing an iterative process and showing it converges when started on a sufficiently approximate solution. For our case, the proof presented in [12] is particularly well suited. It leads to a quadratically convergent algorithm that requires little storage and a small operation count per step. On the other hand, the Newton method relies on having a good approximate solution. The algorithm can be used as the basis of a continuation method. Notice that the method

does not rely on indirect methods such as [34] and that it is generally applicable (i.e., one can take any system and let the continuation run). The method of [34] requires continuing high period orbits, which is problematic in systems with several harmonics ([30, 40]).

We also note that, in order to have an effective part D), the discretization used has to be such that it allows the evaluation of the norms involved. As indicated above, the KAM theorem requires derivatives of rather high order, so it seems that a Fourier discretization could be effective if we consider norms that can be read from the Fourier coefficients. This is particularly effective because the functional equations used in part A) lead to functions whose maximal domain is a complex strip (the domain is invariant under an irrational translation), which are the natural domains of convergence of Fourier series.

Part D) is in principle straightforward since the number of operations is rather small. As mentioned before, it can be made fully rigorous using interval arithmetic.

The goal of this paper is to implement this strategy for conformally symplectic mappings and obtain concrete results for an emblematic example that has been considered many times in the literature. One caveat is that for part D), we have not implemented interval arithmetic, but have performed the calculations with more than 100 digits of precision and with several precisions.

1.1. Organization of the paper. This paper is organized as follows. In Section 2, we present some standard preliminaries, including norms, Cauchy estimates, the Diophantine inequality, the solution of the cohomology equation, the definition of conformally symplectic systems, the introduction and properties of the dissipative standard map.

In Section 3 we state a very explicit KAM theorem in an a-posteriori format, Theorem 10 which implements part A) of the strategy indicated above. The statement of Theorem 10 includes the explicit formulation of the smallness conditions on the parameters ensuring the existence of an exact solution of the invariance equation. Such conditions depend on a set of constants, whose explicit expression is given in Appendix B). The proof of Theorem 10 is reviewed in Section 4. The proof follows closely the proof in [12], but we take advantage that we will consider a specific model in which the tori are one-dimensional and such that the symplectic form is the standard one. Some of the most straightforward calculations have been relegated to Appendix B).

2. PRELIMINARIES

In this Section, we collect several notions that play a role in our results. We will describe the models, some of their properties and describe the examples. The material in Section 2.1 concerns standard properties of analytic functions and can be used mainly as a reference for the notation. In Section 2.2 we introduce conformally symplectic systems, which are the main geometric assumption in our results. In Section 2.3 we introduce the concrete model we will study and which has been widely investigated in the literature.

2.1. Norms and preliminary Lemmas. In this Section we need to specify the norms (see Section 2.1.1), to estimate the composition of functions (see Section 2.1.2), to bound derivatives (see Section 2.1.3), to introduce Diophantine numbers (see Section 2.1.4), and to give estimates of a cohomology equation associated to the linearization of the invariance equation (see Section 2.1.5).

2.1.1. *Norms.* For a vector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$, we define its norm as

$$\|v\| = |v_1| + |v_2| .$$

For a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}^2$, we define its norm as

$$\|A\| = \max \left\{ |a_{11}| + |a_{21}|, |a_{12}| + |a_{22}| \right\} .$$

To define the norm of functions and vector functions, we start by introducing for $\rho > 0$ the following complex extensions of a torus \mathbb{T} , of a set B and of the manifold $\mathcal{M} = B \times \mathbb{T}$:

$$\begin{aligned} \mathbb{T}_\rho &\equiv \{z = x + iy \in \mathbb{C}/\mathbb{Z} : x \in \mathbb{T}, |y| \leq \rho\} , \\ B_\rho &\equiv \{z = x + iy \in \mathbb{C} : x \in B, |y| \leq \rho\} , \\ \mathcal{M}_\rho &= B_\rho \times \mathbb{T}_\rho . \end{aligned}$$

We denote by \mathcal{A}_ρ the set of functions which are analytic in $\text{Int}(\mathbb{T}_\rho)$ and that extend continuously to the boundary of \mathbb{T}_ρ . Within such set, we introduce the norm

$$\|f\|_\rho = \sup_{z \in \mathbb{T}_\rho} |f(z)| .$$

For a vector valued function $f = (f_1, f_2, \dots, f_n)$, $n \geq 1$, we define the norm

$$\|f\|_\rho = \|f_1\|_\rho + \|f_2\|_\rho + \dots + \|f_n\|_\rho . \tag{2.1}$$

For an $n_1 \times n_2$ matrix valued function F we define

$$\|F\|_\rho = \sum_{i=1}^{n_1} \sup_{j=1, \dots, n_2} \|F_{ij}\|_\rho . \quad (2.2)$$

Notice that if F is a matrix valued function and f is a vector valued function, then one has

$$\|F f\|_\rho \leq \|F\|_\rho \|f\|_\rho .$$

2.1.2. Composition Lemma. Composition of two functions is an important operation in dynamical systems. Indeed, our main functional equation, see (2.10) below, involves composition.

Lemma 2. *Let $F \in \mathcal{A}_\mathcal{C}$ be an analytic function on a domain $\mathcal{C} \subset \mathbb{C} \times \mathbb{C}/\mathbb{Z}$.*

Assume that the function g is such that $g(\mathbb{T}_\rho) \subset \mathcal{C}$ and $g \in \mathcal{A}_\rho$ with $\rho > 0$. Then, $F \circ g \in \mathcal{A}_\rho$ and

$$\|F \circ g\|_\rho \leq \|F\|_{\mathcal{A}_\mathcal{C}} ,$$

where $\|F\|_{\mathcal{A}_\mathcal{C}} = \sup_{z \in \mathcal{C}} |F(z)|$.

If, furthermore, we have that $\text{dist}(g(\mathcal{A}_\rho), \mathbb{C} \setminus \mathcal{C}) = \eta > 0$, then we have:

- *For all $h \in \mathcal{A}_\rho$ with $\|h\|_\rho < \eta/4$, we can define $F \circ (g + h)$.*
- *We have:*

$$\|F \circ (g + h) - F \circ g\|_\rho \leq \sup_{z, \text{dist}(z, \mathcal{C}) \leq \eta/4} (|DF(z)|) \|h\|_\rho ,$$

$$\|F \circ (g + h) - F \circ g - DF \circ g h\|_\rho \leq \frac{1}{2} \sup_{z, \text{dist}(z, \mathcal{C}) \leq \eta/4} (|D^2F(z)|) \|h\|_\rho^2 .$$

2.1.3. Cauchy estimates on the derivatives. Estimates on the derivatives will be needed throughout the whole proof of the main result (Theorem 10).

Lemma 3. *For a function $h \in \mathcal{A}_\rho$, we have the following estimate on the first derivative on a smaller domain:*

$$\|Dh\|_{\rho-\delta} \leq C_c \delta^{-1} \|h\|_\rho , \quad C_c = 1 , \quad (2.3)$$

where $0 < \delta < \rho$. For the ℓ -th order derivatives with $\ell \geq 1$, one has:

$$\|D^\ell h\|_{\rho-\delta} \leq C_{c,\ell} \delta^{-\ell} \|h\|_\rho , \quad C_{c,\ell} = \ell! (2\pi)^{-1} .$$

Notice that the Cauchy constant C_c might assume different values, if one adopts different norms with respect to (2.1), (2.2) (this is why we keep a symbol for such constant).

2.1.4. *Diophantine numbers.* The following definition is standard in number theory and appears frequently in KAM theory.

Definition 4. *Let $\omega \in \mathbb{R}$, $\tau \geq 1$, $\nu \geq 1$. We say that ω is Diophantine of class τ and constant ν , if the following inequality is satisfied:*

$$|\omega k - q| \geq \nu |k|^{-\tau}, \quad q \in \mathbb{Z}, \quad k \in \mathbb{Z} \setminus \{0\}. \quad (2.4)$$

The set of Diophantine numbers satisfying (2.4) is denoted by $\mathcal{D}(\nu, \tau)$. The union over $\nu > 0$ of the sets $\mathcal{D}(\nu, \tau)$ has full Lebesgue measure in \mathbb{R} .

2.1.5. *Estimates on the cohomology equation.* Given any Lebesgue measurable function η , we consider the following cohomology equation:

$$\varphi(\theta + \omega) - \lambda \varphi(\theta) = \eta(\theta), \quad \theta \in \mathbb{T}. \quad (2.5)$$

The solution of an equation of the form (2.5) will be an essential ingredient of the proof, see e.g. (4.5) below. The two following Lemmas show that there is one Lebesgue measurable function φ , which is the solution of (2.5). Precisely, Lemma 5 applies for $|\lambda| \neq 1$, $\omega \in \mathbb{R}$ and it provides a non-uniform estimate on the solution, while Lemma 6 applies to any λ and any ω Diophantine, and it provides a uniform estimate on the solution.

Lemma 5. *Assume $|\lambda| \neq 1$, $\omega \in \mathbb{R}$. Then, given any Lebesgue measurable function η , there is one Lebesgue measurable function φ satisfying (2.5). Furthermore, the following estimate holds:*

$$\|\varphi\|_\rho \leq \frac{1}{\left| |\lambda| - 1 \right|} \|\eta\|_\rho.$$

Moreover, one can bound the derivatives of φ with respect to λ as

$$\|D_\lambda^j \varphi\|_\rho \leq \frac{j!}{\left| |\lambda| - 1 \right|^{j+1}} \|\eta\|_\rho, \quad j \geq 1.$$

Lemma 6. *Consider (2.5) for $\lambda \in [A_0, A_0^{-1}]$ for some $0 < A_0 < 1$ and let $\omega \in \mathcal{D}(\nu, \tau)$. Assume that $\eta \in \mathcal{A}_\rho$, $\rho > 0$ and that*

$$\int_{\mathbb{T}} \eta(\theta) d\theta = 0.$$

Then, there is one and only one solution of (2.5) with zero average: $\int_{\mathbb{T}} \varphi(\theta) d\theta = 0$. Furthermore, if $\varphi \in \mathcal{A}_{\rho-\delta}$ for $0 < \delta < \rho$, then we have

$$\|\varphi\|_{\rho-\delta} \leq C_0 \nu^{-1} \delta^{-\tau} \|\eta\|_\rho, \quad (2.6)$$

where

$$C_0 = \frac{1}{(2\pi)^\tau} \frac{\pi}{2^\tau(1+\lambda)} \sqrt{\frac{\Gamma(2\tau+1)}{3}}. \quad (2.7)$$

The proof of Lemma 6 with the constant C_0 as in (2.7) is given in Appendix A.

2.2. Conformally symplectic systems. In this Section we give the definition of conformally symplectic systems for one-dimensional maps. Indeed, the dissipative standard map that we will introduce in Section 2.3 and that we will consider throughout this paper, is a one-dimensional, conformally symplectic map. A more general definition of a conformally symplectic system in the n -dimensional case is provided in [12].

Definition 7. *Let \mathcal{M} be an analytic symplectic manifold with $\mathcal{M} \equiv B \times \mathbb{T}$, where $B \subseteq \mathbb{R}$ is an open, simply connected domain with a smooth boundary. Let Ω be the symplectic form associated to \mathcal{M} . Let f be a diffeomorphism defined on the phase space \mathcal{M} . The diffeomorphism f is conformally symplectic, if there exists a function $\lambda : \mathcal{M} \rightarrow \mathbb{R}$ such that*

$$f^*\Omega = \lambda\Omega ,$$

where f^* denotes the pull-back of f .

We remark that when $n = 1$, then λ can be a function of the coordinates, while it can be shown that for $n \geq 2$ one can only have that λ is a constant function.

In the following discussion, we will always assume that λ is a constant, as in the model (2.8) below, which is the main goal of the present work.

2.3. A specific model. In this work we consider a specific 1-parameter family f_μ of one-dimensional, conformally symplectic maps, known as the *dissipative standard map*:

$$\begin{aligned} I' &= \lambda I + \mu + \frac{\varepsilon}{2\pi} \sin(2\pi\varphi) , \\ \varphi' &= \varphi + I' , \end{aligned} \quad (2.8)$$

where $I \in B \subseteq \mathbb{R}$ with B as in Definition 7, $\varphi \in \mathbb{T}$, $\varepsilon \in \mathbb{R}_+$, $\lambda \in \mathbb{R}_+$, $\mu \in \mathbb{R}$. This model has been studied both numerically and theoretically in the literature. For example [53, 54, 55] consider the breakdown and conjecture universality properties; [10] studies the breakdown even for complex values of the parameters; [11] studies the invariant bundles near the circles and find scaling properties at breakdown; [9, 14] study the domains of analyticity in the limit of small dissipation.

To fix some terminology, we shall refer to ε as the *perturbing parameter*, to λ as the *dissipative parameter*, and to μ as the *drift parameter*.

Notice that the Jacobian of the mapping (2.8) is equal to λ , so that the mapping is contractive for $\lambda < 1$, expanding for $\lambda > 1$ and it is symplectic for $\lambda = 1$.

We denote by (\cdot, \cdot) the Euclidean scalar product. We remark that if $J = J(x)$ is the matrix representing Ω at x , namely $\Omega_x(u, v) = (u, J(x)v)$ for any $u, v \in \mathbb{R}$, then for the mapping (2.8), J is the following constant matrix:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.9)$$

2.3.1. *Formulation of the problem of an invariant attractor.* We proceed to provide the definition of a KAM attractor with frequency ω .

Having fixed a value of the dissipative parameter, our goal will be to prove the persistence of invariant attractors associated to (2.8) for non-zero values of the perturbing parameter. To this end, we need to require that the frequency of the attractor, say $\omega \in \mathbb{R}$, is Diophantine according to Definition (2.4). We note that this will require adjusting the drift parameter μ .

Definition 8. *Given a family of conformally symplectic maps $f_\mu : \mathcal{M} \rightarrow \mathcal{M}$, a KAM attractor with frequency ω is an invariant torus which can be described by an embedding $K : \mathbb{T} \rightarrow \mathcal{M}$, such that the following invariance equation is satisfied for all $\theta \in \mathbb{T}$:*

$$f_\mu \circ K(\theta) = K(\theta + \omega). \quad (2.10)$$

The equation (2.10) will be the key of our statements. Note that we will think that both the embedding K and the parameter μ are unknowns of (2.10).

Remark 9. (i) *For the dissipative standard map (2.8) the embedding K can be conveniently written as*

$$K(\theta) = \begin{pmatrix} \theta + u(\theta) \\ v(\theta) \end{pmatrix} \quad (2.11)$$

for some continuous, periodic functions $u : \mathbb{T} \rightarrow \mathbb{R}$, $v : \mathbb{T} \rightarrow \mathbb{R}$. Denoting by (I_j, φ_j) the j -th iterate of (2.8), one finds that orbits are characterized by

$$\varphi_{j+1} - (1 + \lambda)\varphi_j + \lambda\varphi_{j-1} = \mu + \frac{\varepsilon}{2\pi} \sin(2\pi\varphi_j).$$

Using (2.11) one obtains that the invariance equation (2.10) in terms of u is

$$u(\theta + \omega) - (1 + \lambda)u(\theta) + \lambda u(\theta - \omega) = \mu + \frac{\varepsilon}{2\pi} \sin(2\pi(\theta + u(\theta))). \quad (2.12)$$

Equation (2.12) can be used to determine the function u and then one can determine the function v appearing in (2.11) by

$$v(\theta) = \omega + u(\theta) - u(\theta - \omega) .$$

(ii) It is interesting to notice that for $\varepsilon = 0$ the embedding can be chosen as $K(\theta) = (\theta, \omega)$. In this case, the mapping (2.8) admits a natural attractor with frequency $\omega = \mu/(1 - \lambda)$. This simple observation highlights the role of the drift μ and its relation to the frequency ω .

The existence of an invariant attractor for $\varepsilon \neq 0$ will be established by fixing the frequency ω and determining a solution (K, μ) (equivalently (u, μ) according to Remark 9), satisfying the invariance equation (2.10) (equivalently (2.12)) for a fixed value of the dissipative parameter λ . The focus of this paper will be in giving explicit estimates and showing that the hypotheses of the theorem are satisfied numerically in a concrete example for explicit values of ε, λ . In particular, we will verify numerically that the estimates of the theorem are satisfied taking a numerically computed solution as the approximate solution. The computation of the solution is described in Section 7.1. The verification of the estimates on these numerical solutions is presented in Section 7.2.

3. A KAM THEOREM

In this Section we state the main mathematical result, Theorem 10, which is a KAM result in the a-posteriori format described in Theorem Format 1. Theorem 10 specifies some condition numbers to be measured in the approximate solution. It shows that, if there is a function K_0 and a number μ_0 that, when substituted in (2.10), give a residual (measured in a norm that we specify) which is smaller than a function of the condition numbers, then, there is a solution of (2.10) close (in some norm that we specify) to K_0, μ_0 .

We also note that the method of proof, which is based on constructing an iterative procedure, leads to a very efficient algorithm. Later, we will describe the implementation of the algorithm and the verification of the estimates required in Theorem 10.

For an embedding $K_0 = K_0(\theta)$ and a frequency ω , we start by introducing some auxiliary quantities defined as follows:

$$\begin{aligned} M_0(\theta) &\equiv [DK_0(\theta) \mid J^{-1} \circ K_0(\theta) DK_0(\theta)N_0(\theta)] , \\ S_0(\theta) &\equiv ((DK_0N_0) \circ T_\omega)^\top(\theta)Df_{\mu_0} \circ K_0(\theta)J^{-1} \circ K_0(\theta)DK_0(\theta)N_0(\theta) , \\ N_0(\theta) &\equiv (DK_0(\theta)^\top DK_0(\theta))^{-1} , \end{aligned} \tag{3.1}$$

where the superscript \top denotes the transposition and T_ω denotes the shift by ω : for a function $P = P(\theta)$, then $(P \circ T_\omega)(\theta) = P(\theta + \omega)$.

The following Theorem provides a constructive version of Theorem 20 in [12] for mappings as those introduced in Definition 7; in particular, it applies to the dissipative standard map (2.8).

It is quite important that the condition numbers in Theorem 10 are properties of the approximate solution, not global properties of the map. The condition numbers can be computed from the approximate solution by taking derivatives, performing algebraic operations and averaging.

