SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME FOURTH ORDER NON-FREDHOLM OPERATORS

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Abstract: We study solvability of some linear nonhomogeneous elliptic problems and establish that under reasonable technical conditions the convergence in $L^2(\mathbb{R}^d)$ of their right sides implies the existence and the convergence in $H^4(\mathbb{R}^d)$ of the solutions. The problems contain the squares of the sums of second order non- Fredholm differential operators and we use the methods of the spectral and scattering theory for Schrödinger type operators. We especially emphasize that here we deal with the fourth order operators in contrast to the second order operators in [29] and investigate the dependence of the solvability conditions on the dimension of our problem when the constant $a = 0$. We also consider the case of solvability with a single potential in an arbitrary dimension.

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1. Introduction

Consider the problem

$$-\Delta u + V(x)u - au = f, \tag{1.1}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ tends to 0 at infinity (it is well known that if $V(x) \to \infty$ as $|x| \to \infty$, it leads only to the discreteness of the spectrum). For $a \geq 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of equation (1.1) contains the origin. Consequently, this operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimensions of its kernel and the codimension of its image are not finite. The present work is devoted to the studies of certiain properties of the operators of this kind. Let us recall that elliptic equations containing non Fredholm operators were treated extensively in recent years (see [14], [23], [24], [25], [26], [27], [28], also [6]) along with their potential applications to the theory of reaction-diffusion equations (see [8], [9]). Non-Fredholm operators are also very significant when studying wave systems with an infinite number of localized traveling waves (see [1]). In particular, when $a = 0$ the operator A satisfies the Fredholm property in certain properly chosen weighted spaces (see [2], [3], [4], [5], [6], [10], [11], [12], [13]). However, the case of $a \neq 0$ is considerably different and the method developed in these works cannot be applied.

One of the important questions about problems with non-Fredholm operators concerns their solvability. We address it in the following setting. Let f_n be a sequence of functions in the image of the operator A, such that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Denote by u_n a sequence of functions from $H^2(\mathbb{R}^d)$ such that

$$Au_n = f_n, \ n \in \mathbb{N}.$$

Since the operator A does not satisfy the Fredholm property, the sequence u_n may not be convergent. Let us call a sequence u_n the solution in the sense of sequences of the equation $Au = f$ if $Au_n \to f$ (see [23]). If such sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this equation. Solution in the sense of sequences is equivalent in this case to the usual solution. However, in the case of the non-Fredholm operators, this convergence may not hold or it can occur in some weaker sense. In such case, solution in the sense of sequences may not imply the existence of the usual solution. In the present work we will find sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, the conditions on sequences f_n under which the corresponding sequences u_n are strongly convergent. Solvability in the sense of sequences for the sums of non-Fredholm Schrödinger type operators was studied in [29]. In the first part of the work we consider such operators squared, namely

$$\{-\Delta_x + V(x) - \Delta_y + U(y)\}^2 u - a^2 u = f(x, y), \quad x, y \in \mathbb{R}^3, \tag{1.2}$$

with the constant $a > 0$. The operator

$$H_{U,\,V} := \{-\Delta_x + V(x) - \Delta_y + U(y)\}^2 : H^4(\mathbb{R}^6) \to L^2(\mathbb{R}^6) \tag{1.3}$$

under the technical conditions on the scalar potential functions $V(x)$ and $U(y)$ stated below. Here and throughout the article the Laplace operators Δ_x and Δ_y are with respect to the x and y variables respectively, such that cumulatively $\Delta = \Delta_x + \Delta_y$. Similarly for the gradients, ∇_x and ∇_y are with respect to the x and y variables respectively. In the applications the sum of the two Schrödinger type operators has the physical meaning of the resulting hamiltonian of the two non-interacting quantum particles.

The boundedness of the gradient of a solution for the bi-harmonic equation was established in [18]. The behavior near the boundary of solutions to the Dirichlet problem for the biharmonic operator was studied in [19]. Article [20] is devoted to the Dirichlet problem in Lipschitz domains for higher order elliptic systems with rough coefficients. Solvability conditions for a linearized Cahn-Hilliard equation were obtained in [25].

The scalar potential functions involved in operator (1.3) are assumed to be shallow and short-range, satisfying the assumptions analogous to the ones of [26] and [27]. We also add a few extra regularity conditions.

Assumption 1. *The potential functions $V(x), U(y) : \mathbb{R}^3 \to \mathbb{R}$ satisfy the estimates*

$$|V(x)| \le \frac{C}{1+|x|^{3.5+\varepsilon}}, \quad |U(y)| \le \frac{C}{1+|y|^{3.5+\varepsilon}}$$

with some $\varepsilon > 0$ and $x, y \in \mathbb{R}^3$ a.e. such that

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1, \tag{1.4}$$

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|U\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|U\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \tag{1.5}$$

and

$$\sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi, \quad \sqrt{c_{HLS}}\|U\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi.$$

Moreover, $|\nabla_x V(x)|, \ \Delta_x V(x), \ |\nabla_y U(y)|, \ \Delta_y U(y) \in L^\infty(\mathbb{R}^3)$.

Here and further down C denotes a finite positive constant and c_{HLS} given on p.98 of [17] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\right| \le c_{HLS}\|f_1\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^2, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

The norm of a function $f_1 \in L^p(\mathbb{R}^d), \ 1 \le p \le \infty, \ d \in \mathbb{N}$ is designated as $\|f_1\|_{L^p(\mathbb{R}^d)}$.

Proposition. *The function $V(x) = \dfrac{C}{1+|x|^4}$, where C is small enough satisfies Assumption 1.*

Proof. A straightforward computation yields

$$|\nabla_x V(x)| = \frac{4C|x|^3}{(1+|x|^4)^2} \in L^\infty(\mathbb{R}^3)$$

and

$$\Delta_x V(x) = -4C\frac{5|x|^2 - 3|x|^6}{(1+|x|^4)^3} \in L^\infty(\mathbb{R}^3)$$

as well. ■

Let us denote the inner product of two functions as

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx, \tag{1.6}$$

with a slight abuse of notations when these functions are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x)$ is bounded, like for example the functions of the continuos spectrum of the Schrödinger operators discussed below (see Corollary 2.2 of [27]), then the integral in the right side of (1.6) makes sense. We use the spaces $H^2(\mathbb{R}^d)$ and $H^4(\mathbb{R}^d)$ equipped with the norms

$$\|u\|^2_{H^2(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|\Delta u\|^2_{L^2(\mathbb{R}^d)} \tag{1.7}$$

and

$$\|u\|^2_{H^4(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|\Delta^2 u\|^2_{L^2(\mathbb{R}^d)} \tag{1.8}$$

respectively. Throughout the work, the sphere of radius $r > 0$ in $\mathbb{R}^d$ centered at the origin will be designated by S^d_r, the unit sphere is denoted by S^d and $|S^d|$ stands for its Lebesgue measure. By means of Lemma 2.3 of [27], under Assumption 1 above on the scalar potentials, operator (1.3) considered as acting in $L^2(\mathbb{R}^6)$ with domain $H^4(\mathbb{R}^6)$ is self-adjoint and is unitarily equivalent to $\{-\Delta_x - \Delta_y\}^2$ on $L^2(\mathbb{R}^6)$ via the product of the wave operators (see [16], [22])

$$\Omega^\pm_V := s - \lim_{t\to\mp\infty} e^{it(-\Delta_x + V(x))}e^{it\Delta_x}, \quad \Omega^\pm_U := s - \lim_{t\to\mp\infty} e^{it(-\Delta_y + U(y))}e^{it\Delta_y},$$

