An outlook of shape functions and $f(R)$ gravity for Morris-Thorne wormhole solutions

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Recent investigations into wormhole solutions in modified gravity have shown that the primary topological features of such solutions is the "shape function", which determines the topological structure of these wormholes. In particular, $f(R)$ theories of gravity allow us to understand such models and to understand the nature of such solutions by extending General relativity into an arbitrary class of functions of the Ricci scalar. In this paper, we will discuss shape functions of different models and the nature of energy conditions for such models by finding out the energy condition parameters.

I. INTRODUCTION

$f(R)$ theories of gravity are derived by perturbing the approach to GR. Such theories are theories that extend the Einstein-Hilbert action into arbitrary functions of the Ricci scalar [14]. Such theories are in general, a part of family of such modified theories built of curvature invariants C_n, where C_1 is R, C_2 is $R_{\mu\nu}R^{\mu\nu}$, C_3 is $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ and so on. $f(R)$ theories of gravity are made of a modified Einstein-Hilbert action of the form

$$S = \frac{1}{2\kappa}\int f(R)\, \mathrm{d}^4x\sqrt{-g} + S_{\mathrm{matter}}[g_{\mu\nu},\ \Psi] \tag{1}$$

Where Ψ are the matter fields on $(M,\ g)$. In the metric formalism, we derive the field equations for our theory of gravity by directly varying the action with respect to the metric, which is different than the approach in the Palatini formalism where we consider the metric and the connection as two independent terms. The general form of field equations for $f(R)$ gravity is of the form

$$f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) - [\nabla_\mu\nabla\nu - g_{\mu\nu}\Box]f'(R) = T_{\mu\nu} \tag{2}$$

This can be written in terms of the "effective" energy momentum tensor $T^{eff}_{\mu\nu}$, which is a combination of $T^{c}_{\mu\nu}$ and $\tilde{T}^{m}_{\mu\nu}$ as:

$$G_{\mu\nu} = T^{eff}_{\mu\nu}$$

Wormholes are topological structures joining two points of same or two different universes. Such geometric solutions lack a singularity or a horizon, and are defined in terms of the shape function $b(r)$ and the redshift function $\chi(r)$. Predicted initially in GR, such solutions require a violation of the Null Energy condition (NEC), characterised by their requirement of "exotic matter" that contribute to a negative nature of the energy momentum tensor. However, modified theories of gravity allow us to understand wormholes without exotic matter.

The kind of wormholes we wish to understand are spherically symmetric and have a static metric, i.e. $\partial_0 g_{\mu\nu} = 0$. Such wormholes would have a metric of the form

$$ds^2 = -\chi(r)dt^2 + dr^2\left(1 - \frac{b(r)}{r}\right)^{-1} + r^2 d\Omega^2 \tag{3}$$

With $\chi(r)$ being the redshift function and $b(r)$ the shape function. The shape function determines the geometry of the wormhole and satisfy the following conditions:

1. $b(r_0) = r_0$,
2. $\left(1 - \frac{b(r)}{r}\right) > 0$ for $r > r_0$,
3. $b;(r_0) < 1$ (flare-out condition),
4. $\lim_{r\to\infty}\frac{b(r)}{r} = 0$ (asymptotic flatness condition)

In the $f(R)$ background, we consider different forms of $f(R)$ models along with $b(r)$ and $\chi(r)$ for determining wormhole solutions. In this paper, our interest is in understanding a case of Nojiri-Odintsov, Amendole et al, and in deriving $f(R)$ from a chosen shape function.

The basic research of wormholes in GR has a feature of violating the energy conditions. This is because a negative energy-density is an elementary complication of wormholes. In $f(R)$ gravity, we can instead ignore this feature and define the physical properties of wormholes using a purely geometric correction. One of the first research showed that this was possible was by Starobinskii. We will study the boundaries of energy-conditions in $f(R)$ wormholes.

The energy conditions are built on the nature of terms appearing in $T_{\mu\nu}$ [3]. The following are the energy conditions for a perfect fluid:

- Null energy condition (NEC): $\rho + p_r \geq 0$, $\rho + p_t \geq 0$,
- Strong energy condition (SEC): $\rho + p_r + 2p_t \geq 0$,
- Weak energy condition (WEC): $\rho \geq 0$, $\rho + p_r \geq 0$, $\rho + p_t \geq 0$
- Dominant energy condition (DEC): $\rho \geq 0$, $\rho + |p_r| \geq 0$, $\rho - |p_r| \geq 0$

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Where ρ represents the energy-density, p_r denotes the radial pressure and p_t the tangential pressure. The energy-momentum tensor of perfect fluids is of the form

$$T_{\mu\nu} = (\rho + p_t u_\mu u_\nu) - p_t g_{\mu\nu} + (p_r - p_t) X_\mu X_\nu \tag{4}$$

The trace of $T_{\mu\nu}$ is of the form $T = p_r - \rho + 2p_t$. The terms that we will calculate for wormhole solutions are of the form $\rho + p_r$, $\rho + p_t$, $\rho - |p_r|$, $\rho - |p_t|$, $\rho + p_r + 2p_t$ and $p_t - p_r$, which will help us in understanding the energy conditions for each model.