Theorem 10. *Consider a family $f_\mu : \mathcal{M} \rightarrow \mathcal{M}$ of conformally symplectic mappings, defined on the manifold $\mathcal{M} \equiv B \times \mathbb{T}$ with $B \subseteq \mathbb{R}$ an open, simply connected domain with a smooth boundary. Let the mappings f_μ be analytic on an open connected domain $\mathcal{C} \subset \mathbb{C} \times \mathbb{C}/\mathbb{Z}$. Let the following assumptions be satisfied.*

H1 *Let $\omega \in \mathcal{D}(\nu, \tau)$ as in (2.4).*

H2 *There exists an approximate solution (K_0, μ_0) with $K_0 \in \mathcal{A}_{\rho_0}$ for some $\rho_0 > 0$ and with $\mu_0 \in \Lambda$, $\Lambda \subset \mathbb{R}$ open. Let (K_0, μ_0) be such that (2.10) is satisfied up to an error function $E_0 = E_0(\theta)$, namely*

$$f_{\mu_0} \circ K_0(\theta) - K_0(\theta + \omega) = E_0(\theta) .$$

Let ε_0 denote the size of the error function, i.e.

$$\varepsilon_0 \equiv \|E_0\|_{\rho_0} .$$

H3 *Assume that the following non-degeneracy condition holds:*

$$\det \begin{pmatrix} \bar{S}_0 & \overline{S_0(B_{b0})^0} + \overline{\tilde{A}_0^{(1)}} \\ \lambda - 1 & \overline{\tilde{A}_0^{(2)}} \end{pmatrix} \neq 0 ,$$

where S_0 is given in (3.1), $\tilde{A}_0^{(1)}$, $\tilde{A}_0^{(2)}$ denote the first and second elements of the vector $\tilde{A}_0 \equiv M_0^{-1} \circ T_\omega D_\mu f_{\mu_0} \circ K_0$, $(B_{b0})^0$ is the solution (with zero average in the $\lambda = 1$ case) of the equation $\lambda(B_{b0})^0 - (B_{b0})^0 \circ T_\omega = -(\tilde{A}_0^{(2)})^0$, where $(\tilde{A}_0^{(2)})^0$ denotes the zero average part of $\tilde{A}_0^{(2)}$. Denote by \mathcal{T}_0 the twist constant defined as

$$\mathcal{T}_0 \equiv \left\| \begin{pmatrix} \bar{S}_0 & \overline{S_0(B_{b0})^0} + \overline{\tilde{A}_0^{(1)}} \\ \lambda - 1 & \overline{\tilde{A}_0^{(2)}} \end{pmatrix}^{-1} \right\| .$$

H4 *Assume there exists $\zeta > 0$, so that*

$$\text{dist}(\mu_0, \partial\Lambda) \geq \zeta , \quad \text{dist}(K_0(\mathbb{T}_{\rho_0}), \partial\mathcal{C}) \geq \zeta .$$

H5 Let $0 < \delta_0 < \rho_0$. Let $\kappa_\mu \equiv 4C_{\sigma_0}$ with C_{σ_0} constant (whose explicit expression is given in Appendix B). Let the quantities $Q_0, Q_{\mu_0}, Q_{z\mu_0}, Q_{\mu\mu_0}, Q_{E_0}$ be defined as

$$\begin{aligned} Q_0 &\equiv \sup_{z \in \mathcal{C}} |Df_{\mu_0}(z)| , \\ Q_{\mu_0} &\equiv \sup_{z \in \mathcal{C}} |D_\mu f_{\mu_0}(z)| , \\ Q_{z\mu_0} &\equiv \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu Df_\mu(z)| , \\ Q_{\mu\mu_0} &\equiv \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu^2 f_\mu(z)| , \\ Q_{E_0} &\equiv \frac{1}{2} \max \left\{ \|D^2 E_0\|_{\rho_0 - \delta_0}, \|DD_\mu E_0\|_{\rho_0 - \delta_0}, \|D_\mu^2 E_0\|_{\rho_0 - \delta_0} \right\} . \end{aligned} \quad (3.2)$$

Assume that ε_0 satisfies the following smallness conditions for suitable real constants $C_{\eta_0}, C_{\mathcal{E}_0}, C_{d_0}, C_{\sigma_0}, C_\sigma, C_{W_0}, C_W, C_{\mathcal{R}}$ (see Appendix B for their explicit expressions):

$$C_{\eta_0} \nu^{-1} \delta_0^{-\tau} \varepsilon_0 < \zeta , \quad (3.3)$$

$$2^{3\tau+4} C_{\mathcal{E}_0} \nu^{-2} \delta_0^{-2\tau} \varepsilon_0 \leq 1 , \quad (3.4)$$

$$4C_{d_0} \nu^{-1} \delta_0^{-\tau} \varepsilon_0 < \zeta , \quad (3.5)$$

$$4C_{\sigma_0} \varepsilon_0 < \zeta , \quad (3.6)$$

$$\|N_0\|_{\rho_0} (2\|DK_0\|_{\rho_0} + D_K) D_K < 1 \quad (3.7)$$

$$4Q_{z\mu_0} C_{\sigma_0} \varepsilon_0 < Q_0 , \quad (3.8)$$

$$4Q_{\mu\mu_0} C_{\sigma_0} \varepsilon_0 < Q_{\mu_0} , \quad (3.9)$$

$$C_\sigma D_K \leq C_{\sigma_0} , \quad (3.10)$$

$$D_K (C_{W_0} + \|M_0\|_{\rho_0} C_W + C_W D_K) \leq C_{d_0} , \quad (3.11)$$

$$D_K \left(C_W C_c \nu \delta_0^{-1+\tau} + C_{\mathcal{R}} \right) \leq C_{\mathcal{E}_0} , \quad (3.12)$$

where D_K is defined as

$$D_K \equiv 4C_{d_0} C_c \nu^{-1} \delta_0^{-\tau-1} \varepsilon_0 \quad (3.13)$$

and with C_c as in (2.3).

Then, there exists an exact solution (K_e, μ_e) of (2.10) such that

$$f_{\mu_e} \circ K_e - K_e \circ T_\omega = 0 .$$

The quantities K_e, μ_e are close to the approximate solution, since one has

$$\begin{aligned} \|K_e - K_0\|_{\rho_0 - \delta_0} &\leq 4C_{d_0} \nu^{-1} \delta_0^{-\tau} \|E_0\|_{\rho_0} , \\ |\mu_e - \mu_0| &\leq 4C_{\sigma_0} \|E_0\|_{\rho_0} . \end{aligned} \quad (3.14)$$

The explicit expressions of the constants entering in the conditions (3.3)-(3.12) are obtained by implementing constructively the KAM proof presented in [12] and sketched in Section 4. In Section 7 the family f_μ will be taken as the dissipative standard map defined in (2.8); then, the explicit expressions for the constants - provided in Appendix B - will allow us to compute concrete values for ε_0 , once we fix the frequency ω and the conformal factor λ . Therefore, the conditions (3.3)-(3.12) will allow us to obtain a lower bound on the perturbing parameter, ensuring the existence of an invariant attractor with fixed frequency ω and for a given conformal factor λ .

Remark 11. *For any value of λ with $|\lambda| < 1$, Theorem 10 also ensures that the quasi-periodic solution provided by the manifold $K_e(\mathbb{T}^n)$ is a local attractor and that the dynamics on this attractor is analytically conjugated to a rigid rotation. Indeed, according to [13], the diagrams in a neighborhood is analytically conjugated to a rotation and homothety.*

Remark 12. *One question that has been posed to us several times is how it is possible to use the computer to verify hypotheses that involve irrational numbers and indeed the Diophantine properties. After all, the standard computer numbers are only rational numbers.*

The answer is that the a-posteriori theorem uses the Diophantine properties and that this theorem is indeed given a traditional proof. To verify the hypothesis, we compute numerically $\|f_\mu \circ K - K \circ T_{\omega_0}\|$ where ω_0 is indeed a rational number.

It is clear that for $\xi \in (\omega_0, \omega)$:

$$\begin{aligned} \|f_\mu \circ K - K \circ T_\omega\| &\leq \|f_\mu \circ K - K \circ T_{\omega_0}\| + \|K \circ T_{\omega_0} - K \circ T_\omega\| \\ &\leq \|f_\mu \circ K - K \circ T_{\omega_0}\| + \|DK \circ T_\xi\| |\omega - \omega_0|. \end{aligned}$$

In our case, we see that $|\omega - \omega_0| \leq 10^{-100}$ and that DK is a number of order 1. Hence, the last term does not affect the final result much.

Of course, implementing interval arithmetic, one can also use an interval that contains the desired frequency and obtain estimates for the error of invariance valid uniformly for all ω in this interval.

4. SKETCH OF THE PROOF OF THEOREM 10

We note that in the statement of Theorem 10 (and in the subsequent text) all the constants are given explicitly (see appendix B). There are only a few dozen of conditions

to check; all these conditions are easy algebraic expressions. Even if this is cumbersome for a human, a computer calculates them in a very short time.

The proof of Theorem 10 is presented in detail in [12]. However, in [12] the proof was given for a general case and no explicit estimates on the constants were provided. On the contrary, in Section 5 we give concrete expressions in view of the application to (2.8).

Before providing the lengthy and detailed proof given in Section 5, we proceed to outline a sketch of the proof of Theorem 10 by splitting it in 5 main steps, which are needed to be performed in order to get the solution of the invariance equation. The constructive version of the proof, which will be developed in Section 5, yields explicit expressions for the constants appearing in the smallness conditions (3.3)-(3.12); being a long list, such constants are given in Appendix B.

We anticipate that it is easy to see that in the one-dimensional case of the mapping (2.8) all invariant curves are Lagrangian; this observation will simplify the proof presented in Section 5 with respect to that developed in [12]. However, in the general n -dimensional case the following remark gives a practical characterization of the Lagrangian character of invariant tori.

Remark 13. *Let f be a conformally symplectic map defined on an n -dimensional manifold $B_n \times \mathbb{T}^n$, where $B_n \subseteq \mathbb{R}^n$ is an open, simply connected domain with smooth boundary. Then, invariant tori are Lagrangian, namely if $|\lambda| \neq 1$ and K satisfies (2.10), then one has*

$$K^*\Omega = 0 . \tag{4.1}$$

Moreover, if f is symplectic and ω is irrational, then the n -dimensional torus is Lagrangian.

Below it is a description of the main steps required to prove Theorem 10.

Step 1: on the initial approximate solution and its linearization.

Let (K_0, μ_0) be an approximate solution of the invariance equation (2.10) and let $E_0 = E_0(\theta)$ be the associated error function. In coordinates, the Lagrangian condition (4.1) becomes

$$DK_0^\top(\theta) J \circ K_0(\theta) DK_0(\theta) = 0 ,$$

which shows that the tangent space can be decomposed as $\text{Range} \left(DK_0(\theta) \right) \oplus \text{Range} \left(J^{-1} \circ K_0(\theta) DK_0(\theta) \right)$.

Therefore, one can show that, up to a remainder function $R_0 = R_0(\theta)$, the quantity $Df_{\mu_0} \circ K_0$ is conjugate to an upper diagonal matrix with constant diagonals and the following identity is satisfied:

$$Df_{\mu_0} \circ K_0(\theta) M_0(\theta) = M_0(\theta + \omega) \begin{pmatrix} \text{Id} & S_0(\theta) \\ 0 & \lambda \text{Id} \end{pmatrix} + R_0(\theta) \quad (4.2)$$

with M_0 and S_0 as in (3.1).

Then, we proceed to find some corrections W_0 and σ_0 such that, setting $K_1 = K_0 + M_0 W_0$, $\mu_1 = \mu_0 + \sigma_0$, one has that the new approximation (K_1, μ_1) satisfies the following invariance equation:

$$f_{\mu_1} \circ K_1(\theta) - K_1(\theta + \omega) = E_1(\theta) \quad (4.3)$$

for some error function $E_1 = E_1(\theta)$. The requirement on E_1 is that its norm is quadratically smaller than the norm of the initial approximation E_0 . This can be obtained provided that the following equation is satisfied:

$$Df_{\mu_0} \circ K_0(\theta) M_0(\theta) W_0(\theta) - M_0(\theta + \omega) W_0(\theta + \omega) + D_{\mu} f_{\mu_0} \circ K_0(\theta) \sigma_0 = -E_0(\theta) . \quad (4.4)$$

Step 2: determination of the new approximation.

The corrections (W_0, σ_0) in Step 1 are determined as follows. Using (4.2), (4.4) and neglecting higher order terms, one obtains two cohomology equations with constant coefficients for W_0 and σ_0 . More precisely, writing W_0 in components as $W_0 = (W_0^{(1)}, W_0^{(2)})$, such cohomological equations are given by

$$\begin{aligned} W_0^{(1)}(\theta) - W_0^{(1)}(\theta + \omega) &= -\tilde{E}_0^{(1)}(\theta) - S_0(\theta) W_0^{(2)}(\theta) - \tilde{A}_0^{(1)}(\theta) \sigma_0 , \\ \lambda W_0^{(2)}(\theta) - W_0^{(2)}(\theta + \omega) &= -\tilde{E}_0^{(2)}(\theta) - \tilde{A}_0^{(2)}(\theta) \sigma_0 \end{aligned} \quad (4.5)$$

with S_0 given in (3.1), while \tilde{E}_0, \tilde{A}_0 are defined as

$$\begin{aligned} \tilde{E}_0 &\equiv (\tilde{E}_0^{(1)}, \tilde{E}_0^{(2)}) \equiv M_0^{-1} \circ T_{\omega} E_0 , \\ \tilde{A}_0 &\equiv M_0^{-1} \circ T_{\omega} D_{\mu} f_{\mu_0} \circ K_0 , \end{aligned} \quad (4.6)$$

where we denote by $\tilde{A}_0^{(1)}, \tilde{A}_0^{(2)}$ the first and second elements of the vector \tilde{A}_0 .

We remark that the first equation in (4.5) involves small divisors. In fact, the Fourier expansion of the l.h.s. of the first equation in (4.5) is given by

$$W_0^{(1)}(\theta) - W_0^{(1)}(\theta + \omega) = \sum_{k \in \mathbb{Z}} \widehat{W}_{0,k}^{(1)} e^{2\pi i k \theta} (1 - e^{2\pi i k \omega}) .$$

Then, we notice that for $k = 0$ there appears the zero factor $1 - e^{2\pi ik\omega} = 0$. On the other hand, the second equation in (4.5) is always solvable for any $|\lambda| \neq 1$ by a contraction mapping argument.

Let us split $W_0^{(2)}$ as $W_0^{(2)} = \overline{W_0^{(2)}} + (W_0^{(2)})^0$, where the first term denotes the average of $W_0^{(2)}$ and the second term the zero-average part. We remark that the average of $W_0^{(1)}$ can be set to zero without loss of generality. On the other hand, computing the averages of the cohomological equations (4.5), one can determine $\overline{W_0^{(2)}}$, σ_0 by solving the system of equations

$$\begin{pmatrix} \overline{S_0} & \overline{S_0(B_{b0})^0} + \overline{\tilde{A}_0^{(1)}} \\ \lambda - 1 & \overline{\tilde{A}_0^{(2)}} \end{pmatrix} \begin{pmatrix} \overline{W_0^{(2)}} \\ \sigma_0 \end{pmatrix} = \begin{pmatrix} -\overline{S_0(B_{a0})^0} - \overline{\tilde{E}_0^{(1)}} \\ -\overline{\tilde{E}_0^{(2)}} \end{pmatrix}, \quad (4.7)$$

where we have split $(W_0^{(2)})^0$ as $(W_0^{(2)})^0 = (B_{a0})^0 + \sigma_0(B_{b0})^0$, where $(B_{a0})^0$, $(B_{b0})^0$ are the zero average solutions of

$$\begin{aligned} \lambda(B_{a0})^0 - (B_{a0})^0 \circ T_\omega &= -(\tilde{E}_0^{(2)})^0, \\ \lambda(B_{b0})^0 - (B_{b0})^0 \circ T_\omega &= -(\tilde{A}_0^{(2)})^0 \end{aligned} \quad (4.8)$$

with $(\tilde{E}_0^{(2)})^0$, $(\tilde{A}_0^{(2)})^0$ denoting the zero average parts of $\tilde{E}_0^{(2)}$, $\tilde{A}_0^{(2)}$. After solving (4.7), one can proceed to solve (4.5) for the zero average parts of $W_0^{(1)}$, $W_0^{(2)}$.

Step 3: on the quadratic convergence of the iterative step.

Once we have determined the correction (W_0, σ_0) as in step 2, we show that the new solution $K_1 = K_0 + M_0 W_0$, $\mu_1 = \mu_0 + \sigma_0$ satisfies the invariance equation with an error quadratically smaller with respect to the error at the previous step. Precisely, for $0 < \delta_0 < \rho_0$ one can prove that the error term E_1 in (4.3) associated to (K_1, μ_1) satisfies an inequality of the form

$$\|E_1\|_{\rho_0 - \delta_0} \leq C' \nu^{-1} \delta_0^{-\tau} \|E_0\|_{\rho_0}^2,$$

for some constant $C' > 0$ (of course, in this constructive version of the Theorem we will assume explicit values of the new error).

Step 4: on the analytic convergence of the sequence of approximate solutions.

The procedure outlined in steps 1–3 can be iterated to get a sequence of approximate solutions, say $\{K_j, \mu_j\}$ with $j \geq 0$. The convergence of the sequence of approximate solutions to the true solution of the invariance equation (2.10) is obtained through an analytic smoothing, which provides the convergence of the iterative step to the exact solution. It is worth noticing that the sequence of approximate solutions is constructed

in domains that are smaller than the original domains. However, one can suitably choose the loss of analyticity domains, so that the exact solution is defined in a non-empty domain.

Step 5: on the local uniqueness of the solution.

According to [12], if there exist two solutions (K_a, μ_a) , (K_b, μ_b) close enough, then there exists $s \in \mathbb{R}$ such that for all $\theta \in \mathbb{T}$:

$$\begin{aligned} K_b(\theta) &= K_a(\theta + s) , \\ \mu_a &= \mu_b . \end{aligned}$$

We refer to [12] for the proof of the uniqueness of the solution.

5. A CONSTRUCTIVE VERSION OF THE PROOF OF THEOREM 10

In this Section we prove Theorem 10, providing explicit expressions for all the constants involved. Such quantities will depend on the norm of the mapping, the initial parameters and the norm of the initial approximation.