with the limits here understood in the strong L^2 sense (see e.g. [21] p.34, [7] p.90). Hence, operator (1.3) has no nontrivial $L^2(\mathbb{R}^6)$ eigenfunctions. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Therefore, operator (1.3) does not satisfy the Fredholm property. On the contrary, the operator

$$h_{u,\,v} := -\Delta_x + V(x) - \Delta_y + U(y) + a$$

considered as acting in $L^2(\mathbb{R}^6)$ with domain $H^2(\mathbb{R}^6)$ satisfies the Fredholm property, has only the essential spectrum, which fills the interval $[a, +\infty)$, such that the inverse $h^{-1}_{u,\,v} : L^2(\mathbb{R}^6) \to H^2(\mathbb{R}^6)$ is bounded. The functions of the continuos spectrum of the first operator involved in (1.3) are the solutions of the Schrödinger equation

$$[-\Delta_x + V(x)]\varphi_k(x) = k^2\varphi_k(x), \quad k \in \mathbb{R}^3,$$

in the integral form the Lippmann-Schwinger equation (see e.g. [21] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{1.9}$$

for the perturbed plane waves and the orthogonality conditions

$$(\varphi_k(x), \varphi_{k_1}(x))_{L^2(\mathbb{R}^3)} = \delta(k - k_1), \ k, k_1 \in \mathbb{R}^3.$$

The integral operator involved in (1.9)

$$(Q\varphi)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi)(y)dy, \quad \varphi(x) \in L^\infty(\mathbb{R}^3).$$

Let us consider $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$. Its norm $\|Q\|_\infty < 1$ under Assumption 1 via Lemma 2.1 of [27]. In fact, this norm is bounded above by the k-independent quantity, which is the left side of inequality (1.4). Similarly, for the second operator involved in (1.3) the functions of its continuous spectrum solve

$$[-\Delta_y + U(y)]\eta_q(y) = q^2\eta_q(y), \quad q \in \mathbb{R}^3,$$

in the integral formulation

$$\eta_q(y) = \frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|}(U\eta_q)(z)dz, \tag{1.10}$$

such that the orthogonality conditions $(\eta_q(y), \eta_{q_1}(y))_{L^2(\mathbb{R}^3)} = \delta(q - q_1), \ q, q_1 \in \mathbb{R}^3$ hold. $\eta_0(y)$ will correspond to the case of $q = 0$. The integral operator involved in (1.10) is

$$(P\eta)(y) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|}(U\eta)(z)dz, \quad \eta(y) \in L^\infty(\mathbb{R}^3).$$

For $P : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ its norm $\|P\|_\infty < 1$ under Assumption 1 by means of Lemma 2.1 of [27]. As before, this norm can be estimated from above by the q-independent quantity $I(U)$, which is the left side of inequality (1.5). Let us denote by the double tilde sign the generalized Fourier transform with the product of these functions of the continuous spectrum

$$\tilde{\tilde{f}}(k, q) := (f(x, y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)}, \quad k, q \in \mathbb{R}^3. \tag{1.11}$$

(1.11) is a unitary transform on $L^2(\mathbb{R}^6)$. Our first main proposition is as follows.

Theorem 2. *Let Assumption 1 hold,* $a > 0$ *and* $f(x, y) \in L^2(\mathbb{R}^6)$*. Assume also that* $|x|f(x, y), \ |y|f(x, y) \in L^1(\mathbb{R}^6)$*. Then problem (1.2) has a unique solution* $u(x, y) \in H^4(\mathbb{R}^6)$ *if and only if*

$$(f(x, y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0, \quad (k, q) \in S^6_{\sqrt{a}} \quad a.e. \tag{1.12}$$

In the very special case when the scalar potential functions $V(x)$ and $U(y)$ vanish identically in $\mathbb{R}^3$, condition (1.12) gives us the orthogonality to the products of the corresponding standard Fourier harmonics. Then we turn our attention to the issue of the solvability in the sense of sequences for our equation. The corresponding sequence of approximate equations with $n \in \mathbb{N}$ is given by

$$\{-\Delta_x + V(x) - \Delta_y + U(y)\}^2 u_n - a^2 u_n = f_n(x, y), \quad x, y \in \mathbb{R}^3, \tag{1.13}$$

with the constant $a > 0$ and the right sides converge to the right side of (1.2) in $L^2(\mathbb{R}^6)$ as $n \to \infty$.

Theorem 3. *Let Assumption 1 hold,* $a > 0,\ n \in \mathbb{N}$ *and* $f_n(x, y) \in L^2(\mathbb{R}^6)$*, such that* $f_n(x, y) \to f(x, y)$ *in* $L^2(\mathbb{R}^6)$ *as* $n \to \infty$*. Let in addition* $|x|f_n(x, y),\ |y|f_n(x, y) \in L^1(\mathbb{R}^6),\ n \in \mathbb{N}$*, such that* $|x|f_n(x, y) \to |x|f(x, y),\ |y|f_n(x, y) \to |y|f(x, y)$ *in* $L^1(\mathbb{R}^6)$ *as* $n \to \infty$ *and the orthogonality relations*

$$(f_n(x, y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0, \quad (k, q) \in S^6_{\sqrt{a}} \quad a.e. \tag{1.14}$$

hold for all $n \in \mathbb{N}$*. Then problems (1.2) and (1.13) admit unique solutions* $u(x, y) \in H^4(\mathbb{R}^6)$ *and* $u_n(x, y) \in H^4(\mathbb{R}^6)$ *respectively, such that* $u_n(x, y) \to u(x, y)$ *in* $H^4(\mathbb{R}^6)$ *as* $n \to \infty$*.*

The second part of the article is devoted to the studies of the equation

$$\{-\Delta_x - \Delta_y + U(y)\}^2 u - a^2 u = \phi(x, y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^3, \tag{1.15}$$

where $d \in \mathbb{N}$, the constant $a > 0$ and the scalar potential function involved in (1.15) is shallow and short-range under Assumption 1 above. The more singular case of $a = 0$ will be treated in the Appendix below in higher dimensions. The operator

$$L_U := \{-\Delta_x - \Delta_y + U(y)\}^2 : H^4(\mathbb{R}^{d+3}) \to L^2(\mathbb{R}^{d+3}). \tag{1.16}$$

Similarly to (1.3), under the given assumptions operator (1.16) considered as acting in $L^2(\mathbb{R}^{d+3})$ with domain $H^4(\mathbb{R}^{d+3})$ is self-adjoint and is unitarily equivalent to $\{-\Delta_x - \Delta_y\}^2$. Thus, operator (1.16) does not have nontrivial $L^2(\mathbb{R}^{d+3})$ eigenfunctions. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Therefore, operator (1.16) is non Fredholm. On the contrary, the operator

$$l_U := -\Delta_x - \Delta_y + U(y) + a$$

considered as acting in $L^2(\mathbb{R}^{d+3})$ with domain $H^2(\mathbb{R}^{d+3})$ satisfies the Fredholm property, has only the essential spectrum, which fills the interval $[a, +\infty)$, such that the inverse ${l_U}^{-1} : L^2(\mathbb{R}^{d+3}) \to H^2(\mathbb{R}^{d+3})$ is bounded. Let us consider another generalized Fourier transform with the standard Fourier harmonics and the perturbed plane waves

$$\tilde{\hat{\phi}}(k, q) := \left(\phi(x, y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})}, \quad k \in \mathbb{R}^d, \quad q \in \mathbb{R}^3. \tag{1.17}$$

(1.17) is a unitary transform on $L^2(\mathbb{R}^{d+3})$. We have the following statement.