The Einstein field equations for the metric would be in terms of ρ, p_t and p_r as follows:

$$\rho + F\frac{b'(r)}{r^2} - \left(1 - \frac{b(r)}{r}\right) F'\chi'(r) - \mathcal{H} \tag{5}$$

$$p_t = F\frac{b(r) - rb'(r)}{2r^3} - \frac{F'}{r}\left(1 - \frac{b(r)}{r}\right) + F\left(1 - \frac{b(r)}{r}\right) \left(\chi''(r) - \frac{(rb'(r)\chi'(r))}{2r^2 - 2rb(r)} + \frac{\chi'(r)}{r} + \chi'^2(r)\right) + \mathcal{H} \tag{6}$$

$$p_r = -F\frac{b(r)}{r^3} + 2\left(1 - \frac{b(r)}{r}\right)\frac{\chi'(r)F}{r} - \left(1 - \frac{b(r)}{r}\right) \left(F'' - \frac{F'(rb'(r) - b(r))}{2r^2 - 2r^2\left(1 - \frac{b(r)}{r}\right)}\right) + \mathcal{H} \tag{7}$$

Where $\mathcal{H} = \frac{1}{4}(FR + \Box F + T)$, $F(R)$ is $\mathrm{d}f(R)/\mathrm{d}R$ and the prime denotes $\partial/\partial r$.

II. NOJIRI-ODINTSOV WORMHOLE SOLUTIONS

Consider the form of $f(R)$ gravity introduced by Nojiri and Odintsov [8], given by

$$f(R) = R + \alpha R^m - \beta R^{-n} \tag{8}$$

This model has many implications in cosmology. Late time acceleration of Friedmann-Robertson cosmologies with such an $f(R)$ background was analyzed by Cao *et al*.

Following [7], we introduce a shape function of the form

$$b(r) = r_0\left(\frac{x^r}{x^{r_0}}\right) \tag{9}$$

with $0 < x < 1$. It is clear that the above shape function satisfies the required conditions for shape functions seen in section I.

We will start by deriving the field equations for this model in terms of ρ, p_r and p_t, which would allow us to calculate the nature of the energy conditions for the model. Recall that if a model always has a non-negative value of $\rho + p_t$ and $\rho + p_r$, it satisfies NEC, while if it always has a non-negative $\rho + p_r + 2p_t$ and a non-negative energy-density ρ, it satisfies SEC and WEC respectively. In our present theory, following [7], we will split the theory into different cases based on the values of the parameters α, β, m, n and x.

Following the required values for NEC, SEC and DEC, we can analyse the nature of the geometry of the model. Following our required physical parameters for calculating the energy conditions, we first consider different classes of this model, by varying the parameters α, β, m, n and x.

A. Recovering GR: $f(R) = R$

Our model, defined by (8) recovers GR when $\alpha = \beta = 0$. In this case, the following would be the nature of the energy conditions:

- Null energy condition (NEC): $\rho + p_r \geq 0$: clearly not satisfied, following the calculations for ρ and p_r. In the tangential pressure component of NEC, we would consider $\rho + p_t \geq 0$. Clearly, this is also not the case, since there would be a negative value of this quantity some point. Therefore, NEC is violated.

- Strong energy condition (SEC): By finding $\rho + p_r + 2p_t \geq 0$, we see that SEC is also violated.

- Dominant energy condition (DEC): The DEC in the radial and tangential components are found to be negative, implying a violation of DEC by the model.

- The anisotropy parameter shows that the geometry of the model is repulsive due to a positive value of it.

Therefore, in the case of the Nojiri-Odintsov model (8) being equivalent to GR, we see that the model is filled with exotic matter.

B. When we set $\beta = 0$

We set the base $x = 0.5$ and $m = 2$. With these conditions, we can plot the energy conditions w.r.t α to derive the following results:

- Null energy condition (NEC): The radial component of the NEC is found to be preserved. However, for suitable values of α, the tangential component of NEC is found to be violated.

- Strong energy condition (SEC): This is found to be violated for r greater than r_0.

- Dominant energy condition (DEC): In both the radial and the tangential components of DEC, the values observed are negative.
- Again, the anisotropy parameter has positive values in certain regions (interestingly at $r = r_0$), and therefore implies a repulsive nature of the model.

Again, we see that this model too is filled with exotic matter as a result of the violation of NEC and DEC and the nature of the anisotropy parameter.