According to the sketch of the proof given in Section 4, we proceed to compute the estimates of the following quantities:

- (i) estimate of the remainder $R_0 = R_0(\theta)$ appearing in (4.2) (see Section 5.1 and compare with step 1 of Section 4);
- (ii) estimates for the corrections (W_0, σ_0) defined as the solutions of the cohomological equations (4.5) (see Section 5.2 and compare with step 2 of Section 4);
- (iii) quadratic estimates on the convergence of the iterative step, which implies to bound the norm of $D\mathcal{E} M_0 W_0 + D_\mu \mathcal{E} \sigma_0 + E_0$, where \mathcal{E} is the error functional $\mathcal{E}[K, \mu] \equiv f_\mu \circ K - K \circ T_\omega$ (see Section 5.3);
- (iv) thanks to the results in (iii), we will be able to give quadratic estimates of the error associated to the new solution, say $\mathcal{E}[K + \Delta, \mu + \sigma]$, in terms of the square of the norm of E_0 (see Section 5.4 and compare with (4.3) and step 3 of Section 4);
- (v) proof of the analytic convergence of the sequence of approximate solutions to the true solution of the invariance equation (see Section 5.5 and compare with step 4 of Section 4).

5.1. Estimate on the error R_0 . Any torus associated to a one-dimensional map is always Lagrangian; this leads to a simplification of the expression of the remainder function

in (4.2). In fact, recalling the definition of M_0 in (3.1), it turns out that

$$R_0(\theta) = DE_0(\theta) . \quad (5.1)$$

Using Cauchy estimates, a bound on R_0 in (5.1) is given by

$$\|R_0\|_{\rho_0-\delta_0} \leq C_c \delta_0^{-1} \|E_0\|_{\rho_0} \quad (5.2)$$

with C_c as in (2.3).

5.2. Estimates for the increment in the steps. We proceed to give estimates for the corrections W_0 and σ_0 , which satisfy the equations (4.5).

Lemma 14. *Let $K_0 \in \mathcal{A}_{\rho_0+\delta_0}$, $K_0(\mathbb{T}_{\rho_0}) \subset \text{domain}(f_\mu)$, $\text{dist}(K_0(\mathbb{T}_{\rho_0}), \partial(\text{domain}(f_\mu))) \geq \zeta > 0$ with ρ_0, δ_0, ζ as in Theorem 10. For any $|\lambda| \neq 1$ we have*

$$\begin{aligned} \|W_0\|_{\rho_0-\delta_0} &\leq C_{W_0} \nu^{-1} \delta_0^{-\tau} \|E_0\|_{\rho_0} , \\ |\sigma_0| &\leq C_{\sigma_0} \|E_0\|_{\rho_0} , \end{aligned}$$

where

$$\begin{aligned} C_{\sigma_0} &\equiv \mathcal{T}_0 \left[|\lambda - 1| \left(\frac{1}{\left| |\lambda| - 1 \right|} \|S_0\|_{\rho_0} + 1 \right) + \|S_0\|_{\rho_0} \right] \|M_0^{-1}\|_{\rho_0} , \\ \bar{C}_{W_{20}} &\equiv 2 \mathcal{T}_0 \left(\frac{1}{\left| |\lambda| - 1 \right|} \|S_0\|_{\rho_0} + 1 \right) Q_{\mu_0} \|M_0^{-1}\|_{\rho_0}^2 , \\ C_{W_{20}} &\equiv \frac{1}{\left| |\lambda| - 1 \right|} \left(1 + C_{\sigma_0} Q_{\mu_0} \right) \|M_0^{-1}\|_{\rho_0} , \\ C_{W_{10}} &\equiv C_0 \left[\|S_0\|_{\rho_0} (C_{W_{20}} + \bar{C}_{W_{20}}) + \|M_0^{-1}\|_{\rho_0} + Q_{\mu_0} \|M_0^{-1}\|_{\rho_0} C_{\sigma_0} \right] , \\ C_{W_0} &\equiv C_{W_{10}} + (C_{W_{20}} + \bar{C}_{W_{20}}) \nu \delta_0^\tau . \end{aligned} \quad (5.3)$$

Proof. Let Q_{μ_0} be an upper bound on the norm of $D_\mu f_{\mu_0}$ as in (3.2). Let \tilde{A}_0 be defined as in (4.6); then, we have:

$$\|\tilde{A}_0\|_{\rho_0} \leq Q_{\mu_0} \|M_0^{-1}\|_{\rho_0} .$$

Recalling the definition of S_0 in (3.1), we obtain

$$\|S_0\|_{\rho_0} \leq J_e Q_0 \|DK_0\|_{\rho_0}^2 \|N_0\|_{\rho_0}^2 \leq C_c^2 J_e Q_0 \|K_0\|_{\rho_0+\delta_0}^2 \|N_0\|_{\rho_0}^2 \delta_0^{-2} ,$$

where we used the estimate $\|DK_0\|_{\rho_0} \leq C_c \|K_0\|_{\rho_0+\delta_0} \delta_0^{-1}$ and where J_e denotes the norm of the symplectic matrix J in (2.9) (the norm of J^{-1} is again bounded by J_e). Notice that, with the choice of the norms in Section 2.1.1 it is $J_e = 1$. We notice that, recalling the definition of S_0 and M_0 in (3.1), one can compute directly the functions and evaluate their norm.

For any $|\lambda| \neq 1$, we have the estimates given below, which follow from (4.6), (4.7), (4.8):

$$\begin{aligned}
|\overline{W}_0^{(2)}| &\leq \mathcal{T}_0 \left(\|\overline{S}_0(B_{b0})^0 + \overline{A}_0^{(1)}\|_{\rho_0} \|\overline{E}_0^{(2)}\|_{\rho_0} + \|\overline{S}_0(B_{a0})^0 + \overline{E}_0^{(1)}\|_{\rho_0} \|\overline{A}_0^{(2)}\|_{\rho_0} \right) \\
&\leq \mathcal{T}_0 \left[\left(\frac{1}{\left| |\lambda| - 1 \right|} \|S_0\|_{\rho_0} + 1 \right) Q_{\mu 0} \|M_0^{-1}\|_{\rho_0}^2 \|E_0\|_{\rho_0} \right. \\
&\quad \left. + \left(\frac{1}{\left| |\lambda| - 1 \right|} \|S_0\|_{\rho_0} + 1 \right) \|M_0^{-1}\|_{\rho_0}^2 \|E_0\|_{\rho_0} Q_{\mu 0} \right] \\
&\leq 2\mathcal{T}_0 \left(\frac{1}{\left| |\lambda| - 1 \right|} \|S_0\|_{\rho_0} + 1 \right) Q_{\mu 0} \|M_0^{-1}\|_{\rho_0}^2 \|E_0\|_{\rho_0} \\
&\equiv \overline{C}_{W_{20}} \|E_0\|_{\rho_0} , \\
|\sigma_0| &\leq \mathcal{T}_0 \left(|\lambda - 1| \|\overline{S}_0(B_{a0})^0 + \overline{E}_0^{(1)}\|_{\rho_0} + \|S_0\|_{\rho_0} \|\overline{E}_0^{(2)}\|_{\rho_0} \right) \\
&\leq \mathcal{T}_0 \left[|\lambda - 1| \left(\frac{1}{\left| |\lambda| - 1 \right|} \|S_0\|_{\rho_0} + 1 \right) + \|S_0\|_{\rho_0} \right] \|M_0^{-1}\|_{\rho_0} \|E_0\|_{\rho_0} \\
&\equiv C_{\sigma 0} \|E_0\|_{\rho_0}
\end{aligned}$$

with $\overline{C}_{W_{20}}$, $C_{\sigma 0}$ as in (5.3). Then, using Lemma 5 and Lemma 6, we have:

$$\begin{aligned}
\|(W_0^{(2)})^0\|_{\rho_0} &\leq \frac{1}{\left| |\lambda| - 1 \right|} (\|M_0^{-1}\|_{\rho_0} \|E_0\|_{\rho_0} + Q_{\mu 0} \|M_0^{-1}\|_{\rho_0} |\sigma_0|) \leq C_{W_{20}} \|E_0\|_{\rho_0} , \\
\|W_0^{(1)}\|_{\rho_0 - \delta_0} &\leq C_0 \nu^{-1} \delta_0^{-\tau} \left(\|S_0\|_{\rho_0} \|W_0^{(2)}\|_{\rho_0} + \|M_0^{-1}\|_{\rho_0} \|E_0\|_{\rho_0} + Q_{\mu 0} \|M_0^{-1}\|_{\rho_0} |\sigma_0| \right) \\
&\leq C_{W_{10}} \nu^{-1} \delta_0^{-\tau} \|E_0\|_{\rho_0}
\end{aligned}$$

with $C_{W_{10}}$, $C_{W_{20}}$ as in (5.3). In conclusion, we obtain:

$$\begin{aligned}
\|W_0\|_{\rho_0 - \delta_0} &\leq \|W_0^{(1)}\|_{\rho_0 - \delta_0} + \|W_0^{(2)}\|_{\rho_0} \leq C_{W_{10}} \nu^{-1} \delta_0^{-\tau} \|E_0\|_{\rho_0} + (C_{W_{20}} + \overline{C}_{W_{20}}) \|E_0\|_{\rho_0} \\
&\equiv C_{W_0} \nu^{-1} \delta_0^{-\tau} \|E_0\|_{\rho_0} ,
\end{aligned}$$

with C_{W_0} as in (5.3). □

Remark 15. *Let us define the error functional*

$$\mathcal{E}[K_0, \mu_0] \equiv f_{\mu_0} \circ K_0 - K_0 \circ T_\omega .$$

Let

$$(\Delta_0, \sigma_0) = -\eta[K_0, \mu_0]E_0 ,$$

where $\Delta_0 = -(\eta[K_0, \mu_0]E_0)_1$, $\sigma_0 = -(\eta[K_0, \mu_0]E_0)_2$. Then, using that $\Delta_0 = M_0 W_0$, one has:

$$\begin{aligned}
\|\eta[K_0, \mu_0]E_0\|_{\rho_0 - \delta_0} &\leq \|M_0\|_{\rho_0} \|W_0\|_{\rho_0 - \delta_0} + |\sigma_0| \\
&\leq C_{\eta 0} \nu^{-1} \delta_0^{-\tau} \|E_0\|_{\rho_0} ,
\end{aligned}$$

where

$$C_{\eta_0} \equiv C_{W_0} \|M_0\|_{\rho_0} + C_{\sigma_0} \nu \delta_0^\tau . \quad (5.4)$$

5.3. Estimates for the convergence of the iterative step. In this Section we give quadratic estimates on the norm of $D\mathcal{E}[K_0, \mu_0]\Delta_0 + D_\mu\mathcal{E}[K_0, \mu_0]\sigma_0 + E_0$ with $\Delta_0 \equiv M_0W_0$; these estimates are needed to bound the error of the new approximate solution as it will be done in Section 5.4.

Lemma 16. *We have the following estimate:*

$$\|E_0 + D\mathcal{E}[K_0, \mu_0]\Delta_0 + D_\mu\mathcal{E}[K_0, \mu_0]\sigma_0\|_{\rho_0-\delta_0} \leq C_c C_{W_0} \nu^{-1} \delta_0^{-1-\tau} \|E_0\|_{\rho_0}^2 . \quad (5.5)$$

Proof. Taking into account that $W_0 = M_0^{-1}\Delta_0$, from the relation (7.15) in [12] we have that,

$$E_0 + D\mathcal{E}[K_0, \mu_0]\Delta_0 + D_\mu\mathcal{E}[K_0, \mu_0]\sigma_0 = R_0W_0 .$$

From Lemma 14 and (5.2), we obtain that

$$\begin{aligned} \|E_0 + D\mathcal{E}[K_0, \mu_0]\Delta_0 + D_\mu\mathcal{E}[K_0, \mu_0]\sigma_0\|_{\rho_0-\delta_0} &\leq \|R_0\|_{\rho_0-\delta_0} \|W_0\|_{\rho_0-\delta_0} \\ &\leq C_c C_{W_0} \nu^{-1} \delta_0^{-1-\tau} \|E_0\|_{\rho_0}^2 . \end{aligned}$$

In conclusion, we have (5.5). \square

5.4. Estimates for the error of the new solution. We proceed to bound the error corresponding to the new approximate solution.

Lemma 17. *Let $\eta[K_0, \mu_0]$ be as in Lemma 16 and let $\zeta > 0$ be such that*

$$\begin{aligned} \text{dist}(\mu_0, \partial\Lambda) &\geq \zeta , \\ \text{dist}(K_0(\mathbb{T}_{\rho_0}), \partial\mathcal{C}) &\geq \zeta . \end{aligned}$$

Assume that

$$C_{\eta_0} \nu^{-1} \delta_0^{-\tau} \|E_0\|_{\rho_0} < \zeta < 1 \quad (5.6)$$

with C_{η_0} as in (5.4). Then, we obtain the following estimate for the error:

$$\|\mathcal{E}[K_0 + \Delta_0, \mu_0 + \sigma_0]\|_{\rho_0-\delta_0} \leq C_{\mathcal{E}0} \nu^{-2} \delta_0^{-2\tau} \|E_0\|_{\rho_0}^2 ,$$

where

$$C_{\mathcal{E}0} \equiv C_c C_{W_0} \nu \delta_0^{-1+\tau} + C_{\mathcal{R}0} \quad (5.7)$$

with

$$C_{\mathcal{R}0} \equiv Q_{E0} (\|M_0\|_{\rho_0}^2 C_{W_0}^2 + C_{\sigma_0}^2 \nu^2 \delta_0^{2\tau}) . \quad (5.8)$$

Proof. Define the remainder of the Taylor series expansion as

$$\mathcal{R}[(K_0, \mu_0), (K'_0, \mu'_0)] \equiv \mathcal{E}[K'_0, \mu'_0] - \mathcal{E}[K_0, \mu_0] - D\mathcal{E}[K_0, \mu_0](K'_0 - K_0) - D_\mu \mathcal{E}[K_0, \mu_0](\mu'_0 - \mu_0).$$

Then, we can write

$$\mathcal{E}[K_0 + \Delta_0, \mu_0 + \sigma_0] = E_0 + D\mathcal{E}[K_0, \mu_0]\Delta_0 + D_\mu \mathcal{E}[K_0, \mu_0]\sigma_0 + \mathcal{R}[(K_0, \mu_0), (K_0 + \Delta_0, \mu_0 + \sigma_0)].$$

From Lemma 14 and the definition of Q_{E_0} in (3.2), we obtain

$$\begin{aligned} \|\mathcal{R}\|_{\rho_0 - \delta_0} &\leq Q_{E_0} \left(\|\Delta_0\|_{\rho_0 - \delta_0}^2 + |\sigma_0|^2 \right) \leq Q_{E_0} \left[\|M_0\|_{\rho_0}^2 (C_{W_0} \nu^{-1} \delta_0^{-\tau} \|E_0\|_{\rho_0})^2 + (C_{\sigma_0} \|E_0\|_{\rho_0})^2 \right] \\ &\equiv C_{\mathcal{R}0} \nu^{-2} \delta_0^{-2\tau} \|E_0\|_{\rho_0}^2, \end{aligned}$$

with $C_{\mathcal{R}0}$ as in (5.8). Then, from Lemma 16 we conclude that

$$\begin{aligned} \|\mathcal{E}[K_0 + \Delta_0, \mu_0 + \sigma_0]\|_{\rho_0 - \delta_0} &\leq C_c C_{W_0} \nu^{-1} \delta_0^{-1-\tau} \|E_0\|_{\rho_0}^2 + C_{\mathcal{R}0} \nu^{-2} \delta_0^{-2\tau} \|E_0\|_{\rho_0}^2 \\ &\leq C_{\mathcal{E}0} \nu^{-2} \delta_0^{-2\tau} \|E_0\|_{\rho_0}^2 \end{aligned}$$

with $C_{\mathcal{E}0}$ as in (5.7). Notice that (5.6) guarantees that

$$\|\Delta_0\|_{\rho_0 - \delta_0} < \zeta, \quad |\sigma_0| < \zeta.$$

□

5.5. Analytic convergence. In this Section we prove that if we start with a small enough error, it is possible to repeat indefinitely the algorithm and that iterating the algorithm, we obtain a sequence of approximate solutions which converge to the true solution of the invariance equation (2.10).