Theorem 4. *Let the potential function $U(y)$ satisfy Assumption 1, $a > 0$ and additionally $\phi(x, y) \in L^2(\mathbb{R}^{d+3}),\ |x|\phi(x, y),\ |y|\phi(x, y) \in L^1(\mathbb{R}^{d+3}),\ d \in \mathbb{N}$. Then problem (1.15) possesses a unique solution $u(x, y) \in H^4(\mathbb{R}^{d+3})$ if and only if*

$$\left(\phi(x, y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} = 0, \quad (k, q) \in S^{d+3}_{\sqrt{a}} \quad a.e. \tag{1.18}$$

Our final main proposition is devoted to the issue of the solvability in the sense of sequences for our problem. The corresponding sequence of approximate equations with $n \in \mathbb{N}$ is given by

$$\{-\Delta_x - \Delta_y + U(y)\}^2 u_n - a^2 u_n = \phi_n(x, y), \quad x \in \mathbb{R}^d, \quad d \in \mathbb{N}, \quad y \in \mathbb{R}^3, \tag{1.19}$$

where the right sides converge to the right side of (1.15) in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$.

Theorem 5. *Let the potential function $U(y)$ satisfy Assumption 1, $a > 0,\ n \in \mathbb{N}$ and $\phi_n(x, y) \in L^2(\mathbb{R}^{d+3}),\ d \in \mathbb{N}$, such that $\phi_n(x, y) \to \phi(x, y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$. Let in addition $|x|\phi_n(x, y),\ |y|\phi_n(x, y) \in L^1(\mathbb{R}^{d+3})$, such that*

$$|x|\phi_n(x, y) \to |x|\phi(x, y),\ |y|\phi_n(x, y) \to |y|\phi(x, y)$$

in $L^1(\mathbb{R}^{d+3})$ as $n \to \infty$ and the orthogonality relations

$$\left(\phi_n(x, y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} = 0, \quad (k, q) \in S^{d+3}_{\sqrt{a}} \quad a.e. \tag{1.20}$$

hold for all $n \in \mathbb{N}$. Then problems (1.15) and (1.19) admit unique solutions $u(x, y) \in H^4(\mathbb{R}^{d+3})$ and $u_n(x, y) \in H^4(\mathbb{R}^{d+3})$ respectively, such that $u_n(x, y) \to u(x, y)$ in $H^4(\mathbb{R}^{d+3})$ as $n \to \infty$.

Remark 1. *Let us note that (1.12), (1.14), (1.18), (1.20) are the orthogonality conditions containing the functions of the continuous spectrum of our Schrödinger operators, as distinct from the Limiting Absorption Principle in which one orthogonalizes to the standard Fourier harmonics (see e.g. Lemma 2.3 and Proposition 2.4 of [15]).*

Remark 2. *In Theorems 2-5 above we assume that the right sides of our equations belong to L^1 after the multiplication by $|x|$ or $|y|$. In the case of the Poisson equation this condition can be weakened (see Lemma 3.3 of [30]).*

Remark 3. *It is worth noting that the proofs of our theorems with a single potential will be identical in the presence of the scalar potential potential $V(x)$ when $U(y)$ vanishes in the whole space.*

We proceed to the proof of our statements.

2. Solvability in the sense of sequences with two potentials

Proof of Theorem 2. First of all, let us observe that it is sufficient to solve equation (1.2) in $H^2(\mathbb{R}^6)$, since this solution will belong to $H^4(\mathbb{R}^6)$ as well. Indeed, it can be easily shown that

$$\Delta^2 u + [V^2(x) + U^2(y)]u - [\Delta_x V(x) + \Delta_y U(y)]u - 2[V(x) + U(y)]\Delta u-$$

$$-2\nabla_x V(x).\nabla_x u - 2\nabla_y U(y).\nabla_y u + 2V(x)U(y)u - a^2 u = f(x,y), \tag{2.1}$$

with $u(x,y)$ a solution of (1.2) belonging to $H^2(\mathbb{R}^6)$. The dot symbol in the fifth and the sixth terms in the left side of (2.1) and throughout the article denotes the standard scalar product of two vectors in $\mathbb{R}^3$. Evidently, all the terms in the left side of (2.1) starting from the second one are square integrable since according to Assumption 1 our scalar potential functions are bounded along with

$$|\nabla_x V(x)|,\ |\nabla_y U(y)|,\ \Delta_x V(x),\ \Delta_y U(y)$$

and $u(x,y) \in H^2(\mathbb{R}^6)$. The right side of (2.1) is square integrable as well as assumed. Therefore, $\Delta^2 u(x,y) \in L^2(\mathbb{R}^6)$, which yields that $u(x,y) \in H^4(\mathbb{R}^6)$.

To show the uniqueness of solutions for our equation, we suppose that problem (1.2) admits two solutions $u_1(x,y), u_2(x,y) \in H^4(\mathbb{R}^6)$. Then their difference $w(x,y) := u_1(x,y) - u_2(x,y) \in H^4(\mathbb{R}^6)$ solves the equation

$$H_{U,\,V} w = a^2 w.$$

But the operator $H_{U,\,V} : H^4(\mathbb{R}^6) \to L^2(\mathbb{R}^6)$ has no nontrivial eigenfunctions as discussed above. Therefore, $w(x,y)$ vanishes in $\mathbb{R}^6$.

Let us apply the generalized Fourier transform (1.11) to both sides of problem (1.2). This yields

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{(k^2+q^2)^2 - a^2}.$$

Hence

$$\tilde{\tilde{u}}(k,q) = \tilde{\tilde{g}}_1(k,q) + \tilde{\tilde{g}}_2(k,q), \tag{2.2}$$

where

$$\tilde{\tilde{g}}_1(k,q) := \frac{\tilde{\tilde{f}}(k,q)}{2a(k^2+q^2-a)}, \quad \tilde{\tilde{g}}_2(k,q) := -\frac{\tilde{\tilde{f}}(k,q)}{2a(k^2+q^2+a)}.$$

It is worth noting that in the right side of (2.2) the first term $\tilde{\tilde{g}}_1(k,q)$ appeared in [26]. The second term there $\tilde{\tilde{g}}_2(k,q)$ is the new one which reflects the presence of the fourth order operator. Evidently, the functions $g_1(x,y)$ and $g_2(x,y)$ satisfy the equations

$$\{-\Delta_x + V(x) - \Delta_y + U(y)\}g_1 - ag_1 = \frac{1}{2a}f(x,y) \tag{2.3}$$

and

$$\{-\Delta_x + V(x) - \Delta_y + U(y)\}g_2 + ag_2 = -\frac{1}{2a}f(x,y) \tag{2.4}$$

respectively. The operator involved in the left side of problem (2.4) has a bounded inverse $h_{u,\,v}^{-1} : L^2(\mathbb{R}^6) \to H^2(\mathbb{R}^6)$ as discussed above and the right side of (2.4) is square integrable as assumed. Therefore, equation (2.4) admits a unique solution $g_2(x,y) \in H^2(\mathbb{R}^6)$. By means of the part a) of Theorem 3 of [26], under the given conditions equation (2.3) has a unique solution $g_1(x,y) \in H^2(\mathbb{R}^6)$ if and only if orthogonality condition (1.12) holds. Note that the solvability of problem (2.3) in $L^2(\mathbb{R}^6)$ is equivalent to its solvability in $H^2(\mathbb{R}^6)$ since the right side of (2.3) is square integrable and the scalar potentials involved in (2.3) are bounded according to the one of our assumptions. ∎

Let us turn our attention to the solvability in the sense of sequences for our equation in the case of two scalar potentials.