C. Setting $\alpha = 0$

Our assumption following the paper in consideration is setting $x = 0.5$ as before, $n = 4$ and plotting the energy conditions w.r.t β. Interestingly, this model has some intriguing features based on NEC and DEC.

- Null energy condition (NEC): The energy

  density is positive, while the radial and tangential components of NEC are positive, therefore preserving NEC at all times.
- Strong energy condition (SEC): Thec SEC term $\rho + p_r + 2p_t \geq 0$ is positive for all values of $\beta > 20$, and is therefore preserved by this case of our model (8).
- Dominant energy condition (DEC): Both the radial and tangential components of DEC are negative, implying that DEC is not preserved.
- The anisotropy parameter in this case is negative, indicating an attractive framework of the model.

This is a case of an elementary wormhole solution without exotic matter. At the same time, this also implies that it is possible to have a wormhole with non-exotic matter while maintaining SEC and NEC, with a violation of DEC with an attractive nature of the model.

D. The case of non-zero parameters

Our final scenario is considering m, n, α and β being non-zero. Considering the case of the base $x = 0.5$, $m = 2$ and $n = 4$ as considered before individually, we find the following results:

- Null energy condition (NEC): The energy density ρ is positive, while NEC in both the radial and tangential components is positive, implying a preservation of NEC throughout the model.
- Strong energy condition (SEC): The SEC term is again found to be always positive, therefore implying a preservation of SEC too.
- Dominant energy condition (DEC): DEC is again found to be violated in both the radial and tangential components.
- The anisotropy parameter is again found to be negative, implying that the model has an attractive nature.

This model is interesting for a number of reasons – firstly, it is straightforward to look at the nature of the energy conditions. We started by considering an $f(R)$ theory of gravity, and then we considered a shape function (9). Using these results, we were able to understand different energy conditions. Next, we were able to analyse the energy conditions and understand the nature of the wormhole solution in question – here, in each case we found a violation of DEC in both the components. However, we showed that a wormhole solution with a satisfied NEC and SEC with a violation of DEC can exist without exotic matter.

We will now investigate into wormhole solutions with a Tsujikawa and Amendola et al type $f(R)$ background. We will follow [20] and [9] and analyse the energy conditions with the exponential shape function in consideration along with the given $f(R)$ models.

The following are tables associated with the four different cases of Nojiri-Odintsov wormhole solutions examined above. Note that SEC does not have radial/tangential components – the following values imply that SEC is violated in general instead of a particular form of violation or preservation of SEC in radial or tangential components. In the case of Table-1, we analyse the nature of the energy conditions and their violations in Nojiri-Odintsov wormhole solutions in section A. It is to be noted that the anisotropy parameter does not have radial or tangential components and has therefore been left empty in those rows.

Table-1 shows that in the case of $\alpha, \beta = 0$, NEC in radial component and SEC and DEC are violated, while the model has a positive anisotropy parameter, indicating that the model has exotic matter in it. Table-2 shows that in the case of $\beta = 0$ all the energy conditions are violated, with the model exhibiting a positive anisotropy parameter.

Table-3 depicts the case of $\alpha = 0$, where NEC in both radial and tangential components and SEC are preserved. DEC in this case is negative in regions of the wormhole, implying a violation of DEC in both radial and tangential components. In this case, the anisotropy parameter is negative, meaning that the model has an attractive nature. The case of $\alpha, \beta \neq 0$ is depicted in Table-4, where we observe that NEC in both radial and tangential components are preserved, while SEC is also preserved. DEC however, is still violated in both the radial and tangential components, with the anisotropy parameter being negative, again indicating that the nature of the model is attractive.

	NEC	SEC	DEC	Anisotropy parameter
Radial	Violated	Violated	Violated	
Tangential	Preserved	Violated	Violated	
				Positive

Table-1: Violation of energy conditions in the case of the Nojiri-Odintsov model (8) with shape function (9) and $f(R) = R$. In this case, the equation of state (EOS) is positive.

	NEC	SEC	DEC	Anisotropy parameter
Radial	Violated	Violated	Violated	
Tangential	Violated	Violated	Violated	
				Positive

Table-2: Violation of energy conditions in the case of the Nojiri-Odintsov model (8) with shape function (9) and $f(R) = R + \alpha R^m$.

	NEC	SEC	DEC	Anisotropy parameter
Radial	Preserved	Preserved	Violated	
Tangential	Preserved	Preserved	Violated	
				Negative

Table-3: Violation of energy conditions in the case of the Nojiri-Odintsov model (8) with shape function (9) and $f(R) = R - \beta R^{-n}$.

	NEC	SEC	DEC	Anisotropy parameter
Radial	Preserved	Preserved	Violated	
Tangential	Preserved	Preserved	Violated	
				Negative

Table-4: Violation of energy conditions in the case of the Nojiri-Odintsov model (8) with shape function (9) and $f(R) = R + \alpha R^m - \beta R^{-n}$.