Again, let (K_0, μ_0) be the initial approximate solution with $K_0 \in \mathcal{A}_{\rho_0}$ for some $\rho_0 > 0$ as in Theorem 10 and define the sequence of parameters $\{\delta_h\}$, $\{\rho_h\}$, $h \geq 0$, as

$$\delta_h \equiv \frac{\rho_0}{2^{h+2}}, \quad \rho_{h+1} \equiv \rho_h - \delta_h, \quad h \geq 0.$$

With this choice of parameters the domain of analyticity where the true solution is defined will be a non-empty domain with size ρ_∞ given by

$$\rho_\infty = \rho_0 - \sum_{j=0}^{\infty} \frac{\rho_0}{2^{j+2}} = \rho_0 - \frac{\rho_0}{2} > 0.$$

Let (K_h, μ_h) , $h \geq 1$, be the approximate solution constructed by finding at each step the corrections (W_h, σ_h) solving the analogous of the cohomological equations (4.5) for $h = 0$. To make the notation precise, all quantities associated to (K_h, μ_h) will carry a subindex h , indicating the step of the algorithm. Define

$$\varepsilon_h \equiv \|\mathcal{E}(K_h, \mu_h)\|_{\rho_h},$$

and let us introduce the following quantities:

$$d_h \equiv \|\Delta_h\|_{\rho_h}, \quad v_h \equiv \|D\Delta_h\|_{\rho_h}, \quad s_h \equiv |\sigma_h| .$$

By Lemma 14 we have the following inequalities:

$$\begin{aligned} d_h &\leq C_{dh} \nu^{-1} \delta_h^{-\tau} \varepsilon_h , \\ v_h &\leq C_{dh} C_c \nu^{-1} \delta_h^{-\tau-1} \varepsilon_h , \\ s_h &\leq C_{\sigma h} \varepsilon_h , \end{aligned}$$

where

$$C_{dh} \equiv C_{Wh} \|M_h\|_{\rho_h} \tag{5.9}$$

where the quantities C_{Wh} , $C_{\sigma h}$ are obtained through the following expressions:

$$\begin{aligned} C_{\sigma h} &\equiv \mathcal{T}_h \left[|\lambda - 1| \left(\frac{1}{\left| |\lambda| - 1 \right|} \|S_h\|_{\rho_h} + 1 \right) + \|S_h\|_{\rho_h} \right] \|M_h^{-1}\|_{\rho_h} , \\ \bar{C}_{W_2h} &\equiv 2\mathcal{T}_h \left(\frac{1}{\left| |\lambda| - 1 \right|} \|S_h\|_{\rho_h} + 1 \right) Q_{\mu h} \|M_h^{-1}\|_{\rho_h}^2 , \\ C_{W_2h} &\equiv \frac{1}{\left| |\lambda| - 1 \right|} \left(1 + C_{\sigma h} Q_{\mu h} \right) \|M_h^{-1}\|_{\rho_h} , \\ C_{W_1h} &\equiv C_0 \left[\|S_h\|_{\rho_h} (C_{W_2h} + \bar{C}_{W_2h}) + \|M_h^{-1}\|_{\rho_h} + Q_{\mu h} \|M_h^{-1}\|_{\rho_h} C_{\sigma h} \right] , \\ C_{Wh} &\equiv C_{W_1h} + (C_{W_2h} + \bar{C}_{W_2h}) \nu \delta_h^\tau . \end{aligned} \tag{5.10}$$

Remark 18. *By Lemma 17 one has*

$$\varepsilon_{h+1} \leq C_{\mathcal{E}h} \nu^{-2} \delta_h^{-2\tau} \varepsilon_h^2 ,$$

where $C_{\mathcal{E}h}$ is defined as

$$C_{\mathcal{E}h} \equiv C_c C_{Wh} \nu \delta_h^{-1+\tau} + C_{\mathcal{R}h} \tag{5.11}$$

with

$$C_{\mathcal{R}h} \equiv Q_{\mathcal{E}h} \left(\|M_h\|_{\rho_h}^2 C_{Wh}^2 + C_{\sigma h}^2 \nu^2 \delta_h^{2\tau} \right) \tag{5.12}$$

and

$$Q_{\mathcal{E}h} \equiv \frac{1}{2} \max \left\{ \|D^2 E_h\|_{\rho_h - \delta_h}, \|DD_\mu E_h\|_{\rho_h - \delta_h}, \|D_\mu^2 E_h\|_{\rho_h - \delta_h} \right\} .$$

The results of Theorem 10 are based on the following proposition.

Proposition 19. *Let the constants C_{d0} , $C_{\sigma 0}$, $C_{\mathcal{E}0}$ be as in (5.9), (5.10), (5.11) with $h = 0$. Define the following quantities:*

$$\kappa_K \equiv 4C_{d0} \nu^{-1} \delta_0^{-\tau} , \quad \kappa_\mu \equiv 4C_{\sigma 0} , \quad \kappa_0 \equiv 2^{2\tau+1} C_{\mathcal{E}0} \nu^{-2} \delta_0^{-2\tau} . \tag{5.13}$$

Assume that the following conditions are satisfied:

$$2^{\tau+3} \kappa_0 \varepsilon_0 \leq 1, \quad (5.14)$$

$$\kappa_K \varepsilon_0 < \zeta, \quad (5.15)$$

$$\kappa_\mu \varepsilon_0 < \zeta, \quad (5.16)$$

$$\|N_0\|_{\rho_0} (2\|DK_0\|_{\rho_0} + D_K) D_K < 1 \quad (5.17)$$

$$C_\sigma D_K \leq C_{\sigma_0}, \quad (5.18)$$

$$D_K \left(C_W C_c \nu \delta_0^{-1+\tau} + C_{\mathcal{R}} \right) \leq C_{\varepsilon_0}, \quad (5.19)$$

$$D_K (C_{W_0} + \|M_0\|_{\rho_0} C_W + C_W D_K) \leq C_{d_0}, \quad (5.20)$$

$$4Q_{z\mu_0} C_{\sigma_0} \varepsilon_0 < Q_0, \quad (5.21)$$

$$4Q_{\mu\mu_0} C_{\sigma_0} \varepsilon_0 < Q_{\mu_0}, \quad (5.22)$$

where the constants C_σ , C_W , $C_{\mathcal{R}}$, C_{W_0} , D_K are defined in Appendix B. Then, for all integers $h \geq 0$ the following inequalities $(p1; h)$, $(p2; h)$, $(p3; h)$ hold:

$(p1; h)$

$$\begin{aligned} \|K_h - K_0\|_{\rho_h} &\leq \kappa_K \varepsilon_0 < \zeta, \\ |\mu_h - \mu_0| &\leq \kappa_\mu \varepsilon_0 < \zeta; \end{aligned} \quad (5.23)$$

$(p2; h)$

$$\varepsilon_h \leq (\kappa_0 \varepsilon_0)^{2^h - 1} \varepsilon_0;$$

$(p3; h)$

$$\begin{aligned} C_{dh} &\leq 2C_{d_0}, \\ C_{\sigma h} &\leq 2C_{\sigma_0}, \\ C_{\varepsilon h} &\leq 2C_{\varepsilon_0}. \end{aligned} \quad (5.24)$$

The proof of Proposition 19 is quite long (see Section 6), but it is well structured and broken into small steps that can be easily verified. However, it allows to give the proof of Theorem 10 by *analytic smoothing*: at each step, the corrections (W_h, σ_h) allow to construct increasingly approximate solutions, defined on smaller analyticity domains. The loss of domain is such that the exact solution is defined on a domain with positive radius of analyticity.

Proof. (of Theorem 10) The inequalities (3.14) follow directly from (5.23) and (5.13). The condition (3.3) follows from (5.6) of Lemma 17, while the conditions (3.4)-(3.12) follow from (5.14)-(5.22) of Proposition 19. \square

6. PROOF OF PROPOSITION 19

The proof of Proposition 19 proceeds by induction. We start by noticing that $(p1; 0)$, $(p2; 0)$ and $(p3; 0)$ are trivial. Let $H \in \mathbb{Z}_+$ and assume that $(p1; h)$, $(p2; h)$, $(p3; h)$ are true for $h = 1, \dots, H$. Then, by Lemma 17 we obtain the Taylor estimate

$$\begin{aligned} \varepsilon_h &= \|\mathcal{E}(K_{h-1} + \Delta_{h-1}, \mu_{h-1} + \sigma_{h-1})\|_{\rho_h} \\ &\leq C_{\mathcal{E}, h-1} \nu^{-2} \delta_{h-1}^{-2\tau} \varepsilon_{h-1}^2 \\ &\leq 2C_{\mathcal{E}0} \nu^{-2} \delta_{h-1}^{-2\tau} \varepsilon_{h-1}^2, \end{aligned} \quad (6.1)$$

where

$$C_{\mathcal{E}, h-1} \leq 2C_{\mathcal{E}0},$$

due to $(p3; h)$ for $h = 1, \dots, H$. The estimate (6.1) allows to have a bound of ε_h , $h = 1, \dots, H$, in terms of ε_0 :

$$\begin{aligned} \varepsilon_h &\leq 2C_{\mathcal{E}0} \nu^{-2} \delta_0^{-2\tau} 2^{2\tau(h-1)} \varepsilon_{h-1}^2 \\ &\leq (2C_{\mathcal{E}0} \nu^{-2} \delta_0^{-2\tau}) 2^{2\tau(h-1)} (2C_{\mathcal{E}0} \nu^{-2} \delta_0^{-2\tau} 2^{2\tau(h-2)} \varepsilon_{h-2}^2)^2 \\ &\leq (2C_{\mathcal{E}0} \nu^{-2} \delta_0^{-2\tau})^{1+2+\dots+2^{h-1}} 2^{2\tau((h-1)+2(h-2)+\dots+2^{h-2})} \varepsilon_0^{2^h} \\ &\leq (2C_{\mathcal{E}0} \nu^{-2} \delta_0^{-2\tau})^{2^h-1} 2^{2\tau(2^h-(h+1))} \varepsilon_0^{2^h} \\ &\leq (2C_{\mathcal{E}0} \nu^{-2} \delta_0^{-2\tau} 2^{2\tau} \varepsilon_0)^{2^h-1} \varepsilon_0. \end{aligned}$$

In Sections 6.1, 6.2, 6.3, we will prove $(p1; H+1)$, $(p2; H+1)$ and $(p3; H+1)$, assuming the induction assumption $(p1; h)$, $(p2; h)$, $(p3; h)$ for $h = 1, \dots, H$. To get such result, we need the following Lemma, which gives bounds on the quantities entering the estimates needed to prove the inductive assumption.

Lemma 20. *Assume that $(p1; h)$, $(p2; h)$, $(p3; h)$ hold for $h = 1, \dots, H$. For $H \in \mathbb{Z}_+$ the following inequality holds:*

$$\|DK_{H+1} - DK_0\|_{\rho_{H+1}} \leq D_K, \quad (6.2)$$

where (see (3.13))

$$D_K \equiv 4C_{d0} C_c \nu^{-1} \delta_0^{-\tau-1} \varepsilon_0,$$

where C_{d0} is as in (5.9) and provided that

$$2^{\tau+1} \kappa_0 \varepsilon_0 \leq \frac{1}{2} \quad (6.3)$$

with κ_0 as in (5.13). Furthermore, under the inequality:

$$\|N_0\|_{\rho_0} (2\|DK_0\|_{\rho_0} + D_K) D_K < 1, \quad (6.4)$$

the following relations hold for $0 \leq h \leq H + 1$:

$$\|N_h - N_0\|_{\rho_h} \leq C_N D_K, \quad (6.5)$$

$$\|M_h - M_0\|_{\rho_h} \leq C_M D_K, \quad (6.6)$$

$$\|M_h^{-1} - M_0^{-1}\|_{\rho_h} \leq C_{Minv} D_K, \quad (6.7)$$

where C_N, C_M, C_{Minv} are defined as follows:

$$\begin{aligned} C_N &\equiv \|N_0\|_{\rho_0}^2 \frac{2\|DK_0\|_{\rho_0} + D_K}{1 - \|N_0\|_{\rho_0} D_K (2\|DK_0\|_{\rho_0} + D_K)}, \\ C_M &\equiv 1 + J_e \left[C_N (\|DK_0\|_{\rho_0} + D_K) + \|N_0\|_{\rho_0} \right], \\ C_{Minv} &\equiv C_N (\|DK_0\|_{\rho_0} + D_K) + \|N_0\|_{\rho_0} + J_e, \end{aligned} \quad (6.8)$$

and where J_e is an upper bound on the norm of the matrix J in (2.9).

Proof. We start by proving (6.2):

$$\begin{aligned} \|DK_{H+1} - DK_0\|_{\rho_{H+1}} &\leq \sum_{j=0}^H v_j \\ &\leq \sum_{j=0}^H C_{dj} C_c \nu^{-1} \delta_j^{-\tau-1} \varepsilon_j \\ &\leq D_K \end{aligned} \quad (6.9)$$

with D_K as in (3.13) and provided that (6.3) holds.

The proof of (6.5) is obtained as follows. From the relations

$$\begin{aligned} DK_h &= DK_0 + \tilde{K}_h, \\ DK_h^\top &= DK_0^\top + \tilde{K}_h^\top, \\ \tilde{K}_h &= \sum_{j=0}^{h-1} D\Delta_j, \end{aligned} \quad (6.10)$$

we obtain

$$\begin{aligned} N_h &= (DK_h^\top DK_h)^{-1} = \left((DK_0^\top + \tilde{K}_h^\top)(DK_0 + \tilde{K}_h) \right)^{-1} \\ &= (DK_0^\top DK_0 + \tilde{K}_h^\top DK_0 + DK_0^\top \tilde{K}_h + \tilde{K}_h^\top \tilde{K}_h)^{-1} \\ &= (DK_0^\top DK_0)^{-1} \left(1 + (DK_0^\top DK_0)^{-1} (\tilde{K}_h^\top DK_0 + DK_0^\top \tilde{K}_h + \tilde{K}_h^\top \tilde{K}_h) \right)^{-1} \\ &= N_0 (1 + \chi_h)^{-1}, \end{aligned}$$

having set $\chi_h \equiv N_0(\tilde{K}_h^\top DK_0 + DK_0^\top \tilde{K}_h + \tilde{K}_h^\top \tilde{K}_h)$. Under the inequality (6.4), ensuring that $\|\chi_h\|_{\rho_h} < 1$ and using (6.9), we have the following bound:

$$\|(1 + \chi_h)^{-1} - 1\| \leq \frac{\|\chi_h\|}{1 - \|\chi_h\|},$$

which leads to

$$\begin{aligned} \|N_h - N_0\|_{\rho_h} &\leq \|N_0\|_{\rho_0} \|(1 + \chi_h)^{-1} - 1\|_{\rho_h} \\ &\leq \|N_0\|_{\rho_0} \frac{\|\chi_h\|_{\rho_h}}{1 - \|\chi_h\|_{\rho_h}} \\ &\leq C_N D_K \end{aligned} \tag{6.11}$$

with C_N as in (6.8).

The proof of (6.6) is obtained starting from the identity

$$M_h - M_0 = (DK_h - DK_0 \mid J^{-1} \circ K_h \ DK_h \ N_h - J^{-1} \circ K_0 \ DK_0 \ N_0).$$

Then, one has

$$\|M_h - M_0\|_{\rho_h} \leq \|DK_h - DK_0\|_{\rho_h} + J_e \|DK_h N_h - DK_0 N_0\|_{\rho_h}.$$

From

$$DK_h N_h - DK_0 N_0 = DK_h N_h - DK_h N_0 + DK_h N_0 - DK_0 N_0$$

and from (6.11), we obtain that

$$\begin{aligned} \|DK_h N_h - DK_0 N_0\|_{\rho_h} &\leq \|DK_h\|_{\rho_h} \|N_h - N_0\|_{\rho_h} + \|N_0\|_{\rho_0} \|DK_h - DK_0\|_{\rho_h} \\ &\leq (C_N \|DK_h\|_{\rho_h} + \|N_0\|_{\rho_0}) D_K. \end{aligned} \tag{6.12}$$

Finally, we have⁴

$$\|M_h - M_0\|_{\rho_h} \leq C_M D_K$$

with C_M as in (6.8).

The proof of (6.7) is obtained as follows. We have that the inverse of the matrix M_h can be written as⁵

$$M_h^{-1}(\theta) = \left(DK_h \ N_h^\top \mid (J \circ K_h)^\top \ DK_h \right)^\top = \begin{pmatrix} N_h DK_h^\top \\ DK_h^\top (J \circ K_h) \end{pmatrix}.$$

⁴Notice that we can bound $\|DK_h\|_{\rho_h}$ as $\|DK_h\|_{\rho_h} \leq \|DK_0\|_{\rho_0} + \|DK_h - DK_0\|_{\rho_h}$.

⁵The matrix M_h^{-1} is given by taking the transpose of the matrix obtained juxtaposing the $2n \times n$ matrices (in the generic n -dimensional case) $DK_h N_h^\top$ and $(J \circ K_h)^\top DK_h$ or, equivalently, by constructing the matrix whose first n rows are given by the $n \times 2n$ matrix $N_h DK_h^\top$ and the second n rows are given by the $n \times 2n$ matrix $DK_h^\top (J \circ K_h)$.

Indeed, one can verify that due to the Lagrangian character, $M_h^{-1}M_h = \text{Id}$. By computing the inverse of M_0 in an analogous way, one has

$$M_h^{-1} - M_0^{-1} = \begin{pmatrix} N_h DK_h^\top - N_0 DK_0^\top \\ DK_h^\top (J \circ K_h) - DK_0^\top (J \circ K_0) \end{pmatrix}. \quad (6.13)$$

The bound for the first row $N_h DK_h^\top - N_0 DK_0^\top$ is obtained as in (6.12), while the second row is bounded by $J_e DK$. This yields (6.7) with C_{Inv} as in (6.8). \square

We are now in the position to continue with the proof of $(p1; H + 1)$, $(p2; H + 1)$, $(p3; H + 1)$ to which we devote the rest of this Section.

6.1. Proof of $(p1; H + 1)$. Using the inequality $j + 1 \leq 2^j$, one has

$$\begin{aligned} \|K_{H+1} - K_0\|_{\rho_{H+1}} &\leq \sum_{j=0}^H d_j \leq \sum_{j=0}^H (C_{dj} \nu^{-1} \delta_j^{-\tau} \varepsilon_j) \\ &\leq 4C_{d0} \nu^{-1} \delta_0^{-\tau} \varepsilon_0, \end{aligned}$$

assuming that ε_0 satisfies (5.14). In conclusion, we have

$$\|K_{H+1} - K_0\|_{\rho_{H+1}} \leq \kappa_K \varepsilon_0$$

with κ_K as in (5.13). Moreover, we have:

$$\begin{aligned} |\mu_{H+1} - \mu_0| &\leq \sum_{j=0}^H s_j \leq \sum_{j=0}^H C_{\sigma_j} \varepsilon_j \\ &\leq 2C_{\sigma_0} \sum_{j=0}^H (\kappa_0 \varepsilon_0)^{2^j - 1} \varepsilon_0; \end{aligned}$$

assuming that ε_0 satisfies (5.14), we conclude that

$$|\mu_{H+1} - \mu_0| \leq 4C_{\sigma_0} \varepsilon_0 = \kappa_\mu \varepsilon_0,$$

with κ_μ as in (5.13). We take ε_0 small enough so that (5.15) and (5.16) are satisfied, which provide $(p1; H + 1)$.

6.2. Proof of $(p2; H + 1)$. Having proven $(p1; H + 1)$, we use the Taylor estimate (6.1) with $H + 1$ in place of h to obtain $(p2; H + 1)$:

$$\varepsilon_{H+1} \leq (2C_{\varepsilon_0} \nu^{-2} \delta_0^{-2\tau} 2^{2\tau} \varepsilon_0)^{2^{H+1} - 1} \varepsilon_0 = (\kappa_0 \varepsilon_0)^{2^{H+1} - 1} \varepsilon_0,$$

due to the definition of κ_0 in (5.13).

6.3. **Proof of $(p3; H + 1)$.** The proof of $(p3; H + 1)$ is rather cumbersome and needs several auxiliary results. Given the inductive assumption, we want to prove that

$$C_{d,H+1} \leq 2C_{d0} , \quad C_{\sigma,H+1} \leq 2C_{\sigma0} , \quad C_{\varepsilon,H+1} \leq 2C_{\varepsilon0} . \quad (6.14)$$

First, we estimate $|\mathcal{T}_h - \mathcal{T}_0|$ as described in Section 6.3.1.