Proof of Theorem 3. First of all, let us demonstrate that if $u(x,y)$ and $u_n(x,y),\ n \in \mathbb{N}$ are the unique $H^4(\mathbb{R}^6)$ solutions of (1.2) and (1.13) respectively and $u_n(x,y) \to u(x,y)$ in $H^2(\mathbb{R}^6)$ as $n \to \infty$, then we have $u_n(x,y) \to u(x,y)$ in $H^4(\mathbb{R}^6)$ as $n \to \infty$ as well. Indeed, (1.2) and (1.13) yield that for $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^3$

$$\{-\Delta_x + V(x) - \Delta_y + U(y)\}^2(u_n - u) - a^2(u_n - u) = f_n(x,y) - f(x,y).$$

Hence

$$\Delta^2(u_n - u) + [V^2(x) + U^2(y)](u_n - u) - [\Delta_x V(x) + \Delta_y U(y)](u_n - u) -$$
$$-2[V(x) + U(y)]\Delta(u_n - u) - 2\nabla_x V(x).\nabla_x(u_n - u) - 2\nabla_y U(y).\nabla_y(u_n - u) +$$
$$+2V(x)U(y)(u_n - u) - a^2(u_n - u) = f_n(x,y) - f(x,y). \tag{2.5}$$

Since $u_n(x,y) \to u(x,y)$ in $H^2(\mathbb{R}^6)$ as $n \to \infty$ as assumed, we have here

$$u_n(x,y) \to u(x,y), \quad \nabla_x u_n(x,y) \to \nabla_x u(x,y), \quad \nabla_y u_n(x,y) \to \nabla_y u(x,y),$$
$$\Delta u_n(x,y) \to \Delta u(x,y)$$

in $L^2(\mathbb{R}^6)$ as $n \to \infty$ and

$$V(x), \quad U(y), \quad |\nabla_x V(x)|, \quad \Delta_x V(x), \quad |\nabla_y U(y)|, \quad \Delta_y U(y)$$

are bounded functions due to Assumption 1 above. Therefore, all the terms in the left side of identity (2.5) starting from the second one tend to zero in $L^2(\mathbb{R}^6)$ as

$n \to \infty$. The right side of (2.5) converges to zero in $L^2(\mathbb{R}^6)$ as $n \to \infty$ as assumed. Hence, $\Delta^2 u_n \to \Delta^2 u$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$. By means of norm definition (1.8) we obtain that $u_n(x,y) \to u(x,y)$ in $H^4(\mathbb{R}^6)$ as $n \to \infty$.

By virtue of Theorem 2 above, under the given conditions equation (1.13) admits a unique solution $u_n(x,y) \in H^4(\mathbb{R}^6), \ n \in \mathbb{N}$. Let us recall formula (2.5) in the proof of Theorem 2 of [29]. Hence, under the stated assumptions we arrive at the limiting orthogonality relation

$$(f(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0, \quad (k,q) \in S^6_{\sqrt{a}} \quad a.e.$$

Then by means of Theorem 2 above problem (1.2) possesses a unique solution $u(x,y) \in H^4(\mathbb{R}^6)$. Let us apply the generalized Fourier transform (1.11) to both sides of problems (1.2) and (1.13). This yields the representation (2.2) as in the proof of Theorem 2 above, where the functions $g_1(x,y), \ g_2(x,y) \in H^2(\mathbb{R}^6)$ under the given conditions are the unique solutions of equations (2.3) and (2.4) respectively. Similarly,

$$\tilde{\tilde{u}}_n(k,q) = \tilde{\tilde{g}}_{1,n}(k,q) + \tilde{\tilde{g}}_{2,n}(k,q), \quad n \in \mathbb{N}, \tag{2.6}$$

where

$$\tilde{\tilde{g}}_{1,n}(k,q) := \frac{\tilde{\tilde{f}}_n(k,q)}{2a(k^2+q^2-a)}, \quad \tilde{\tilde{g}}_{2,n}(k,q) := -\frac{\tilde{\tilde{f}}_n(k,q)}{2a(k^2+q^2+a)}.$$

Apparently, the functions $g_{1,n}(x,y)$ and $g_{2,n}(x,y)$ solve the equations

$$\{-\Delta_x + V(x) - \Delta_y + U(y)\}g_{1,n} - ag_{1,n} = \frac{1}{2a} f_n(x,y) \tag{2.7}$$

and

$$\{-\Delta_x + V(x) - \Delta_y + U(y)\}g_{2,n} + ag_{2,n} = -\frac{1}{2a} f_n(x,y) \tag{2.8}$$

respectively. Since the operator involved in the left side of (2.8) has a bounded inverse $h_{u,v}^{-1} : L^2(\mathbb{R}^6) \to H^2(\mathbb{R}^6)$, such that its norm $\|h_{u,v}^{-1}\| < \infty$ as discussed above and the right side of (2.8) belongs to $L^2(\mathbb{R}^6)$ as assumed, (2.8) admits a unique solution $g_{2,n}(x,y) \in H^2(\mathbb{R}^6)$. Because $f_n(x,y) \to f(x,y)$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$ via the one of our assumptions, we have

$$\|g_{2,n} - g_2\|_{H^2(\mathbb{R}^6)} \le \frac{1}{2a}\|h_{u,v}^{-1}\|\|f_n - f\|_{L^2(\mathbb{R}^6)} \to 0, \quad n \to \infty,$$

such that $g_{2,n}(x,y) \to g_2(x,y)$ in $H^2(\mathbb{R}^6)$ as $n \to \infty$. By virtue of the result of the part a) of Theorem 2 of [29], we have that equation (2.7) possesses a unique solution $g_{1,n}(x,y) \in H^2(\mathbb{R}^6)$, such that $g_{1,n}(x,y) \to g_1(x,y)$ in $H^2(\mathbb{R}^6)$ as $n \to \infty$. Using formulas (2.6) and (2.2) considered in the x, y space, we easily arrive at $\|u_n(x,y) - u(x,y)\|_{H^2(\mathbb{R}^6)} \le$

$$\le \|g_{1,n}(x,y) - g_1(x,y)\|_{H^2(\mathbb{R}^6)} + \|g_{2,n}(x,y) - g_2(x,y)\|_{H^2(\mathbb{R}^6)} \to 0$$

as $n \to \infty$. Therefore, $u_n(x,y) \to u(x,y)$ in $H^4(\mathbb{R}^6)$ as $n \to \infty$ as discussed above. ■

In the last main section of the article we treat the case when the free Laplacian is added to our three dimensional Schrödinger operator.

3. Solvability in the sense of sequences with Laplacian and a single potential

Proof of Theorem 4. First of all, we show that it is sufficient to solve problem (1.15) in $H^2(\mathbb{R}^{d+3})$, because such solution will belong to $H^4(\mathbb{R}^{d+3})$ as well. Apparently,

$$\Delta^2 u + U^2(y)u - 2U(y)\Delta u - u\Delta_y U(y) - 2\nabla_y U(y).\nabla_y u - a^2 u = \phi(x,y), \tag{3.1}$$

where $u(x,y)$ is a solution of (1.15), which belongs to $H^2(\mathbb{R}^{d+3})$. Clearly, all the terms in the left side of (3.1) starting from the second one are square integrable because by means of Assumption 1 our scalar potential function is bounded along with $|\nabla_y U(y)|$ and $\Delta_y U(y)$ and $u(x,y) \in H^2(\mathbb{R}^{d+3})$. The right side of (3.1) is square integrable as well as assumed. Hence, $\Delta^2 u \in L^2(\mathbb{R}^{d+3})$, which implies that $u(x,y) \in H^4(\mathbb{R}^{d+3})$.

To establish the uniqueness of solutions for our equation, we suppose that (1.15) possesses two solutions $u_1(x,y),\ u_2(x,y) \in H^4(\mathbb{R}^{d+3})$. Then their difference $w(x,y) := u_1(x,y) - u_2(x,y) \in H^4(\mathbb{R}^{d+3})$ satisfies the equation

$$L_U w = a^2 w.$$

Apparently, the operator $L_U : H^4(\mathbb{R}^{d+3}) \to L^2(\mathbb{R}^{d+3})$ has no nontrivial eigenfunctions as discussed above. Thus, $w(x,y)$ vanishes in $\mathbb{R}^{d+3}$.

We apply the generalized Fourier transform (1.17) to both sides of problem (1.15) and obtain

$$\tilde{\hat{u}}(k,q) = \tilde{\hat{G}}_1(k,q) + \tilde{\hat{G}}_2(k,q), \tag{3.2}$$

where

$$\tilde{\hat{G}}_1(k,q) := \frac{\tilde{\hat{\phi}}(k,q)}{2a(k^2+q^2-a)}, \quad \tilde{\hat{G}}_2(k,q) := -\frac{\tilde{\hat{\phi}}(k,q)}{2a(k^2+q^2+a)}.$$

Clearly, the functions $G_1(x,y)$ and $G_2(x,y)$ solve the equations

$$\{-\Delta_x - \Delta_y + U(y)\}G_1 - aG_1 = \frac{1}{2a}\phi(x,y) \tag{3.3}$$

and

$$\{-\Delta_x - \Delta_y + U(y)\}G_2 + aG_2 = -\frac{1}{2a}\phi(x,y) \tag{3.4}$$

respectively. The operator involved in the left side of equation (3.4) has a bounded inverse $l_U^{-1} : L^2(\mathbb{R}^{d+3}) \to H^2(\mathbb{R}^{d+3})$ as discussed above and the right side of (3.4) is square integrable due to the one of our assumptions. Hence, problem (3.4) possesses a unique solution $G_2(x,y) \in H^2(\mathbb{R}^{d+3})$. By virtue of the part a) of Theorem 6 of [26], under the given assumptions equation (3.3) admits a unique solution $G_1(x,y) \in H^2(\mathbb{R}^{d+3})$ if and only if orthogonality relation (1.18) holds. Evidently, the solvability of equation (3.3) in $L^2(\mathbb{R}^{d+3})$ is equivalent to its solvability in $H^2(\mathbb{R}^{d+3})$ because the right side of (3.3) is square integrable and the scalar potential involved in (3.3) is bounded due to our assumptions. ■

We finish the main part of the work with establishing the solvability in the sense of sequences for our problem when the free Laplacian is added to a three dimensional Schrödinger operator.

Proof of Theorem 5. First of all we establablish that if $u(x,y)$ and $u_n(x,y),\ n\in\mathbb{N}$ are the unique $H^4(\mathbb{R}^{d+3})$ solutions of equations (1.15) and (1.19) respectively and $u_n(x,y)\to u(x,y)$ in $H^2(\mathbb{R}^{d+3})$ as $n\to\infty$, then $u_n(x,y)\to u(x,y)$ in $H^4(\mathbb{R}^{d+3})$ as $n\to\infty$ as well. Clearly, (1.15) and (1.19) imply that for $n\in\mathbb{N}$ and $x\in\mathbb{R}^d,\ y\in\mathbb{R}^3,\ d\in\mathbb{N}$

$$\{-\Delta_x-\Delta_y+U(y)\}^2(u_n-u)-a^2(u_n-u)=\phi_n(x,y)-\phi(x,y).$$

Hence

$$\Delta^2(u_n-u)+U^2(y)(u_n-u)-2U(y)\Delta(u_n-u)-(u_n-u)\Delta_yU(y)-$$
$$-2\nabla_yU(y).\nabla_y(u_n-u)-a^2(u_n-u)=\phi_n(x,y)-\phi(x,y). \tag{3.5}$$

The fact that $u_n(x,y)\to u(x,y)$ in $H^2(\mathbb{R}^{d+3})$ as $n\to\infty$ as assumed implies that

$$u_n(x,y)\to u(x,y),\quad \nabla_yu_n(x,y)\to\nabla_yu(x,y),\quad \Delta u_n(x,y)\to\Delta u(x,y)$$

in $L^2(\mathbb{R}^{d+3})$ as $n\to\infty$ and

$$U(y),\quad |\nabla_yU(y)|,\quad \Delta_yU(y)$$

are bounded functions via Assumption 1. Thus, all the terms in the left side of (3.5) starting from the second one converge to zero in $L^2(\mathbb{R}^{d+3})$ as $n\to\infty$. The right side of (3.5) tends to zero in $L^2(\mathbb{R}^{d+3})$ as $n\to\infty$ via the one of our assumptions. Therefore, $\Delta^2u_n(x,y)\to\Delta^2u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n\to\infty$. By virtue of norm definition (1.8) we have that $u_n(x,y)\to u(x,y)$ in $H^4(\mathbb{R}^{d+3})$ as $n\to\infty$.

By means of Theorem 4 above, under our assumptions problem (1.19) has a unique solution $u_n(x,y)\in H^4(\mathbb{R}^{d+3}),\ n\in\mathbb{N}$. We recall formula (3.6) in the proof of Theorem 3 of [29]. Thus, under the given conditions we obtain the limiting orthogonality relation

$$\left(\phi(x,y),\frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y))\right)_{L^2(\mathbb{R}^{d+3})}=0,\quad (k,q)\in S^{d+3}_{\sqrt{a}}\quad a.e.$$

Therefore, by virtue of Theorem 4 above equation (1.15) admits a unique solution $u(x,y) \in H^4(\mathbb{R}^{d+3})$. We apply the generalized Fourier transform (1.17) to both sides of equations (1.15) and (1.19). This gives us the representation (3.2) given in the proof of Theorem 4, where the functions $G_1(x,y),\ G_2(x,y) \in H^2(\mathbb{R}^{d+3})$ under our assumptions are the unique solutions of problems (3.3) and (3.4) respectively. Apparently,

$$\tilde{\hat{u}}_n(k,q) = \tilde{\hat{G}}_{1,n}(k,q) + \tilde{\hat{G}}_{2,n}(k,q), \quad n \in \mathbb{N}, \tag{3.6}$$

where

$$\tilde{\hat{G}}_{1,n}(k,q) := \frac{\tilde{\hat{\phi}}_n(k,q)}{2a(k^2+q^2-a)}, \quad \tilde{\hat{G}}_{2,n}(k,q) := -\frac{\tilde{\hat{\phi}}_n(k,q)}{2a(k^2+q^2+a)}.$$

Evidently, the functions $G_{1,n}(x,y)$ and $G_{2,n}(x,y)$ satisfy the equations

$$\{-\Delta_x - \Delta_y + U(y)\}G_{1,n} - aG_{1,n} = \frac{1}{2a}\phi_n(x,y) \tag{3.7}$$

and

$$\{-\Delta_x - \Delta_y + U(y)\}G_{2,n} + aG_{2,n} = -\frac{1}{2a}\phi_n(x,y) \tag{3.8}$$

respectively. Because the operator involved in the left side of (3.8) has a bounded inverse $l_U^{-1} : L^2(\mathbb{R}^{d+3}) \to H^2(\mathbb{R}^{d+3})$, such that its norm $\|l_U^{-1}\| < \infty$ as discussed above and the right side of (3.8) is square integrable due to the one of our assumptions, (3.8) has a unique solution $G_{2,n}(x,y) \in H^2(\mathbb{R}^{d+3})$. Since $\phi_n(x,y) \to \phi(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$ as assumed, we obtain

$$\|G_{2,n} - G_2\|_{H^2(\mathbb{R}^{d+3})} \leq \frac{1}{2a}\|l_U^{-1}\|\|\phi_n - \phi\|_{L^2(\mathbb{R}^{d+3})} \to 0, \quad n \to \infty.$$