6.3.1. *Estimate on $|\mathcal{T}_h - \mathcal{T}_0|$.* Before describing the proof of $(p3; H + 1)$ we need the following auxiliary result.

Lemma 21. *Assume that $(p1; h)$, $(p2; h)$, $(p3; h)$ hold for $h = 1, \dots, H$ and that the condition (6.4) of Lemma 20 is valid together with*

$$\begin{aligned} 4Q_{z\mu0}C_{\sigma0}\varepsilon_0 &< Q_0 , \\ 4Q_{\mu\mu0}C_{\sigma0}\varepsilon_0 &< Q_{\mu0} . \end{aligned} \quad (6.15)$$

Let $\mathcal{T}_0, \mathcal{T}_h$ be defined as

$$\mathcal{T}_0 \equiv \|\tau_0\|_{\rho_0} , \quad \mathcal{T}_h \equiv \|\tau_h\|_{\rho_h} .$$

For $h \in \mathbb{N}$, $h = 1, \dots, H$, the following inequality holds:

$$|\mathcal{T}_h - \mathcal{T}_0| \leq C_T D_K , \quad (6.16)$$

where C_T is defined as

$$C_T \equiv \frac{\mathcal{T}_0^2}{1 - \mathcal{T}_0 C_\tau} \max \left\{ C_S, C_{SB} + 2C_{Minv} Q_{\mu0} \right\} \quad (6.17)$$

with

$$\begin{aligned} C_S &\equiv 2J_e Q_0 \left\{ \left(\|N_0\|_{\rho_0} + C_N D_K \right) \left[D_K (\|N_0\|_{\rho_0} + C_N D_K) \right. \right. \\ &\quad + \|DK_0\|_{\rho_0} \|N_0\|_{\rho_0} + \|DK_0\|_{\rho_0} C_N D_K \left. \right] + C_N \|DK_0\|_{\rho_0} \left[D_K (\|N_0\|_{\rho_0} + C_N D_K) \right. \\ &\quad + \|DK_0\|_{\rho_0} \|N_0\|_{\rho_0} + \|DK_0\|_{\rho_0} C_N D_K \left. \right] + \|N_0\|_{\rho_0} \|DK_0\|_{\rho_0} (\|N_0\|_{\rho_0} + C_N D_K) \\ &\quad \left. + C_N \|N_0\|_{\rho_0} \|DK_0\|_{\rho_0}^2 \right\} , \\ C_{SB} &\equiv \frac{1}{\left| |\lambda| - 1 \right|} Q_{\mu0} \|M_0^{-1}\|_{\rho_0} C_S + 2J_e Q_0 \|N_0\|_{\rho_0}^2 \|DK_0\|_{\rho_0}^2 \frac{1}{\left| |\lambda| - 1 \right|} C_{Minv} Q_{\mu0} \\ &\quad + 2C_S \frac{1}{\left| |\lambda| - 1 \right|} C_{Minv} Q_{\mu0} D_K , \\ C_\tau &\equiv \max \left\{ C_S, C_{SB} + 2C_{Minv} Q_{\mu0} \right\} D_K . \end{aligned}$$

Proof. Let τ_0 and τ_h be defined as

$$\tau_0 \equiv \begin{pmatrix} \bar{S}_0 & \overline{S_0(B_{b0})^0} + \overline{\tilde{A}_0^{(1)}} \\ \lambda - 1 & \overline{\tilde{A}_0^{(2)}} \end{pmatrix}^{-1}$$

and

$$\tau_h \equiv \begin{pmatrix} \bar{S}_h & \overline{S_h(B_{bh})^0} + \overline{\tilde{A}_h^{(1)}} \\ \lambda - 1 & \overline{\tilde{A}_h^{(2)}} \end{pmatrix}^{-1}.$$

Then, we obtain

$$\mathcal{T}_h \leq \|\tau_0\|_{\rho_0} + \|\tilde{\tau}_h\|_{\rho_h} = \mathcal{T}_0 + \|\tilde{\tau}_h\|_{\rho_h},$$

where $\tilde{\tau}_h \equiv \tau_h - \tau_0 = \tau_0^2 \left[\left(I + \tau_0(\tau_h^{-1} - \tau_0^{-1}) \right)^{-1} (\tau_0^{-1} - \tau_h^{-1}) \right]$, so that we have the estimate

$$|\mathcal{T}_h - \mathcal{T}_0| \leq \|\tilde{\tau}_h\|_{\rho_h} \quad (6.18)$$

with

$$\|\tilde{\tau}_h\|_{\rho_h} \leq \frac{\mathcal{T}_0^2}{1 - \mathcal{T}_0 C_\tau} C_\tau, \quad (6.19)$$

where C_τ is a bound on $\tau_h^{-1} - \tau_0^{-1}$, say

$$\|\tau_h^{-1} - \tau_0^{-1}\|_{\rho_h} \equiv \left\| \begin{pmatrix} \bar{S}_h - \bar{S}_0 & \overline{S_h(B_{bh})^0} + \overline{\tilde{A}_h^{(1)}} - (\overline{S_0(B_{b0})^0} + \overline{\tilde{A}_0^{(1)}}) \\ 0 & \overline{\tilde{A}_h^{(2)}} - \overline{\tilde{A}_0^{(2)}} \end{pmatrix} \right\|_{\rho_h} \leq C_\tau. \quad (6.20)$$

To obtain an expression for C_τ , we bound term by term the matrix appearing in (6.20).

We start to estimate the first element of the matrix appearing in (6.20), namely $\|\bar{S}_h - \bar{S}_0\|_{\rho_h}$. From (3.1) we have that S_h is defined by

$$S_h = N_h(\theta + \omega)^\top DK_h(\theta + \omega)^\top Df_{\mu_h} \circ K_h(\theta) J^{-1} \circ K_h(\theta) DK_h(\theta) N_h(\theta).$$

Then, we bound $Df_{\mu_h} \circ K_h$ with

$$\begin{aligned} \sup_{z \in \mathcal{C}} |Df_{\mu_h}(z)| &\leq \sup_{z \in \mathcal{C}} |Df_{\mu_0}(z)| + \sup_{z \in \mathcal{C}} |Df_{\mu_h}(z) - Df_{\mu_0}(z)| \\ &\leq Q_0 + \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu Df_\mu(z)| |\mu_h - \mu_0| \\ &\leq Q_0 + 4Q_{z\mu_0} C_{\sigma_0} \varepsilon_0 \leq 2Q_0, \end{aligned}$$

if (5.21) holds. Notice that we have used $(p_1; H+1)$ to bound $\mu_h - \mu_0$ for $h = 1, \dots, H+1$.

Finally, we obtain

$$\begin{aligned} \|S_h - S_0\|_{\rho_h} &\leq 2Q_0 \|N_h(\theta + \omega)^\top DK_h(\theta + \omega)^\top J^{-1} \circ K_h(\theta) DK_h(\theta) N_h(\theta) \\ &\quad - N_0(\theta + \omega)^\top DK_0(\theta + \omega)^\top J^{-1} \circ K_0(\theta) DK_0(\theta) N_0(\theta)\|_{\rho_h}. \end{aligned}$$

Setting $\tilde{N}_h = N_h - N_0$ and writing DK_h as $DK_h = DK_h - DK_0 + DK_0$, one obtains

$$\begin{aligned} \|S_h - S_0\|_{\rho_h} &\leq 2Q_0 \|(N_0 + \tilde{N}_h)(\theta + \omega)^\top (DK_h - DK_0 + DK_0)(\theta + \omega)^\top \\ &\quad J^{-1} \circ K_h(\theta) (DK_h - DK_0 + DK_0)(\theta)(N_0 + \tilde{N}_h)(\theta) \\ &\quad - N_0(\theta + \omega)^\top DK_0(\theta + \omega)^\top J^{-1} \circ K_0(\theta) DK_0(\theta)N_0(\theta)\|_{\rho_h}. \end{aligned}$$

Let us bound \tilde{N}_h using (6.5). Then, using that J is a constant matrix, we have:

$$\begin{aligned} \|S_h - S_0\|_{\rho_h} &\leq 2Q_0 \|[(N_0 + \tilde{N}_h) \circ T_\omega]^\top ((DK_h - DK_0) \circ T_\omega)^\top \\ &\quad + (N_0 \circ T_\omega)^\top (DK_0 \circ T_\omega)^\top + (\tilde{N}_h \circ T_\omega)^\top (DK_0 \circ T_\omega)^\top] J^{-1} \circ K_h(\theta) \\ &\quad [(DK_h - DK_0)(N_0 + \tilde{N}_h) + DK_0N_0 + DK_0\tilde{N}_h] \\ &\quad - (N_0 \circ T_\omega)^\top (DK_0 \circ T_\omega)^\top J^{-1} \circ K_0(\theta) DK_0N_0\|_{\rho_h} \\ &= 2Q_0 \|((N_0 + \tilde{N}_h) \circ T_\omega)^\top ((DK_h - DK_0) \circ T_\omega)^\top \\ &\quad J^{-1} \circ K_h(\theta) [(DK_h - DK_0)(N_0 + \tilde{N}_h) + DK_0N_0 + DK_0\tilde{N}_h] \\ &\quad + (\tilde{N}_h \circ T_\omega)^\top (DK_0 \circ T_\omega)^\top \\ &\quad J^{-1} \circ K_h(\theta) [(DK_h - DK_0)(N_0 + \tilde{N}_h) + DK_0N_0 + DK_0\tilde{N}_h] \\ &\quad + (N_0 \circ T_\omega)^\top (DK_0 \circ T_\omega)^\top J^{-1} \circ K_h(\theta) (DK_h - DK_0)(N_0 + \tilde{N}_h) \\ &\quad + (N_0 \circ T_\omega)^\top (DK_0 \circ T_\omega)^\top J^{-1} \circ K_h(\theta) DK_0\tilde{N}_h\|_{\rho_h} \\ &\leq 2J_e Q_0 \left\{ (\|N_0\|_{\rho_0} + \|\tilde{N}_h\|_{\rho_h}) \|DK_h - DK_0\|_{\rho_h} \right. \\ &\quad \left[\|DK_h - DK_0\|_{\rho_h} (\|N_0\|_{\rho_0} + \|\tilde{N}_h\|_{\rho_h}) + \|DK_0\|_{\rho_0} \|N_0\|_{\rho_0} + \|DK_0\|_{\rho_0} \|\tilde{N}_h\|_{\rho_h} \right] \\ &\quad + \|\tilde{N}_h\|_{\rho_h} \|DK_0\|_{\rho_0} \left[\|DK_h - DK_0\|_{\rho_h} (\|N_0\|_{\rho_0} + \|\tilde{N}_h\|_{\rho_h}) \right. \\ &\quad \left. + \|DK_0\|_{\rho_0} \|N_0\|_{\rho_0} + \|DK_0\|_{\rho_0} \|\tilde{N}_h\|_{\rho_h} \right] \\ &\quad + \|N_0\|_{\rho_0} \|DK_0\|_{\rho_0} \|DK_h - DK_0\|_{\rho_h} (\|N_0\|_{\rho_0} + \|\tilde{N}_h\|_{\rho_h}) \\ &\quad \left. + \|N_0\|_{\rho_0} \|DK_0\|_{\rho_0}^2 \|\tilde{N}_h\|_{\rho_h} \right\}. \end{aligned}$$

Taking into account (6.5), (6.8), we obtain:

$$\|S_h - S_0\|_{\rho_h} \leq C_S D_K \tag{6.21}$$

with

$$\begin{aligned}
C_S &\equiv 2J_e Q_0 \left\{ (\|N_0\|_{\rho_0} + C_N D_K) \left[D_K (\|N_0\|_{\rho_0} + C_N D_K) \right. \right. \\
&\quad \left. \left. + \|DK_0\|_{\rho_0} \|N_0\|_{\rho_0} + \|DK_0\|_{\rho_0} C_N D_K \right] \right. \\
&\quad \left. + C_N \|DK_0\|_{\rho_0} \left[D_K (\|N_0\|_{\rho_0} + C_N D_K) + \|DK_0\|_{\rho_0} \|N_0\|_{\rho_0} + \|DK_0\|_{\rho_0} C_N D_K \right] \right. \\
&\quad \left. + \|N_0\|_{\rho_0} \|DK_0\|_{\rho_0} (\|N_0\|_{\rho_0} + C_N D_K) \right. \\
&\quad \left. + C_N \|N_0\|_{\rho_0} \|DK_0\|_{\rho_0}^2 \right\}. \tag{6.22}
\end{aligned}$$

Now we bound the upper right element of the matrix appearing in (6.20). This computation will give us also a bound on the lower right element of the matrix in (6.20). We start from (see (4.6))

$$\begin{aligned}
\tilde{A}_h &= M_h^{-1} \circ T_\omega D_\mu f_{\mu_h} \circ K_h, \\
\tilde{A}_0 &= M_0^{-1} \circ T_\omega D_\mu f_{\mu_0} \circ K_0
\end{aligned}$$

and the estimate:

$$\sup_{z \in \mathcal{C}} |D_\mu f_{\mu_h}(z)| \leq Q_{\mu_0} + 4Q_{\mu\mu_0} C_{\sigma_0} \varepsilon_0 \leq 2Q_{\mu_0},$$

provided (5.22) holds. Then, we have:

$$\begin{aligned}
\|\tilde{A}_h - \tilde{A}_0\|_{\rho_h} &\leq 2Q_{\mu_0} \|M_h^{-1} - M_0^{-1}\|_{\rho_h} \\
&\leq 2C_{Minv} Q_{\mu_0} D_K.
\end{aligned}$$

Next we estimate $\|\overline{S_h(B_{bh})^0} - \overline{S_0(B_{b0})^0}\|_{\rho_h}$; recall that from (4.8) we have that $(B_{bh})^0$ is the solution of

$$\lambda(B_{bh})^0 - (B_{bh})^0 \circ T_\omega = -(\tilde{A}_h^{(2)})^0, \tag{6.23}$$

while $(B_{b0})^0$ is the solution of

$$\lambda(B_{b0})^0 - (B_{b0})^0 \circ T_\omega = -(\tilde{A}_0^{(2)})^0. \tag{6.24}$$

Expanding (6.23) and (6.24) in Fourier series, we obtain

$$\sum_{j \in \mathbb{Z}} (\hat{B}_{bh})_j^0 (\lambda - e^{2\pi i j \omega}) e^{2\pi i j \theta} = - \sum_{j \in \mathbb{Z}} (\hat{A}_h^{(2)})_j^0 e^{2\pi i j \theta},$$

so that

$$(B_{bh})^0(\theta) = - \sum_{j \in \mathbb{Z}} \frac{(\hat{A}_h^{(2)})_j^0}{\lambda - e^{2\pi i j \omega}} e^{2\pi i j \theta},$$

and similarly

$$(B_{b0})^0(\theta) = - \sum_{j \in \mathbb{Z}} \frac{(\widehat{A}_0^{(2)})_j^0}{\lambda - e^{2\pi i j \theta}} e^{2\pi i j \theta} .$$

In conclusion we have:

$$(B_{bh})^0(\theta) - (B_{b0})^0(\theta) = - \sum_{j \in \mathbb{Z}} \frac{(\widehat{A}_h^{(2)})_j^0 - (\widehat{A}_0^{(2)})_j^0}{\lambda - e^{2\pi i j \omega}} e^{2\pi i j \theta} . \quad (6.25)$$

From (6.25), let us write $(B_{bh})^0$ as

$$(B_{bh})^0 = (B_{b0})^0 + \widetilde{B}_h ,$$

where

$$\widetilde{B}_h \equiv - \sum_{j \in \mathbb{Z}} \frac{(\widehat{A}_h^{(2)})_j^0 - (\widehat{A}_0^{(2)})_j^0}{\lambda - e^{2\pi i j \omega}} e^{2\pi i j \theta} .$$

Let us introduce

$$\widetilde{S}_h \equiv S_h - S_0 ,$$

whose norm can be bounded by (6.21). Then, we have:

$$\begin{aligned} \|\overline{S_h(B_{bh})^0} - \overline{S_0(B_{b0})^0}\| &= \|\overline{(S_0 + \widetilde{S}_h) ((B_{b0})^0 + \widetilde{B}_h) - S_0(B_{b0})^0}\| \\ &= \|\overline{S_0(B_{b0})^0} + \overline{\widetilde{S}_h(B_{b0})^0} + \overline{S_0 \widetilde{B}_h} + \overline{\widetilde{S}_h \widetilde{B}_h} - \overline{S_0(B_{b0})^0}\| \\ &\leq \|(B_{b0})^0\|_{\rho_0} \|\widetilde{S}_h\|_{\rho_h} + \|S_0\|_{\rho_0} \|\widetilde{B}_h\|_{\rho_h} + \|\widetilde{S}_h\|_{\rho_h} \|\widetilde{B}_h\|_{\rho_h} , \end{aligned}$$

where

$$\begin{aligned} \|S_0\|_{\rho_0} &\leq J_e Q_0 \|N_0\|_{\rho_0}^2 \|DK_0\|_{\rho_0}^2 , \\ \|\widetilde{S}_h\|_{\rho_h} &\leq C_S D_K , \\ \|(B_{b0})^0\|_{\rho_0} &\leq \frac{1}{\|\lambda - 1\|} \|\widetilde{A}_0^{(2)}\|_{\rho_0} \leq \frac{1}{\|\lambda - 1\|} Q_{\mu 0} \|M_0^{-1}\|_{\rho_0} , \\ \|\widetilde{B}_h\|_{\rho_h} &\leq \frac{1}{\|\lambda - 1\|} 2C_{Minv} Q_{\mu 0} D_K . \end{aligned}$$

Then, we have:

$$\|\overline{S_h(B_{bh})^0} - \overline{S_0(B_{b0})^0}\| \leq C_{SB} D_K ,$$

where

$$\begin{aligned} C_{SB} &\equiv \frac{1}{\|\lambda - 1\|} Q_{\mu 0} \|M_0^{-1}\|_{\rho_0} C_S \\ &+ 2J_e Q_0 \|N_0\|_{\rho_0}^2 \|DK_0\|_{\rho_0}^2 \frac{1}{\|\lambda - 1\|} C_{Minv} Q_{\mu 0} \\ &+ 2C_S \frac{1}{\|\lambda - 1\|} C_{Minv} Q_{\mu 0} D_K . \end{aligned} \quad (6.26)$$