Hence, $G_{2,n}(x,y) \to G_2(x,y)$ in $H^2(\mathbb{R}^{d+3})$ as $n \to \infty$. By means of the result of the part a) of Theorem 3 of [29], problem (3.7) admits a unique solution $G_{1,n}(x,y) \in H^2(\mathbb{R}^{d+3})$, such that $G_{1,n}(x,y) \to G_1(x,y)$ in $H^2(\mathbb{R}^{d+3})$ as $n \to \infty$. By virtue of formulas (3.6) and (3.2) considered in the x, y space, we easily derive $\|u_n(x,y) - u(x,y)\|_{H^2(\mathbb{R}^{d+3})} \leq$

$$\|G_{1,n}(x,y) - G_1(x,y)\|_{H^2(\mathbb{R}^{d+3})} + \|G_{2,n}(x,y) - G_2(x,y)\|_{H^2(\mathbb{R}^{d+3})} \to 0$$

as $n \to \infty$. This implies that, $u_n(x,y) \to u(x,y)$ in $H^4(\mathbb{R}^{d+3})$ as $n \to \infty$. ■

We would like to emphasize especially that in the applications the sum of the free negative Laplacian and the Schrödinger type operator has the physical meaning of the cumulative hamiltonian of the two non-interacting quantum particles. One of these particles moves freely and another interacts with an external potential.

Let us finish the article by considering equation (1.15) with $a = 0$ in the context of the solvability in the sense of sequences.

4. Appendix

Theorem 6. *Let the potential function $U(y)$ satisfy Assumption 1, $a = 0$ and $\phi(x, y) \in L^2(\mathbb{R}^{d+3}),\ d \in \mathbb{N},\ d \geq 4$.*

a) When $d = 4, 5$, let in addition $|x|\phi(x, y),\ |y|\phi(x, y) \in L^1(\mathbb{R}^{d+3})$. Then problem (1.15) has a unique solution $u(x, y) \in H^4(\mathbb{R}^{d+3})$ if and only if

$$(\phi(x, y), \eta_0(y))_{L^2(\mathbb{R}^{d+3})} = 0. \tag{4.1}$$

b) When $d \geq 6$, let in addition $\phi(x, y) \in L^1(\mathbb{R}^{d+3})$. Then equation (1.15) admits a unique solution $u(x, y) \in H^4(\mathbb{R}^{d+3})$.

Proof. Analogously to the argument in the proof of Theorem 4 above, it would be sufficient here to solve equation (1.15) when $a = 0$ in $H^2(\mathbb{R}^{d+3})$, since this solution will belong to $H^4(\mathbb{R}^{d+3})$. The uniqueness of the solutions of problem (1.15) for $a = 0$ was established in the proof of Theorem 4 as well. Let

$$v := -\Delta_x u - \Delta_y u + U(y)u. \tag{4.2}$$

If it is known that $u(x, y), v(x, y) \in L^2(\mathbb{R}^{d+3})$ then $\Delta u(x, y) \in L^2(\mathbb{R}^{d+3})$, since the scalar potential $U(y)$ is bounded as assumed. Then we have $u(x, y) \in H^2(\mathbb{R}^{d+3})$.

Let us apply the generalized Fourier transform (1.17) to both sides of equation (1.15). This yields

$$\tilde{\hat{u}}(k, q) = \frac{\tilde{\hat{\phi}}(k, q)}{(k^2 + q^2)^2} = \frac{\tilde{\hat{\phi}}(k, q)}{(k^2 + q^2)^2}\chi_{\{k^2+q^2\leq 1\}} + \frac{\tilde{\hat{\phi}}(k, q)}{(k^2 + q^2)^2}\chi_{\{k^2+q^2>1\}}. \tag{4.3}$$

Here and below χ_A stands for the characteristic function of a set $A \subseteq \mathbb{R}^{d+3}$. Clearly, the second term in the right side of (4.3) can be bounded from above in the absolute value by $|\tilde{\hat{\phi}}(k, q)| \in L^2(\mathbb{R}^{d+3})$ due to the one of our assumptions. We apply the generalized Fourier transform (1.17) to both sides of (4.2) and use (4.3) to obtain

$$\tilde{\hat{v}}(k, q) = \frac{\tilde{\hat{\phi}}(k, q)}{k^2 + q^2}\chi_{\{k^2+q^2\leq 1\}} + \frac{\tilde{\hat{\phi}}(k, q)}{k^2 + q^2}\chi_{\{k^2+q^2>1\}}. \tag{4.4}$$

Evidently, the second term in the right side of (4.4) can be estimated from above in the absolute value by $|\tilde{\hat{\phi}}(k, q)| \in L^2(\mathbb{R}^{d+3})$. The first term in the right side of (4.4) can be trivially bounded from above in the absolute value by virtue of (4.3) by

$$\frac{|\tilde{\hat{\phi}}(k, q)|}{(k^2 + q^2)^2} = |\tilde{\hat{u}}(k, q)|.$$

Therefore, if we have $u(x, y) \in L^2(\mathbb{R}^{d+3})$ then $v(x, y) \in L^2(\mathbb{R}^{d+3})$ as well.

Let us first treat the case when the dimension $d \geq 6$. By means of Corollary 2.2 of [27] under the given assumptions we have

$$\|\eta_q(y)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(U)} < \infty, \quad q \in \mathbb{R}^3, \tag{4.5}$$

such that via (1.17) we obtain

$$|\tilde{\hat{\phi}}(k, q)| \leq \frac{1}{(2\pi)^{\frac{d+3}{2}}} \frac{1}{1 - I(U)} \|\phi(x, y)\|_{L^1(\mathbb{R}^{d+3})}.$$

This allows us to estimate the first term in the right side of (4.3) in the norm as

$$\left\| \frac{\tilde{\hat{\phi}}(k, q)}{(k^2 + q^2)^2} \chi_{\{k^2+q^2 \leq 1\}} \right\|_{L^2(\mathbb{R}^{d+3})} \leq \frac{1}{(2\pi)^{\frac{d+3}{2}}} \frac{\|\phi(x, y)\|_{L^1(\mathbb{R}^{d+3})}}{1 - I(U)} \sqrt{\frac{|S^{d+3}|}{d - 5}} < \infty$$

due to our assumptions. By virtue of the argument above this completes the proof of the part b) of our theorem.

Then we turn our attention to the situation when the dimension $d = 4, 5$. Let us use the expansion

$$\tilde{\hat{\phi}}(k, q) = \tilde{\hat{\phi}}(0) + \int_0^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\hat{\phi}}(s, \sigma)}{\partial s} ds. \tag{4.6}$$

Here and below σ will denote the angle variables on the sphere. By means of (4.6) we write the first term in the right side of (4.3) as

$$\frac{\tilde{\hat{\phi}}(0)}{(k^2 + q^2)^2} \chi_{\{k^2+q^2 \leq 1\}} + \frac{\int_0^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\hat{\phi}}(s,\sigma)}{\partial s} ds}{(k^2 + q^2)^2} \chi_{\{k^2+q^2 \leq 1\}}. \tag{4.7}$$

Lemma 12 of [26] yields that under the given conditions $(\nabla_k + \nabla_q)\tilde{\hat{\phi}}(k, q) \in L^\infty(\mathbb{R}^{d+3})$, such that the second term in (4.7) can be bounded from above in the absolute value by

$$\frac{\|(\nabla_k + \nabla_q)\tilde{\hat{\phi}}(k, q)\|_{L^\infty(\mathbb{R}^{d+3})}}{(k^2 + q^2)^{\frac{3}{2}}} \chi_{\{k^2+q^2 \leq 1\}}.$$

Hence

$$\left\| \frac{\int_0^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\hat{\phi}}(s,\sigma)}{\partial s} ds}{(k^2 + q^2)^2} \chi_{\{k^2+q^2 \leq 1\}} \right\|_{L^2(\mathbb{R}^{d+3})} \leq$$

$$\leq \|(\nabla_k + \nabla_q)\tilde{\hat{\phi}}(k, q)\|_{L^\infty(\mathbb{R}^{d+3})} \sqrt{\frac{|S^{d+3}|}{d - 3}} < \infty.$$

It can be easily verified that the first term in (4.7) is square integrable if and only if $\hat{\tilde{\phi}}(0)$ vanishes, which is equivalent to orthogonality condition (4.1). By means of the argument above this completes the proof of the part a) of the theorem. ∎

The final proposition of the article is as follows.