Recalling (6.20), we obtain

$$\begin{aligned} \|\tau_h^{-1} - \tau_0^{-1}\|_{\rho_h} &\leq \max \left\{ \|\overline{S}_h - \overline{S}_0\|_{\rho_h}, \|\overline{S}_h(B_{bh})^0 - \overline{S}_0(B_{b0})^0\|_{\rho_h} + \|\overline{\widetilde{A}}_h^{(1)} - \overline{\widetilde{A}}_0^{(1)}\|_{\rho_h} + \|\overline{\widetilde{A}}_h^{(2)} - \overline{\widetilde{A}}_0^{(2)}\|_{\rho_h} \right\} \\ &\leq \max \left\{ C_S, C_{SB} + 2C_{Minv} Q_{\mu 0} \right\} D_K \equiv C_\tau, \end{aligned} \quad (6.27)$$

where C_τ is defined by the last inequality in (6.27). From (6.18) and (6.19) we get (6.16):

$$|\mathcal{T}_h - \mathcal{T}_0| \leq C_T D_K$$

with

$$C_T \equiv \frac{\mathcal{T}_0^2}{1 - \mathcal{T}_0 C_\tau} \max \left\{ C_S, C_{SB} + 2C_{Minv} Q_{\mu 0} \right\}. \quad (6.28)$$

□

6.3.2. *Proof of $C_{\sigma, H+1} \leq 2C_{\sigma 0}$.* We now prove (6.14) and we begin from the second inequality. We start with the following relations, which are a consequence of (5.10):

$$\begin{aligned} C_{\sigma, H+1} &= \mathcal{T}_{H+1} \left[|\lambda - 1| \left(\frac{1}{|\lambda| - 1} \|S_{H+1}\|_{\rho_{H+1}} + 1 \right) + \|S_{H+1}\|_{\rho_{H+1}} \right] \|M_{H+1}^{-1}\|_{\rho_{H+1}}, \\ C_{\sigma 0} &= \mathcal{T}_0 \left[|\lambda - 1| \left(\frac{1}{|\lambda| - 1} \|S_0\|_{\rho_0} + 1 \right) + \|S_0\|_{\rho_0} \right] \|M_0^{-1}\|_{\rho_0} \end{aligned}$$

with

$$\begin{aligned} \|M_{H+1}^{-1}\|_{\rho_{H+1}} &\leq \|M_0^{-1}\|_{\rho_0} + \|M_{H+1}^{-1} - M_0^{-1}\|_{\rho_{H+1}} \\ &\leq \|M_0^{-1}\|_{\rho_0} + C_{Minv} D_K \end{aligned}$$

with C_{Minv} as in (6.8). We also have

$$\begin{aligned} \|S_{H+1}\|_{\rho_{H+1}} &\leq \|S_0\|_{\rho_0} + \|S_{H+1} - S_0\|_{\rho_{H+1}} \\ &\leq \|S_0\|_{\rho_0} + C_S D_K \end{aligned}$$

with C_S as in (6.22). From the relation

$$\mathcal{T}_{H+1} = \mathcal{T}_0 + (\mathcal{T}_{H+1} - \mathcal{T}_0) \leq \mathcal{T}_0 + C_T D_K$$

with C_T as in (6.28), we obtain:

$$\begin{aligned} C_{\sigma, H+1} &\leq (\mathcal{T}_0 + C_T D_K) \left\{ |\lambda - 1| \left[\frac{1}{|\lambda| - 1} (\|S_0\|_{\rho_0} + C_S D_K) + 1 \right] \right. \\ &\quad \left. + (\|S_0\|_{\rho_0} + C_S D_K) \right\} \left(\|M_0^{-1}\|_{\rho_0} + C_{Minv} D_K \right) \\ &= \mathcal{T}_0 \left[|\lambda - 1| \left(\frac{1}{|\lambda| - 1} \|S_0\|_{\rho_0} + 1 \right) \|M_0^{-1}\|_{\rho_0} + \|S_0\|_{\rho_0} \|M_0^{-1}\|_{\rho_0} \right] + C_\sigma D_K \\ &= C_{\sigma 0} + C_\sigma D_K \\ &\leq 2C_{\sigma 0}, \end{aligned}$$

if

$$\begin{aligned}
C_\sigma &\equiv C_T \left\{ |\lambda - 1| \left[\frac{1}{||\lambda| - 1|} (\|S_0\|_{\rho_0} + C_S D_K) + 1 \right] \right. \\
&\quad \left. + \left(\|S_0\|_{\rho_0} + C_S D_K \right) \right\} \left(\|M_0^{-1}\|_{\rho_0} + C_{Minv} D_K \right) \\
&+ \mathcal{T}_0 \left\{ |\lambda - 1| \left[\frac{1}{||\lambda| - 1|} (\|S_0\|_{\rho_0} + C_S D_K) + 1 \right] C_{Minv} \right. \\
&\quad \left. + |\lambda - 1| \frac{1}{||\lambda| - 1|} \|M_0^{-1}\|_{\rho_0} C_S \right. \\
&\quad \left. + C_S \left(\|M_0^{-1}\|_{\rho_0} + C_{Minv} D_K \right) + C_{Minv} \|S_0\|_{\rho_0} \right\},
\end{aligned} \tag{6.29}$$

and if we require (5.18).

6.3.3. *Proof of $C_{\mathcal{E},H+1} \leq 2C_{\mathcal{E}0}$.* Recall that $\delta_{H+1} = \frac{\delta_0}{2^{H+1}}$ and that from (5.11), (5.12), one has

$$C_{\mathcal{E},H+1} \equiv C_c C_{W,H+1} \nu \delta_{H+1}^{-1+\tau} + C_{\mathcal{R},H+1}.$$

First, it suffices to prove that

$$C_{W,H+1} \leq C_{W0} + C_W D_K, \tag{6.30}$$

for a suitable constant C_W which will be given in (6.34) below.

From (5.10), for $C_{W_2,H+1}$ we have:

$$\begin{aligned}
C_{W_2,H+1} &\leq \frac{1}{||\lambda| - 1|} \left[1 + 2Q_{\mu 0} (C_{\sigma 0} + D_K C_\sigma) \right] (\|M_0^{-1}\|_{\rho_0} + D_K) \\
&\leq C_{W_2 0} + D_K C_{W_2},
\end{aligned}$$

where

$$C_{W_2} \equiv \frac{1}{||\lambda| - 1|} \left[1 + 2Q_{\mu 0} \|M_0^{-1}\|_{\rho_0} C_\sigma + 2Q_{\mu 0} C_{\sigma 0} + 2Q_{\mu 0} C_\sigma D_K \right]. \tag{6.31}$$

Concerning $\bar{C}_{W_2,H+1}$, we have:

$$\begin{aligned}
\bar{C}_{W_2,H+1} &\leq 4(\mathcal{T}_0 + C_T D_K) \left[\frac{1}{||\lambda| - 1|} (\|S_0\|_{\rho_0} + C_S D_K) + 1 \right] Q_{\mu 0} (\|M_0^{-1}\|_{\rho_0} + D_K)^2 \\
&= \bar{C}_{W_2 0} + \bar{C}_{W_2} D_K
\end{aligned}$$

with

$$\begin{aligned}
\bar{C}_{W_2} &\equiv 4C_T \left[\frac{1}{|\lambda| - 1} (\|S_0\|_{\rho_0} + C_S D_K) + 1 \right] Q_{\mu_0} (\|M_0^{-1}\|_{\rho_0} + D_K)^2 \\
&+ 4\mathcal{T}_0 Q_{\mu_0} \frac{1}{|\lambda| - 1} C_S (\|M_0^{-1}\|_{\rho_0} + D_K)^2 \\
&+ 4\mathcal{T}_0 Q_{\mu_0} \left[\frac{1}{|\lambda| - 1} (\|S_0\|_{\rho_0} + C_S D_K) + 1 \right] (D_K + 2\|M_0^{-1}\|_{\rho_0}) .
\end{aligned} \tag{6.32}$$

As for $C_{W_1, H+1}$ we have:

$$\begin{aligned}
C_{W_1, H+1} &\leq C_0 \left[(\|S_0\|_{\rho_0} + C_S D_K) (C_{W_{20}} + C_{W_2} D_K + \bar{C}_{W_{20}} + \bar{C}_{W_2} D_K) \right. \\
&+ \left. \|M_0^{-1}\|_{\rho_0} + D_K + 2Q_{\mu_0} (\|M_0^{-1}\|_{\rho_0} + D_K) (C_{\sigma_0} + D_K C_\sigma) \right] \\
&= C_{W_{10}} + D_K C_{W_1} ,
\end{aligned}$$

where

$$\begin{aligned}
C_{W_1} &\equiv C_0 \left[\|S_0\|_{\rho_0} C_{W_2} + C_S C_{W_{20}} + C_S C_{W_2} D_K + \|S_0\|_{\rho_0} \bar{C}_{W_2} + C_S \bar{C}_{W_{20}} + C_S \bar{C}_{W_2} D_K + 1 \right. \\
&+ \left. 2Q_{\mu_0} \|M_0^{-1}\|_{\rho_0} C_\sigma + 2Q_{\mu_0} C_{\sigma_0} + 2Q_{\mu_0} C_\sigma D_K \right] .
\end{aligned} \tag{6.33}$$

In conclusion, from (5.10) we have:

$$\begin{aligned}
C_{W, H+1} &\equiv (C_{W_{10}} + D_K C_{W_1}) + (C_{W_{20}} + D_K C_{W_2} + \bar{C}_{W_{20}} + D_K \bar{C}_{W_2}) \nu \delta_0^\tau 2^{-\tau(H+1)} \\
&\leq C_{W_0} + C_W D_K
\end{aligned}$$

with

$$C_W \equiv C_{W_1} + C_{W_2} \nu \delta_0^\tau + \bar{C}_{W_2} \nu \delta_0^\tau . \tag{6.34}$$

In order to get $C_{\mathcal{E}, H+1}$ as in (5.11) we estimate $C_{\mathcal{R}, H+1}$. To this end, we use the following inequality:

$$Q_{E, H+1} \leq Q_{E_0} + C_Q D_{2K} , \tag{6.35}$$

for a suitable constant C_Q that will be given later in (6.42) and for D_{2K} defined as

$$D_{2K} \equiv 4C_{d_0} C_c^2 \nu^{-1} \delta_0^{-\tau-2} \varepsilon_0 . \tag{6.36}$$

We postpone for a moment the proof of (6.35) and we rather stress that, as a consequence of (6.35), we obtain:

$$\begin{aligned}
C_{\mathcal{R},H+1} &\leq Q_{E,H+1} (\|M_{H+1}\|_{\rho_{H+1}}^2 C_{W,H+1}^2 + C_{\sigma,H+1}^2 \nu^2 \delta_{H+1}^{2\tau}) \\
&\leq (Q_{E0} + C_Q D_{2K}) \left[(\|M_0\|_{\rho_0} + C_M D_K)^2 (C_{W0} + C_W D_K)^2 \right. \\
&\quad \left. + (C_{\sigma_0} + C_\sigma D_K)^2 \nu^2 \delta_0^{2\tau} 2^{-2\tau(H+1)} \right] \\
&\leq C_{\mathcal{R}0} + C_{\mathcal{R}} D_K ,
\end{aligned}$$

where

$$\begin{aligned}
C_{\mathcal{R}} &\equiv Q_{E0} \left[(2C_M \|M_0\|_{\rho_0} + C_M^2 D_K) (C_{W0} + C_W D_K)^2 + \|M_0\|_{\rho_0}^2 (C_W^2 D_K + 2C_{W0} C_W) \right. \\
&\quad \left. + (C_\sigma^2 D_K + 2C_{\sigma_0} C_\sigma) \nu^2 \delta_0^{2\tau} \right] + C_Q \left[(\|M_0\|_{\rho_0} + C_M D_K)^2 (C_{W0} + C_W D_K)^2 \right. \\
&\quad \left. + (C_{\sigma_0} + C_\sigma D_K)^2 \nu^2 \delta_0^{2\tau} \right] C_c \delta_0^{-1} \tag{6.37}
\end{aligned}$$

with D_{2K} is as in (6.36).

We obtain that

$$\begin{aligned}
C_{\mathcal{E},H+1} &\leq (C_{W0} + C_W D_K) C_c \nu \delta_0^{-1+\tau} 2^{-(1+\tau)(H+1)} + C_{\mathcal{R}0} + C_{\mathcal{R}} D_K \\
&\leq C_{W0} C_c \nu \delta_0^{-1+\tau} + C_{\mathcal{R}0} + D_K \left(C_W C_c \nu \delta_0^{-1+\tau} + C_{\mathcal{R}} \right) \\
&\leq 2C_{\mathcal{E}0} ,
\end{aligned}$$

if (5.19) is satisfied.

Let us conclude by proving (6.35) starting from the definition

$$Q_{E,H+1} \equiv \frac{1}{2} \max \left\{ \|D^2 E_{H+1}\|_{\rho_{H+1}-\delta_{H+1}}, \|DD_\mu E_{H+1}\|_{\rho_{H+1}-\delta_{H+1}}, \|D_\mu^2 E_{H+1}\|_{\rho_{H+1}-\delta_{H+1}} \right\} .$$

We recall that

$$E_{H+1} = \mathcal{E}[K_{H+1}, \mu_{H+1}] = f_{\mu_{H+1}} \circ K_{H+1} - K_{H+1} \circ T_\omega .$$

It is convenient to introduce Δ_H and Ξ_H such that

$$K_{H+1} = K_0 + (K_{H+1} - K_0) \equiv K_0 + \Delta_H , \quad \mu_{H+1} = \mu_0 + \sum_{j=0}^H \sigma_j \equiv \mu_0 + \Xi_H .$$

Then, we have the following bound on E_{H+1} :

$$\begin{aligned}
\|E_{H+1}\|_{\rho_{H+1}-\delta_{H+1}} &= \|(f_{\mu_0} \circ K_0 - K_0 \circ T_\omega) + f_{\mu_{H+1}} \circ K_{H+1} - f_{\mu_0} \circ K_0 \\
&\quad - (K_{H+1} - K_0) \circ T_\omega\|_{\rho_{H+1}-\delta_{H+1}} \\
&\leq \|E_0\|_{\rho_0} + (1 + \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |Df_{\mu_0}(z)|) \|K_{H+1} - K_0\|_{\rho_{H+1}} \\
&\quad + \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu f_{\mu_0}(z)| |\mu_{H+1} - \mu_0| \\
&\leq \|E_0\|_{\rho_0} + (1 + \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |Df_\mu(z)|) \kappa_K \varepsilon_0 \\
&\quad + \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu f_\mu(z)| \kappa_\mu \varepsilon_0 ,
\end{aligned}$$

where we used (5.23).

We now observe that the derivative of $f \circ K$ is given by

$$D(f \circ K) = D(f(K(\theta))) = Df(K(\theta)) DK(\theta) \quad (6.38)$$

and that the second derivative is given by

$$D^2(f \circ K) = D^2(f(K(\theta)))(DK(\theta))^2 + Df(K(\theta)) D^2K(\theta) .$$

Then, one has

$$\begin{aligned}
\|D^2 E_{H+1}\|_{\rho_{H+1}} &\leq \|D^2 E_0\|_{\rho_0} + D_{2K} \\
&+ \|D^2 f_{\mu_0 + \Xi_H}(K_0 + \Delta_H) (DK_0 + D\Delta_H) \\
&- D^2 f_{\mu_0}(K_0) DK_0\|_{\rho_{H+1}} \|DK_0\|_{\rho_0} \\
&+ \|Df_{\mu_0 + \Xi_H}(K_0 + \Delta_H) - Df_{\mu_0}(K_0)\|_{\rho_{H+1}} \|D^2 K_0\|_{\rho_0} \\
&+ \|D^2 f_{\mu_0 + \Xi_H}(K_0 + \Delta_H) (DK_0 + D\Delta_H) D\Delta_H\|_{\rho_{H+1}} \\
&+ \|Df_{\mu_0 + \Xi_H}(K_0 + \Delta_H) D^2 \Delta_H\|_{\rho_{H+1}} \\
&\leq \|D^2 E_0\|_{\rho_0} + D_{2K} + \sup_{z \in \mathcal{C}} |D^3 f_{\mu_0}(z)| \|DK_0\|_{\rho_0}^2 \kappa_K \varepsilon_0 \\
&+ \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu D^2 f_\mu(z)| \|DK_0\|_{\rho_0}^2 \kappa_\mu \varepsilon_0 \\
&+ \sup_{z \in \mathcal{C}} |D^2 f_{\mu_0}(z)| \|DK_0\|_{\rho_0} D_K \\
&+ \sup_{z \in \mathcal{C}} |D^3 f_{\mu_0}(z)| \|DK_0\|_{\rho_0} \kappa_K \varepsilon_0 D_K \\
&+ \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu D^2 f_\mu(z)| \|DK_0\|_{\rho_0} \kappa_\mu \varepsilon_0 D_K \\
&+ \sup_{z \in \mathcal{C}} |D^2 f_{\mu_0}(z)| \|D^2 K_0\|_{\rho_0}^2 \kappa_K \varepsilon_0 \\
&+ \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu Df_\mu(z)| \|D^2 K_0\|_{\rho_0} \kappa_\mu \varepsilon_0 \\
&+ \sup_{z \in \mathcal{C}} |D^2 f_{\mu_0}(z)| (\|DK_0\|_{\rho_0} + D_K) D_K \\
&+ \sup_{z \in \mathcal{C}} |D^3 f_{\mu_0}(z)| (\|DK_0\|_{\rho_0} + D_K) D_K \kappa_K \varepsilon_0 \\
&+ \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu D^2 f_\mu(z)| (\|DK_0\|_{\rho_0} + D_K) D_K \kappa_\mu \varepsilon_0 \\
&+ \sup_{z \in \mathcal{C}} |Df_{\mu_0}(z)| D_{2K} + \sup_{z \in \mathcal{C}} |D^2 f_{\mu_0}(z)| \kappa_K \varepsilon_0 D_{2K} \\
&+ \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu Df_\mu(z)| \kappa_\mu \varepsilon_0 D_{2K} , \tag{6.39}
\end{aligned}$$

with $\|DK_H\|_{\rho_H} \leq \|DK_0\|_{\rho_0} + D_K$, where D_K was estimated as in (3.13), $\|D^2 K_H\|_{\rho_H} \leq \|D^2 K_0\|_{\rho_0} + D_{2K}$, where D_{2K} is defined through the following inequalities:

$$\begin{aligned}
\|D^2 K_{H+1} - D^2 K_0\|_{\rho_{H+1}} &\leq \sum_{j=1}^H \|D^2 \Delta_j\|_{\rho_j} \leq C_c \sum_{j=1}^H \delta_j^{-1} \nu_j \\
&\leq C_c^2 \sum_{j=1}^H C_{dj} \nu^{-1} \delta_j^{-\tau-2} \varepsilon_j \leq 4C_c^2 C_{d0} \nu^{-1} \delta_0^{-\tau-2} \varepsilon_0 = D_{2K} ,
\end{aligned}$$

if (5.14) holds.