Theorem 7. *Let the potential function $U(y)$ satisfy Assumption 1, $a = 0,\ n \in \mathbb{N}$ and $\phi_n(x, y) \in L^2(\mathbb{R}^{d+3}),\ d \in \mathbb{N},\ d \geq 4$, such that $\phi_n(x, y) \to \phi(x, y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$.*

a) When $d = 4, 5$, let in addition $|x|\phi_n(x, y),\ |y|\phi_n(x, y) \in L^1(\mathbb{R}^{d+3})$, such that $|x|\phi_n(x, y) \to |x|\phi(x, y),\ |y|\phi_n(x, y) \to |y|\phi(x, y)$ in $L^1(\mathbb{R}^{d+3})$ as $n \to \infty$ and the orthogonality conditions

$$(\phi_n(x, y), \eta_0(y))_{L^2(\mathbb{R}^{d+3})} = 0 \tag{4.8}$$

hold for all $n \in \mathbb{N}$. Then equations (1.15) and (1.19) possess unique solutions $u(x, y) \in H^4(\mathbb{R}^{d+3})$ and $u_n(x, y) \in H^4(\mathbb{R}^{d+3})$ respectively, such that $u_n(x, y) \to u(x, y)$ in $H^4(\mathbb{R}^{d+3})$ as $n \to \infty$.

b) When $d \geq 6$, let in addition $\phi_n(x, y) \in L^1(\mathbb{R}^{d+3})$, such that $\phi_n(x, y) \to \phi(x, y)$ in $L^1(\mathbb{R}^{d+3})$ as $n \to \infty$. Then problems (1.15) and (1.19) admit unique solutions $u(x, y) \in H^4(\mathbb{R}^{d+3})$ and $u_n(x, y) \in H^4(\mathbb{R}^{d+3})$ respectively, such that $u_n(x, y) \to u(x, y)$ in $H^4(\mathbb{R}^{d+3})$ as $n \to \infty$.

Proof. Our argument will consist out of several steps.

I) Suppose $u(x, y)$ and $u_n(x, y),\ n \in \mathbb{N}$ are the unique $H^4(\mathbb{R}^{d+3})$ solutions of equations (1.15) and (1.19) with $a = 0$ respectively and $u_n(x, y) \to u(x, y)$ in $H^2(\mathbb{R}^{d+3})$ as $n \to \infty$. Then $u_n(x, y) \to u(x, y)$ in $H^4(\mathbb{R}^{d+3})$ as $n \to \infty$ as well, which can be shown similarly to the argument in the proof of Theorem 5 above.

II) Let us introduce

$$v_n := -\Delta_x u_n - \Delta_y u_n + U(y)u_n, \quad n \in \mathbb{N}. \tag{4.9}$$

By means of (4.9) and (4.2), we easily obtain

$$\|\Delta u_n - \Delta u\|_{L^2(\mathbb{R}^{d+3})} \leq \|v_n - v\|_{L^2(\mathbb{R}^{d+3})} + \|U(y)(u_n - u)\|_{L^2(\mathbb{R}^{d+3})}. \tag{4.10}$$

Suppose $v(x, y), v_n(x, y) \in L^2(\mathbb{R}^{d+3}), n \in \mathbb{N}$, such that $v_n(x, y) \to v(x, y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$. Moreover, let $u(x, y), u_n(x, y) \in H^4(\mathbb{R}^{d+3}), n \in \mathbb{N}$, such that $u_n(x, y) \to u(x, y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$. Since $U(y) \in L^\infty(\mathbb{R}^3)$ as assumed, the second term in the right side of (4.10) tends to zero as $n \to \infty$. Thus, $\Delta u_n \to \Delta u$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$, which implies that $u_n(x, y) \to u(x, y)$ in $H^2(\mathbb{R}^{d+3})$ as $n \to \infty$ as well.

III) We apply the generalized Fourier transform (1.17) to both sides of problem (1.19) with $a = 0$. This gives us

$$\tilde{\hat{u}}_n(k,q) = \frac{\tilde{\hat{\phi}}_n(k,q)}{(k^2+q^2)^2}. \tag{4.11}$$

Let us apply the generalized Fourier transform (1.17) to both sides of (4.9) and use (4.11). This yields

$$\tilde{\hat{v}}_n(k,q) = \frac{\tilde{\hat{\phi}}_n(k,q)}{k^2+q^2} = \frac{\tilde{\hat{\phi}}_n(k,q)}{k^2+q^2}\chi_{\{k^2+q^2\le 1\}} + \frac{\tilde{\hat{\phi}}_n(k,q)}{k^2+q^2}\chi_{\{k^2+q^2>1\}}. \tag{4.12}$$

Evidently, the second term in the right side of (4.12) can be bounded from above in the absolute value by $|\tilde{\hat{\phi}}_n(k,q)| \in L^2(\mathbb{R}^{d+3})$ via the one of our assumptions. The first term in the right side of (4.12) can be trivially estimated from above in the absolute value using (4.11) by

$$\frac{|\tilde{\hat{\phi}}_n(k,q)|}{(k^2+q^2)^2} = |\tilde{\hat{u}}_n(k,q)|.$$

Hence, $u_n(x,y) \in H^4(\mathbb{R}^{d+3})$ will imply that $v_n(x,y) \in L^2(\mathbb{R}^{d+3})$. By the similar reasoning via (4.3) and (4.4) we easily deduce that $u(x,y) \in H^4(\mathbb{R}^{d+3})$ yields $v(x,y) \in L^2(\mathbb{R}^{d+3})$.

IV) Formulas (4.4) and (4.12) give us $\tilde{\hat{v}}_n(k,q) - \tilde{\hat{v}}(k,q) =$

$$= \frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{k^2+q^2}\chi_{\{k^2+q^2\le 1\}} + \frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{k^2+q^2}\chi_{\{k^2+q^2>1\}}. \tag{4.13}$$

Clearly, the second term in the right side of (4.13) can be bounded above in the absolute value by $|\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)|$. Thus

$$\left\| \frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{k^2+q^2}\chi_{\{k^2+q^2>1\}} \right\|_{L^2(\mathbb{R}^{d+3})} \le \|\phi_n(x,y) - \phi(x,y)\|_{L^2(\mathbb{R}^{d+3})} \to 0$$

as $n \to \infty$ due to the one of our assumptions. The first term in the right side of (4.13) can be easily estimated from above in the absolute value using (4.11) and (4.3) by

$$\frac{|\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)|}{(k^2+q^2)^2} = |\tilde{\hat{u}}_n(k,q) - \tilde{\hat{u}}(k,q)|.$$

Therefore,

$$\left\| \frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{k^2+q^2}\chi_{\{k^2+q^2\le 1\}} \right\|_{L^2(\mathbb{R}^{d+3})} \le \|u_n(x,y) - u(x,y)\|_{L^2(\mathbb{R}^{d+3})}.$$

Hence, if $u(x,y), u_n(x,y) \in H^4(\mathbb{R}^{d+3}), n \in \mathbb{N}$, such that $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$, then we arrive at $v_n(x,y) \to v(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$ as well.

V) Let us first discuss the situation when $d \geq 6$. By means of the part b) of Theorem 6 above, under the given conditions equations (1.15) and (1.19) admit unique solutions $u(x,y), u_n(x,y) \in H^4(\mathbb{R}^{d+3}), n \in \mathbb{N}$ respectively. By virtue of (4.11) and (4.3), we obtain $\tilde{\hat{u}}_n(k,q) - \tilde{\hat{u}}(k,q) =$

$$= \frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{(k^2+q^2)^2}\chi_{\{k^2+q^2 \leq 1\}} + \frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{(k^2+q^2)^2}\chi_{\{k^2+q^2 > 1\}}. \tag{4.14}$$

Clearly, the second term in the right side of (4.14) can be bounded above in the absolute value by $|\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)|$, such that

$$\left\| \frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{(k^2+q^2)^2}\chi_{\{k^2+q^2 > 1\}} \right\|_{L^2(\mathbb{R}^{d+3})} \leq \|\phi_n(x,y) - \phi(x,y)\|_{L^2(\mathbb{R}^{d+3})} \to 0$$

as $n \to \infty$ via the one of the given conditions. By means of (1.17) along with (4.5) we easily derive

$$|\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)| \leq \frac{1}{(2\pi)^{\frac{d+3}{2}}} \frac{1}{1 - I(U)} \|\phi_n(x,y) - \phi(x,y)\|_{L^1(\mathbb{R}^{d+3})}.$$

Thus

$$\left\| \frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{(k^2+q^2)^2}\chi_{\{k^2+q^2 \leq 1\}} \right\|_{L^2(\mathbb{R}^{d+3})} \leq$$

$$\leq \frac{1}{(2\pi)^{\frac{d+3}{2}}} \frac{1}{1 - I(U)} \sqrt{\frac{|S^{d+3}|}{d-5}} \|\phi_n(x,y) - \phi(x,y)\|_{L^1(\mathbb{R}^{d+3})} \to 0, \quad n \to \infty$$

due to our assumptions. Therefore, by virtue of (4.14) we arrive at $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$. Steps III) and IV) above give us $v(x,y), v_n(x,y) \in L^2(\mathbb{R}^{d+3}), n \in \mathbb{N}$ and $v_n(x,y) \to v(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$. Then by means of step II) we have $u_n(x,y) \to u(x,y)$ in $H^2(\mathbb{R}^{d+3})$ as $n \to \infty$. Finally, step I) above yields $u_n(x,y) \to u(x,y)$ in $H^4(\mathbb{R}^{d+3})$ as $n \to \infty$, which completes the proof of the part b) of our theorem.

VI) Let us proceed to establishing the result of the theorem in the situation when $d = 4, 5$. We use orthogonality relations (4.8) along with inequality (4.5) to obtain

$$|(\phi(x,y), \eta_0(y))_{L^2(\mathbb{R}^{d+3})}| = |(\phi(x,y) - \phi_n(x,y), \eta_0(y))_{L^2(\mathbb{R}^{d+3})}| \leq$$

$$\leq \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(U)} \|\phi_n(x,y) - \phi(x,y)\|_{L^1(\mathbb{R}^{d+3})} \to 0, \quad n \to \infty.$$

Note that under the stated assumptions $\phi_n(x,y) \in L^1(\mathbb{R}^{d+3})$ and $\phi_n(x,y) \to \phi(x,y)$ in $L^1(\mathbb{R}^{d+3})$ as $n \to \infty$ (see the proof of Theorem 3 of [29]). Therefore, the limiting orthogonality condition

$$(\phi(x,y), \eta_0(y))_{L^2(\mathbb{R}^{d+3})} = 0 \tag{4.15}$$

holds. By means of the result of the part a) of Theorem 6 above, equations (1.15) and (1.19) possess unique solutions $u(x,y), u_n(x,y) \in H^4(\mathbb{R}^{d+3}), n \in \mathbb{N}$. Orthogonality relations (4.8) and (4.15) along with definition (1.17) give us

$$\tilde{\hat{\phi}}(0) = 0, \quad \tilde{\hat{\phi}}_n(0) = 0, \quad n \in \mathbb{N}. \tag{4.16}$$

(4.16) enables us to express

$$\tilde{\hat{\phi}}(k,q) = \int_0^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\hat{\phi}}(s,\sigma)}{\partial s} ds, \quad \tilde{\hat{\phi}}_n(k,q) = \int_0^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\hat{\phi}}_n(s,\sigma)}{\partial s} ds, \tag{4.17}$$

where $n \in \mathbb{N}$. The second term in the right side of (4.14) tends to zero in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$ as discussed in step V) above. By virtue of (4.17) we write the first term in the right side of (4.14) as

$$\frac{\int_0^{\sqrt{k^2+q^2}} \left[\frac{\partial \tilde{\hat{\phi}}_n(s,\sigma)}{\partial s} - \frac{\partial \tilde{\hat{\phi}}(s,\sigma)}{\partial s}\right] ds}{(k^2+q^2)^2} \chi_{\{k^2+q^2 \le 1\}}. \tag{4.18}$$

Clearly, (4.18) can be bounded above in the absolute value by

$$\frac{\|(\nabla_k + \nabla_q)(\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q))\|_{L^\infty(\mathbb{R}^{d+3})}}{(k^2+q^2)^{\frac{3}{2}}} \chi_{\{k^2+q^2 \le 1\}}.$$

Note that under the given conditions we have

$$\|(\nabla_k + \nabla_q)(\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q))\|_{L^\infty(\mathbb{R}^{d+3})} \to 0, \quad n \to \infty,$$

which is the result of the part b) of Lemma 5 of [29]. Hence

$$\left\| \frac{\int_0^{\sqrt{k^2+q^2}} \left[\frac{\partial \tilde{\hat{\phi}}_n(s,\sigma)}{\partial s} - \frac{\partial \tilde{\hat{\phi}}(s,\sigma)}{\partial s}\right] ds}{(k^2+q^2)^2} \chi_{\{k^2+q^2 \le 1\}} \right\|_{L^2(\mathbb{R}^{d+3})} \le$$

$$\le \|(\nabla_k + \nabla_q)(\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q))\|_{L^\infty(\mathbb{R}^{d+3})} \sqrt{\frac{|S^{d+3}|}{d-3}} \to 0, \quad n \to \infty.$$

Therefore, $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$. By means of steps III), IV), II) and I) above we obtain $u_n(x,y) \to u(x,y)$ in $H^4(\mathbb{R}^{d+3})$ as $n \to \infty$, which completes the proof of the part a) of the theorem. ■

Remark 4. *Solvability in the sense of sequences for problem (1.15) with $a = 0$ in lower dimensions $d < 4$ will be addressed in our consecutive work.*

Remark 5. *Note that no orthogonality conditions are needed to solve equation (1.15) with $a = 0$ in $H^4(\mathbb{R}^{d+3})$ in higher dimensions $d \geq 6$. As distinct from the present case, when dealing with the sum of the free Laplacian and our three dimensional Schrödinger operator to the first power for $a = 0$, the solvability relations are not needed for all $d \geq 2$ (see Theorem 6 of [26] and Theorem 3 of [29]).*

Remark 6. *Our approach can be extended to the higher, even order elliptic equations. For example, in the case of the sixth order operator $\{-\Delta_x + V(x) - \Delta_y + U(y)\}^3$ we can check for the analog of Assumption 1 of Theorem 3.*

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