In a similar way we obtain the following estimate. Given $f \circ K$, from (6.38) we have

$$DD_\mu(f \circ K) = DD_\mu(f(K(\theta)))DK(\theta) .$$

Then, we have

$$DD_\mu E_{H+1} = DD_\mu E_0 + DD_\mu f_{\mu_0}(K_0) (DK_{H+1} - DK_0) + DD_\mu (f_{\mu_{H+1}}(K_0 + \Delta_H) - f_{\mu_0}(K_0)) DK_{H+1} ,$$

so that

$$\begin{aligned} \|DD_\mu E_{H+1}\|_{\rho_{H+1}} &\leq \|DD_\mu E_0\|_{\rho_0} + \sup_{z \in \mathcal{C}} |DD_\mu f_{\mu_0}(z)| D_K \\ &+ \sup_{z \in \mathcal{C}} |D^2 D_\mu f_{\mu_0}(z)| \|\Delta_H\|_{\rho_{H+1}} (\|DK_0\|_{\rho_0} + \|D\Delta_H\|_{\rho_{H+1}}) \\ &+ \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |DD_\mu^2 f_{\mu_0}(z)| \|\Xi_H\|_{\rho_{H+1}} (\|DK_0\|_{\rho_0} + \|D\Delta_H\|_{\rho_{H+1}}) \\ &\leq \|DD_\mu E_0\|_{\rho_0} + \sup_{z \in \mathcal{C}} |DD_\mu f_{\mu_0}(z)| D_K \\ &+ \sup_{z \in \mathcal{C}} |D^2 D_\mu f_{\mu_0}(z)| \kappa_K \varepsilon_0 (\|DK_0\|_{\rho_0} + D_K) \\ &+ \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |DD_\mu^2 f_{\mu_0}(z)| \kappa_\mu \varepsilon_0 (\|DK_0\|_{\rho_0} + D_K) . \end{aligned} \quad (6.40)$$

Finally, we have:

$$\|D_\mu^2 E_{H+1}\|_{\rho_{H+1}} \leq \|D_\mu^2 E_0\|_{\rho_0} + \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu^3 f_\mu(z)| \kappa_\mu \varepsilon_0 . \quad (6.41)$$

Casting together (6.39), (6.40), (6.41), we obtain:

$$Q_{E, H+1} \leq Q_{E_0} + C_Q D_{2K}$$

with

$$\begin{aligned}
C_Q &\equiv \frac{1}{2} \max \left\{ 1 + \sup_{z \in \mathcal{C}} |D^3 f_{\mu_0}(z)| \|DK_0\|_{\rho_0}^2 C_c^{-2} \delta_0^2 \right. \\
&+ \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu D^2 f_\mu(z)| \|DK_0\|_{\rho_0}^2 \frac{C_{\sigma_0}}{C_{d_0}} C_c^{-2} \delta_0^{\tau+2} \\
&+ \sup_{z \in \mathcal{C}} |D^2 f_{\mu_0}(z)| \|DK_0\|_{\rho_0} C_c^{-1} \delta_0 \\
&+ \sup_{z \in \mathcal{C}} |D^3 f_{\mu_0}(z)| \|DK_0\|_{\rho_0} 4C_{d_0} C_c^{-1} \nu^{-1} \delta_0^{-\tau+1} \varepsilon_0 \\
&+ \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu D^2 f_\mu(z)| \|DK_0\|_{\rho_0} 4C_{\sigma_0} C_c^{-1} \delta_0 \varepsilon_0 \\
&+ \sup_{z \in \mathcal{C}} |D^2 f_{\mu_0}(z)| \|D^2 K_0\|_{\rho_0}^2 C_c^{-2} \delta_0^2 \\
&+ \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu D f_\mu(z)| \|D^2 K_0\|_{\rho_0} \frac{C_{\sigma_0}}{C_{d_0}} C_c^{-2} \nu \delta_0^{\tau+2} \\
&+ \sup_{z \in \mathcal{C}} |D^2 f_{\mu_0}(z)| (\|DK_0\|_{\rho_0} + D_K) C_c^{-1} \delta_0 \\
&+ \sup_{z \in \mathcal{C}} |D^3 f_{\mu_0}(z)| (\|DK_0\|_{\rho_0} + D_K) 4C_{d_0} C_c^{-1} \nu^{-1} \delta_0^{-\tau+1} \varepsilon_0 \\
&+ \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu D^2 f_\mu(z)| (\|DK_0\|_{\rho_0} + D_K) 4C_{\sigma_0} C_c^{-1} \delta_0 \varepsilon_0 \\
&+ \sup_{z \in \mathcal{C}} |D f_{\mu_0}(z)| + \sup_{z \in \mathcal{C}} |D^2 f_{\mu_0}(z)| \kappa_K \varepsilon_0 \\
&+ \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu D f_\mu(z)| \kappa_\mu \varepsilon_0 , \\
&\sup_{z \in \mathcal{C}} |DD_\mu f_{\mu_0}(z)| C_c^{-1} \delta_0 + \sup_{z \in \mathcal{C}} |D^2 D_\mu f_{\mu_0}(z)| C_c^{-2} \delta_0^2 (\|DK_0\|_{\rho_0} + D_K) \\
&+ \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |DD_\mu^2 f_\mu(z)| \frac{C_{\sigma_0}}{C_{d_0}} C_c^{-2} \nu \delta_0^{\tau+2} (\|DK_0\|_{\rho_0} + D_K), \\
&\left. \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu^3 f_\mu(z)| \frac{C_{\sigma_0}}{C_{d_0}} C_c^{-2} \nu \delta_0^{\tau+2} \right\} . \tag{6.42}
\end{aligned}$$

6.3.4. *Proof of $C_{d,H+1} \leq 2C_{d_0}$.* From (5.9), (6.30) we have:

$$\begin{aligned}
C_{d,H+1} &= \|M_{H+1}\|_{\rho_{H+1}} C_{W,H+1} \leq (\|M_0\|_{\rho_0} + D_K)(C_{W_0} + C_W D_K) \\
&\leq C_{d_0} + D_K (C_{W_0} + \|M_0\|_{\rho_0} C_W + C_W D_K) \\
&\leq 2C_{d_0} ,
\end{aligned}$$

if (5.20) is satisfied.

7. KAM ESTIMATES FOR THE STANDARD MAP

In this Section we implement Theorem 10 to obtain explicit estimates on the numerical validation of the golden mean curve of the dissipative standard map (2.8) that are close to the numerical breakdown value. As mentioned in Section 4, we need to start with an approximate solution (K_0, μ_0) , which satisfies the invariance equation (2.10) with an error term E_0 , whose norm on a domain of radius $\rho_0 > 0$ was denoted as ε_0 in Theorem 10.

The construction of the approximate solution (K_0, μ_0) can be obtained by implementing the algorithm described in [12] and reviewed in Section 7.1 below. An estimate on the quantity ε_0 is obtained by imposing the list of conditions (3.3)-(3.12); explicit bounds are given in Section 7.2, using the definitions of the constants provided in Appendix B.

7.1. Construction of the approximate solution. To construct an approximate solution (K_0, μ_0) of the invariance equation (2.10), we make use of the fact that the a-posteriori format described in [12] provides an explicit algorithm, which can be implemented numerically in a very efficient way. Each step of the algorithm is denoted as follows: “ $a \leftarrow b$ ” means that the quantity a is assigned by the quantity b .

Algorithm 22. *Given $K_0 : \mathbb{T} \rightarrow \mathcal{M}$, $\mu_0 \in \mathbb{R}$, we denote by $\lambda \in \mathbb{R}$ the conformal factor for f_{μ_0} . We perform the following computations:*

- 1) $E_0 \leftarrow f_{\mu_0} \circ K_0 - K_0 \circ T_\omega$
- 2) $\alpha \leftarrow DK_0$
- 3) $N_0 \leftarrow [\alpha^\top \alpha]^{-1}$
- 4) $M_0 \leftarrow [\alpha | J^{-1} \circ K_0 \alpha N_0]$
- 5) $\beta \leftarrow M_0^{-1} \circ T_\omega$
- 6) $\tilde{E}_0 \leftarrow \beta E_0$
- 7) $P_0 \leftarrow \alpha N_0$
 $S_0 \leftarrow (P_0 \circ T_\omega)^\top Df_{\mu_0} \circ K_0 J^{-1} \circ K_0 P_0$
 $\tilde{A}_0 \leftarrow M_0^{-1} \circ T_\omega D_\mu f_{\mu_0} \circ K_0$
- 8) $(B_{a0})^0$ solves $\lambda(B_{a0})^0 - (B_{a0})^0 \circ T_\omega = -(\tilde{E}_0^{(2)})^0$
 $(B_{b0})^0$ solves $\lambda(B_{b0})^0 - (B_{b0})^0 \circ T_\omega = -(\tilde{A}_0^{(2)})^0$
- 9) Find $\overline{W}_0^{(2)}$, σ_0 solving

$$0 = -\overline{S}_0 \overline{W}_0^{(2)} - \overline{S}_0 (B_{a0})^0 - \overline{S}_0 (B_{b0})^0 \sigma_0 - \overline{\tilde{E}_0^{(1)}} - \overline{\tilde{A}_0^{(1)}} \sigma_0$$

$$(\lambda - 1) \overline{W}_0^{(2)} = -\overline{\tilde{E}_0^{(2)}} - \overline{\tilde{A}_0^{(2)}} \sigma_0 .$$
- 10) $(W_0^{(2)})^0 = (B_{a0})^0 + \sigma_0 (B_{b0})^0$

- 11) $W_0^{(2)} = (W_0^{(2)})^0 + \overline{W_0^{(2)}}$
- 12) $(W_0^{(1)})^0$ solves $(W_0^{(1)})^0 - (W_0^{(1)})^0 \circ T_\omega = -(S_0 W_0^{(2)})^0 - (\tilde{E}_0^{(1)})^0 - (\tilde{A}_0^{(1)})^0 \sigma_0$
- 13) $K_1 \leftarrow K_0 + M_0 W_0$
 $\mu_0 \leftarrow \mu_0 + \sigma_0$.

Remark 23. *We call attention on the fact that steps 2), 8), 10), 11), 12) involve diagonal operations in the Fourier space. On the contrary, the other steps are diagonal in the real space (while steps 10), 11) are diagonal in both spaces). If we represent a function in discrete points or in Fourier space, then we can compute the other functions by applying the Fast Fourier Transform (FFT). This implies that if we use N Fourier modes to discretize the function, then we need $O(N)$ storage and $O(N \log N)$ operations.*

Next task is to translate the procedure described before into a numerical algorithm that computes invariant tori of (2.8). To this end, we fix the frequency equal to the golden ratio:

$$\omega = \frac{\sqrt{5} - 1}{2} . \quad (7.1)$$

We remark that the golden ratio (7.1) satisfies the Diophantine condition (2.4) with constants $\nu = \frac{2}{3+\sqrt{5}}$, $\tau = 1$.

Then, we start from $(K_0, \mu_0) = (0, 0)$, implement Algorithm 22 using Fast Fourier Transforms and perform a continuation method to get an approximation of the invariant circle close to the breakdown value.

To get closer to breakdown, one needs to implement Algorithm 22 with a sufficient accuracy. The result described in Section 7.2 is obtained making all computations by means of the GNU MPFR Library using 115 significant digits. We use our own extended precision implementation of the classical radix-2 Cooley-Tukey in [26] by using GNU MPFR. We compute 2^{18} Fourier coefficients to discretize the invariant circle and we ask for a tolerance equal to 10^{-46} in the approximation of the analytic norm (2.1), of the invariance equation (2.10) to have convergence.

We fix $\lambda = 0.9$ and (by trial and error to optimize the final result) we select the parameters measuring the size of the domain as $\rho_0 = 3 \cdot 10^{-5}$, $\delta_0 = \rho_0/4$. This choice of ρ_0 is taken to optimize the final result. We will denote by ε_{KAM} the value of the parameter ε after the algorithm has converged to an approximate solution, (K, μ) , and all the estimates of Theorem 10 (precisely (3.3)-(3.4)-(3.5)-(3.6)-(3.7)-(3.8)-(3.9)-(3.10)-(3.11)-(3.12)) have been verified numerically for that approximate solution. In fact,

Table 1 provides the value of ε_{KAM} obtained with 2^{18} Fourier coefficients for different values of ρ_0 . We emphasize that the a-posteriori format of Theorem 10 verifies the solution and does not need to justify how the approximate solution is constructed.

ρ_0	ε_{KAM}	agreement with ε_c	μ
10^{-5}	0.97094171	99.89%	0.06139053
$2 \cdot 10^{-5}$	0.97136363	99.93%	0.06139054
$3 \cdot 10^{-5}$	0.97142178	99.94%	0.06139056
$4 \cdot 10^{-5}$	0.97136363	99.93%	0.06139060
$5 \cdot 10^{-5}$	0.97133318	99.93%	0.06139063
$6 \cdot 10^{-5}$	0.97127502	99.92%	0.06139068
$7 \cdot 10^{-5}$	0.97120503	99.92%	0.06139072
$8 \cdot 10^{-5}$	0.97114973	99.91%	0.06139075
$9 \cdot 10^{-5}$	0.97094171	99.89%	0.06139079
10^{-4}	0.97094171	99.89%	0.06139082
$2 \cdot 10^{-4}$	0.97011584	99.80%	0.06139146

TABLE 1. The analytical estimate ε_{KAM} for the golden mean curve of (2.8) with $\lambda = 0.9$ for different values of the parameter ρ measuring the width of the analyticity domain considered for K .

As Table 2 shows, the higher the number of Fourier coefficients, the better is the result, although the execution time becomes longer. We also notice that the improvement is smaller as the number of Fourier coefficients increases; in particular, the results are very similar when taking 2^{17} and 2^{18} Fourier coefficients.

n. Fourier coefficients	ε_{KAM}	μ	agreement with ε_c	execution time (sec)
2^{13}	0.95730400	0.06140120	98.49%	612.28
2^{14}	0.96512016	0.06139562	99.29%	2015.22
2^{15}	0.96807778	0.06139307	99.60%	3205.34
2^{16}	0.97011583	0.06139161	99.81%	8460.19
2^{17}	0.97094171	0.06139089	99.89%	13375.78
2^{18}	0.97142178	0.06139056	99.94%	38222.48

TABLE 2. The analytical estimate ε_{KAM} for the golden mean curve of (2.8) with $\lambda = 0.9$, $\rho = 3 \cdot 10^{-5}$, as the number of Fourier coefficients of the solution increases.

The output of the construction of the approximate solution via the MPRF program is represented by the analytic norms of the following quantities, which will be used to check the conditions (3.3)-(3.12), needed to implement Theorem 10. All quantities are given with 30 decimal digits:

$$\begin{aligned}
\|M_0\|_{\rho_0} &= 44.9270811990274410452148184267 , \\
\|M_0^{-1}\|_{\rho_0} &= 39.930678840711850152808576113 , \\
\|Df_{\mu_0}\|_{\rho_0} &= 5.07550011737521959347639032433 , \\
\|D^2f_{\mu_0}\|_{\rho_0} &= 12.2074077197778485732557018883 , \\
\|S_0\|_{\rho_0} &= 215.24720762912463716286404004 , \\
\|N_0\|_{\rho_0} &= 156.534312450915756580422752539 , \\
\|N_0^{-1}\|_{\rho_0} &= 591.408362768291837018626059244 , \\
\|DK_0\|_{\rho_0} &= 44.9270811990274410452148184267 , \\
\|D^2K_0\|_{\rho_0} &= 221591.876024617607481468301961 , \\
\|DK_0^{-1}\|_{\rho_0} &= 7032.62976591622436294280767134 , \\
\mathcal{T}_0 &= 7.6434265622376167352649577512 , \\
\|E_0\|_{\rho_0} &= 7.71650351451832566847490849233 \cdot 10^{-36} , \\
\|D^2E_0\|_{\rho_0} &= 5.1576300492851806964395530006 \cdot 10^{-24} . \tag{7.2}
\end{aligned}$$

With reference to the quantities in (3.2), we notice that in the case of the dissipative standard map (2.8) we have $Q_{\mu_0} = 1$ and $Q_{z\mu_0} = Q_{\mu\mu_0} = 0$. We stress that the quantities which require the hardest computation effort is the error E_0 and its derivatives.

7.2. Check of the conditions of Theorem 10 and results. We verify numerically the estimates of the theorem on the existence of the golden mean torus for the dissipative standard map described by equation (2.8) with frequency as in (7.1) and $\lambda = 0.9$. The corresponding breakdown threshold, as computed by means of the Sobolev's method used in [10], or equivalently by means of Greene's technique (see [10], [15]), gives

$$\varepsilon_c = 0.97198 , \tag{7.3}$$

(compare with [10]).

On the other hand, implementing the analytical estimates of Section 5, we obtain that the conditions (3.3)-(3.12), appearing in Theorem 10 are satisfied for a value of the

perturbing parameter equal to

$$\varepsilon_{KAM} = 0.971421780429401935547661013138 . \quad (7.4)$$

The corresponding value of the drift parameter amounts to

$$\mu = 0.061390559555891469231218991051 . \quad (7.5)$$

The result is validated by running the program with different precision on a DELL Machine with an Intel Xeon Processor E5-2643 (Quad Core, 3.30GHz Turbo, 10MB, 8.0 GT/s) and 16GB RAM. Precisely, we provide in Table 3 the results with different significant digits.

digits	ε_{KAM}	execution time (sec)
50	0.97142178	27632.88
60	0.97142178	29027.68
70	0.97142178	30094.44
85	0.97142178	32685.89
100	0.97142178	35390.35
115	0.97142178	38222.48

TABLE 3. The analytical estimate ε_{KAM} for the golden mean curve of (2.8) with $\lambda = 0.9$, $\rho = 3 \cdot 10^{-5}$, number of Fourier coefficients equal to 2^{18} and for different precision of the computation, obtained varying the number of digits as in the first column.

The results shown in Table 3 suggest that the norms provided in (7.2) are robust and, even if we do not implement interval arithmetic, we can conjecture that the values provided in (7.2) are not affected by numerical errors. Below 50 digits of precision, the algorithm does not produce any result, since some quantities are so small that a precision less than 50 digits is not enough. This remark leads us to state the following result.

Theorem 24. *Let us consider the map (2.8) with $\lambda = 0.9$. Let $\rho_0 = 3 \cdot 10^{-5}$, $\delta_0 = \rho_0/4$, $\zeta = 3 \cdot 10^{-5}$; let us fix the frequency as $\omega = \frac{\sqrt{5}-1}{2}$. Assume that the norms of M_0 , M_0^{-1} , Df_{μ_0} , $D^2f_{\mu_0}$, S_0 , N_0 , N_0^{-1} , DK_0 , D^2K_0 , DK_0^{-1} , E_0 , D^2E_0 , and that the twist constant \mathcal{T}_0 are given by the values provided in (7.2).*

Then, there exists an invariant attractor with frequency ω for $\varepsilon \approx \varepsilon_{KAM}$ with ε_{KAM} as in (7.4) and for a value of the drift parameter as in (7.5).

The result stated in Theorem 24 verifies the estimates for ε_{KAM} which is consistent within 99.94% of the numerical value ε_c given in (7.3). This result shows that, beside a world-wide recognized theoretical interest, KAM theory can also provide a constructive effective algorithm to estimate the breakdown value with great accuracy.

APPENDIX A. PROOF OF LEMMA 6

In this appendix, we include the proof of Lemma 6. In the proof we follow the construction of [57] to derive the constant C_0 in (2.7).

Proof. For the proof of the existence of the solution of (2.5) we refer to [12], where the constant C_0 was not made explicit. Here, instead, we provide an explicit estimate of C_0 , which closely follows [56]. Let us expand φ and η in Fourier series as

$$\varphi(\theta) = \sum_{k \in \mathbb{Z}} \hat{\varphi}_k e^{2\pi i k \theta}, \quad \eta(\theta) = \sum_{k \in \mathbb{Z}} \hat{\eta}_k e^{2\pi i k \theta},$$

where $\hat{\varphi}_k, \hat{\eta}_k$ denote the Fourier coefficients. Then, equation (2.5) becomes

$$\sum_{k \in \mathbb{Z}} \hat{\varphi}_k (e^{2\pi i k \omega} - \lambda) e^{2\pi i k \theta} = \sum_{k \in \mathbb{Z}} \hat{\eta}_k e^{2\pi i k \theta},$$

providing

$$\hat{\varphi}_k = \frac{\hat{\eta}_k}{e^{2\pi i k \omega} - \lambda}.$$

Adding the Fourier coefficients, one obtains:

$$\varphi(\theta) = \sum_{k \in \mathbb{Z}} \frac{\hat{\eta}_k}{e^{2\pi i k \omega} - \lambda} e^{2\pi i k \theta}.$$

Let

$$Z_k \equiv \min_{q \in \mathbb{Z}} |\omega k - q|;$$

we have the following inequality

$$\begin{aligned} |e^{2\pi i k \omega} - \lambda|^2 &= (1 - \lambda)^2 \cos^2(\pi k \omega) + (1 + \lambda)^2 \sin^2(\pi k \omega) \\ &\geq (1 + \lambda)^2 \sin^2(\pi k \omega) \geq 4(1 + \lambda)^2 Z_k^2, \end{aligned}$$

where the last inequality comes from noticing that $\sin(x)/x \geq 2/\pi$ for all $0 < x < \frac{\pi}{2}$.

Therefore we obtain:

$$|e^{2\pi i k \omega} - \lambda| \geq 2(1 + \lambda) |Z_k| \geq 2(1 + \lambda) \nu |k|^{-\tau},$$

namely

$$|e^{2\pi i k \omega} - \lambda|^{-1} \leq \frac{1}{2(1 + \lambda)} \nu^{-1} |k|^\tau.$$

Finally, we have

$$\begin{aligned}
\|\varphi\|_{\rho-\delta} &\leq \sum_{k \in \mathbb{Z}} |\hat{\eta}_k| e^{2\pi\rho|k|} \frac{e^{-2\pi\delta|k|}}{|e^{2\pi ik\omega} - \lambda|} \\
&\leq \sqrt{\sum_{k \in \mathbb{Z}} |\hat{\eta}_k|^2 e^{4\pi\rho|k|}} \sqrt{\sum_{k \in \mathbb{Z}} \frac{e^{-4\pi\delta|k|}}{|e^{2\pi ik\omega} - \lambda|^2}} \\
&\leq \|\eta\|_{\rho} \sqrt{F(\delta)},
\end{aligned}$$

where

$$F(\delta) \equiv 4 \sum_{k=1}^{\infty} \frac{e^{-4\pi\delta|k|}}{|e^{2\pi ik\omega} - \lambda|^2}$$

and where we used the estimate (see [56])

$$\sum_{k \in \mathbb{Z}} |\hat{\eta}_k|^2 e^{4\pi\rho|k|} \leq 2\|\eta\|_{\rho}^2.$$

Denoting by Γ the Euler gamma function, using the estimates of [57], one has that

$$F(\delta) \leq \frac{\pi^2 \Gamma(2\tau + 1)}{3\nu^2 (1 + \lambda)^2 (2\delta)^{2\tau} (2\pi)^{2\tau}},$$

which leads to (2.6) with C_0 as in (2.7). □

APPENDIX B. CONSTANTS OF THE KAM THEOREM

The constants entering in the conditions(3.3)-(3.12) of Theorem 10 are defined through the following (long) list. For fast reference, before each constant we provide the label of the formula where the constant was introduced. We note that the constants are given in

an explicit format and evaluating them requires only a few lines of code.

$$\begin{aligned}
(5.3) \quad C_{\sigma 0} &\equiv \mathcal{T}_0 \left[|\lambda - 1| \left(\frac{1}{\left| |\lambda| - 1 \right|} \|S_0\|_{\rho_0} + 1 \right) + \|S_0\|_{\rho_0} \right] \|M_0^{-1}\|_{\rho_0} , \\
(5.3) \quad C_{W_{20}} &\equiv \frac{1}{\left| |\lambda| - 1 \right|} \left(1 + C_{\sigma 0} Q_{\mu 0} \right) \|M_0^{-1}\|_{\rho_0} , \\
(5.3) \quad \bar{C}_{W_{20}} &\equiv 2\mathcal{T}_0 \left(\frac{1}{\left| |\lambda| - 1 \right|} \|S_0\|_{\rho_0} + 1 \right) Q_{\mu 0} \|M_0^{-1}\|_{\rho_0}^2 , \\
(5.3) \quad C_{W_{10}} &\equiv C_0 \left(\|S_0\|_{\rho_0} (C_{W_{20}} + \bar{C}_{W_{20}}) + \|M_0^{-1}\|_{\rho_0} + Q_{\mu 0} \|M_0^{-1}\|_{\rho_0} C_{\sigma 0} \right) , \\
(5.3) \quad C_{W_0} &\equiv C_{W_{10}} + (C_{W_{20}} + \bar{C}_{W_{20}}) \nu \delta_0^\tau , \\
(5.4) \quad C_{\eta 0} &\equiv C_{W_0} \|M_0\|_{\rho_0} + C_{\sigma 0} \nu \delta_0^\tau , \\
(5.8) \quad C_{\mathcal{R}0} &\equiv Q_{E0} (\|M_0\|_{\rho_0}^2 C_{W_0}^2 + C_{\sigma 0}^2 \nu^2 \delta_0^{2\tau}) , \\
(5.7) \quad C_{\mathcal{E}0} &\equiv C_c C_{W_0} \nu \delta_0^{-1+\tau} + C_{\mathcal{R}0} , \\
(5.9) \quad C_{d0} &\equiv C_{W_0} \|M_0\|_{\rho_0} , \\
(5.13) \quad \kappa_0 &\equiv 2^{2\tau+1} C_{\mathcal{E}0} \nu^{-2} \delta_0^{-2\tau} , \\
(5.13) \quad \kappa_K &\equiv 4C_{d0} \nu^{-1} \delta_0^{-\tau} , \\
(5.13) \quad \kappa_\mu &\equiv 4C_{\sigma 0} , \\
(3.13) \quad D_K &\equiv 4C_{d0} C_c \nu^{-1} \delta_0^{-\tau-1} \varepsilon_0 , \\
(6.36) \quad D_{2K} &\equiv 4 C_{d0} C_c^2 \nu^{-1} \delta_0^{-\tau-2} \varepsilon_0 , \\
(6.8) \quad C_N &\equiv \|N_0\|_{\rho_0}^2 \frac{2\|DK_0\|_{\rho_0} + D_K}{1 - \|N_0\|_{\rho_0} D_K (2\|DK_0\|_{\rho_0} + D_K)} , \\
(6.8) \quad C_M &\equiv 1 + J_e \left[C_N (\|DK_0\|_{\rho_0} + D_K) + \|N_0\|_{\rho_0} \right] , \\
(6.8) \quad C_{Minv} &\equiv C_N (\|DK_0\|_{\rho_0} + D_K) + \|N_0\|_{\rho_0} + J_e , \\
(6.22) \quad C_S &\equiv 2J_e Q_0 \left\{ (\|N_0\|_{\rho_0} + C_N D_K) \left[D_K (\|N_0\|_{\rho_0} + C_N D_K) \right. \right. \\
&\quad \left. \left. + \|DK_0\|_{\rho_0} \|N_0\|_{\rho_0} + \|DK_0\|_{\rho_0} C_N D_K \right] \right. \\
&\quad \left. + C_N \|DK_0\|_{\rho_0} \left[D_K (\|N_0\|_{\rho_0} + C_N D_K) + \|DK_0\|_{\rho_0} \|N_0\|_{\rho_0} + \|DK_0\|_{\rho_0} C_N D_K \right] \right. \\
&\quad \left. + \|N_0\|_{\rho_0} \|DK_0\|_{\rho_0} (\|N_0\|_{\rho_0} + C_N D_K) + C_N \|N_0\|_{\rho_0} \|DK_0\|_{\rho_0}^2 \right\} ,
\end{aligned}$$

$$(6.26) \quad C_{SB} \equiv \frac{1}{\left| |\lambda| - 1 \right|} Q_{\mu 0} \|M_0^{-1}\|_{\rho_0} C_S + 2J_e Q_0 \|N_0\|_{\rho_0}^2 \|DK_0\|_{\rho_0}^2 \frac{1}{\left| |\lambda| - 1 \right|} C_{Minv} Q_{\mu 0} \\ + 2C_S \frac{1}{\left| |\lambda| - 1 \right|} C_{Minv} Q_{\mu 0} D_K ,$$

$$(6.27) \quad C_\tau \equiv \max \left\{ C_S, C_{SB} + 2C_{Minv} Q_{\mu 0} \right\} D_K ,$$

$$(6.28) \quad C_T \equiv \frac{\mathcal{T}_0^2}{1 - \mathcal{T}_0 C_\tau} \max \left\{ C_S, C_{SB} + 2C_{Minv} Q_{\mu 0} \right\} ,$$

$$(6.29) \quad C_\sigma \equiv C_T \left\{ |\lambda - 1| \left[\frac{1}{\left| |\lambda| - 1 \right|} (\|S_0\|_{\rho_0} + C_S D_K) + 1 \right] \right. \\ + \left. \left(\|S_0\|_{\rho_0} + C_S D_K \right) \right\} \left(\|M_0^{-1}\|_{\rho_0} + C_{Minv} D_K \right) \\ + \mathcal{T}_0 \left\{ |\lambda - 1| \left[\frac{1}{\left| |\lambda| - 1 \right|} (\|S_0\|_{\rho_0} + C_S D_K) + 1 \right] C_{Minv} \right. \\ + \left. |\lambda - 1| \frac{1}{\left| |\lambda| - 1 \right|} \|M_0^{-1}\|_{\rho_0} C_S + C_S \left(\|M_0^{-1}\|_{\rho_0} + C_{Minv} D_K \right) \right. \\ + \left. C_{Minv} \|S_0\|_{\rho_0} \right\} ,$$

$$(6.32) \quad \bar{C}_{W_2} \equiv 4C_T \left[\frac{1}{\left| |\lambda| - 1 \right|} (\|S_0\|_{\rho_0} + C_S D_K) + 1 \right] Q_{\mu 0} (\|M_0^{-1}\|_{\rho_0} + D_K)^2 \\ + 4\mathcal{T}_0 Q_{\mu 0} \frac{1}{\left| |\lambda| - 1 \right|} C_S (\|M_0^{-1}\|_{\rho_0} + D_K)^2 \\ + 4\mathcal{T}_0 Q_{\mu 0} \left[\frac{1}{\left| |\lambda| - 1 \right|} (\|S_0\|_{\rho_0} + C_S D_K) + 1 \right] (D_K + 2\|M_0^{-1}\|_{\rho_0})$$

$$(6.37) \quad C_{\mathcal{R}} \equiv Q_{E0} \left[(2C_M \|M_0\|_{\rho_0} + C_M^2 D_K) (C_{W0} + C_W D_K)^2 + \|M_0\|_{\rho_0}^2 (C_W^2 D_K + 2C_{W0} C_W) \right. \\ + \left. (C_\sigma^2 D_K + 2C_{\sigma 0} C_\sigma) \nu^2 \delta_0^{2\tau} \right] + C_Q \left[(\|M_0\|_{\rho_0} + C_M D_K)^2 (C_{W0} + C_W D_K)^2 \right. \\ + \left. (C_{\sigma 0} + C_\sigma D_K)^2 \nu^2 \delta_0^{2\tau} \right] C_c \delta_0^{-1} ,$$

$$(6.31) \quad C_{W_2} \equiv \frac{1}{\|\lambda\| - 1} \left[1 + 2Q_{\mu_0} \|M_0^{-1}\|_{\rho_0} C_\sigma + 2Q_{\mu_0} C_{\sigma_0} + 2Q_{\mu_0} C_\sigma D_K \right],$$

$$(6.33) \quad C_{W_1} \equiv C_0 \left[\|S_0\|_{\rho_0} C_{W_2} + C_S C_{W_2 0} + C_S C_{W_2} D_K + \|S_0\|_{\rho_0} \overline{C}_{W_2} \right. \\ \left. + C_S \overline{C}_{W_2 0} + C_S \overline{C}_{W_2} D_K + 1 \right. \\ \left. + 2Q_{\mu_0} \|M_0^{-1}\|_{\rho_0} C_\sigma + 2Q_{\mu_0} C_{\sigma_0} + 2Q_{\mu_0} C_\sigma D_K \right],$$

$$(6.34) \quad C_W \equiv C_{W_1} + C_{W_2} \nu \delta_0^\tau + \overline{C}_{W_2} \nu \delta_0^\tau$$

$$(6.42) C_Q \equiv \frac{1}{2} \max \left\{ 1 + \sup_{z \in \mathcal{C}} |D^3 f_{\mu_0}(z)| \|DK_0\|_{\rho_0}^2 C_c^{-2} \delta_0^2 \right. \\ \left. + \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu D^2 f_\mu(z)| \|DK_0\|_{\rho_0}^2 \frac{C_{\sigma_0}}{C_{d_0}} C_c^{-2} \delta_0^{\tau+2} \right. \\ \left. + \sup_{z \in \mathcal{C}} |D^2 f_{\mu_0}(z)| \|DK_0\|_{\rho_0} C_c^{-1} \delta_0 \right. \\ \left. + \sup_{z \in \mathcal{C}} |D^3 f_{\mu_0}(z)| \|DK_0\|_{\rho_0} 4C_{d_0} C_c^{-1} \nu^{-1} \delta_0^{-\tau+1} \varepsilon_0 \right. \\ \left. + \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu D^2 f_\mu(z)| \|DK_0\|_{\rho_0} 4C_{\sigma_0} C_c^{-1} \delta_0 \varepsilon_0 \right. \\ \left. + \sup_{z \in \mathcal{C}} |D^2 f_{\mu_0}(z)| \|D^2 K_0\|_{\rho_0}^2 C_c^{-2} \delta_0^2 \right. \\ \left. + \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu D f_\mu(z)| \|D^2 K_0\|_{\rho_0} \frac{C_{\sigma_0}}{C_{d_0}} C_c^{-2} \nu \delta_0^{\tau+2} \right. \\ \left. + \sup_{z \in \mathcal{C}} |D^2 f_{\mu_0}(z)| (\|DK_0\|_{\rho_0} + D_K) C_c^{-1} \delta_0 \right. \\ \left. + \sup_{z \in \mathcal{C}} |D^3 f_{\mu_0}(z)| (\|DK_0\|_{\rho_0} + D_K) 4C_{d_0} C_c^{-1} \nu^{-1} \delta_0^{-\tau+1} \varepsilon_0 \right. \\ \left. + \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu D^2 f_\mu(z)| (\|DK_0\|_{\rho_0} + D_K) 4C_{\sigma_0} C_c^{-1} \delta_0 \varepsilon_0 \right. \\ \left. + \sup_{z \in \mathcal{C}} |D f_{\mu_0}(z)| + \sup_{z \in \mathcal{C}} |D^2 f_{\mu_0}(z)| \kappa_K \varepsilon_0 \right. \\ \left. + \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu D f_\mu(z)| \kappa_\mu \varepsilon_0, \right. \\ \left. \sup_{z \in \mathcal{C}} |DD_\mu f_{\mu_0}(z)| C_c^{-1} \delta_0 + \sup_{z \in \mathcal{C}} |D^2 D_\mu f_{\mu_0}(z)| C_c^{-2} \delta_0^2 (\|DK_0\|_{\rho_0} + D_K) \right. \\ \left. + \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |DD_\mu^2 f_\mu(z)| \frac{C_{\sigma_0}}{C_{d_0}} C_c^{-2} \nu \delta_0^{\tau+2} (\|DK_0\|_{\rho_0} + D_K), \right. \\ \left. \sup_{z \in \mathcal{C}, \mu \in \Lambda, |\mu - \mu_0| < 2\kappa_\mu \varepsilon_0} |D_\mu^3 f_\mu(z)| \frac{C_{\sigma_0}}{C_{d_0}} C_c^{-2} \nu \delta_0^{\tau+2} \right\}.$$

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