In this model, we investigated the nature of the energy conditions in Nojiri-Odintsov models with an exponential shape function. We started by understanding the field equations and then by analysing the parameters to find the nature of the violation of energy conditions. We will follow the same algorithm to investigate Tsujikawa and Amendola et al $f(R)$ models using the shape function introduced by Samanta et al in 2018 [20], and we will investigate energy conditions like we did the previous case of Nojiri-Odintsov wormhole solutions.

III. TSUJIKAWA AND AMENDOLA ET AL WORMHOLE SOLUTIONS

Our next candidate for investigating energy conditions for shape functions in $f(R)$ gravity is that of Amendola-Tsujikawa models [16], defined by

$$f(R) = R - \mu R_A \left(\frac{R}{R_A}\right)^p \tag{10}$$

Where μ, R_A and p are positive constants and an exponent, $0 < p < 1$. In this case, following Samanta and Godani 2020 [9], we will consider the shape function of the form

$$b(r) = r \exp\left(-(r - r_0)\right) \tag{11}$$

In this case, we will consider a redshift function $\chi(r) = 1/r$ for simplicity. In our case, the derivatives of $\chi(r)$ do not vanish, and the field equations are as given in [9].

The following would be the results in the case of (10) with a shape function of the form of (11).

- Null energy condition (NEC): For certain values of r, WEC is indefinite. Due to this, we consider values of r greater than that limit. For this, we would see that the NEC in both the radial and tangential components would be positive above certain values. For values lesser than these, both the components of NEC would be indefinite. Therefore, both WEC and NEC have such properties.

- Strong energy condition (SEC): All values of $\rho + p_r + 2p_t$ are negative for $r \geq 1.2$, implying that SEC is violated.

- Dominant energy condition (DEC): The radial component of DEC is indefinite for $r < 1.2$, while the tangential component of DEC is indefinite for values less than or equal to 1.8.

- The anisotropy parameter is further indefinite for all other values from less than 1.2, otherwise negative values, indicating that the geometry of the model is attractive.

The following table summarises the results of our investigation of the shape function (10) together with (11) with a constant variable parameter-type redshift function in terms of NEC, SEC, DEC and the anisotropy parameters – note that each of the energy conditions are only valid for certain regions of the wormhole geometry.

	NEC	SEC	DEC	Anisotropy parameter
Radial	Preserved for r¿l_r^1	Violated for all $r > l_r$	Preserved for $r > l_r^1$	
Tangential	Preserved for $r > l_r^2$	Violated for all $r > l_r$	Preserved for all $r > l_r^2$	Negative for all $r > l_r^1$

Table-4: The nature of energy conditions with model of the form [16] with [9]

Here, l_r^1 is the first radial limit 1.2 considered in the paper [9] and l_r^2 is the second radial limit 1.8 considered. For all other values of r, the energy conditions parameters are either negative or indefinite.

This is the form of Amendola et al $f(R)$ theory with the shape function (11) introduced by Samanta et al (2018). We will now wish to reverse-engineer such theories of gravity – that is, we will consider a shape function and determine the function $f(R)$ corresponding to the theory with that physical setting. We will consider a case of an equation of state relating ρ with the tangential pressure p_t, and then consider a case of a shape function identical to (11). Using our conditions, we will find out the corresponding form of $f(R)$ for the theory.

IV. FINDING $f(R)$ FROM A GIVEN $b(r)$

We wish to find out the function $f(R)$ for a given shape function and investigate the nature of energy conditions for such wormhole solutions. We wil consider a barotropic fluid model, i.e. a model in which p_t is a function of the energy density ρ:

$$p_t = \omega\rho \tag{12}$$

1. We consider the shape function we wish to find the corresponding $f(R)$ of and get the terms ρ, p_t and p_r.

2. We then solve the field equations to find $T = -\rho + p_r + 2p_t$, $\Box F$ and get the value of $f(R)$.

3. We then find out the nature of the energy conditions terms $\rho \pm p_{t,\,r}$, $\rho \pm |p_{t\,r}|$ and $\rho + p_r + 2p_t$.

We will consider two cases of shape functions and derive $f(R)$ for these models.

A. Setting $b(r) = r\exp(-2(r - r_0))$

We will consider a case of a shape function close to (11) [11],

$$b(r) = r\exp(-2(r - r_0)) \tag{13}$$

For the equation of state (12), we would see that $F(R)$ is given by the following:

$$F'\left(1 - \frac{b(r)}{r}\right) - \frac{F}{2r^2}(b(r) - (1 + 2\omega)rb'(r)) = 0 \tag{14}$$

Using (14) for the shape function (13), we would get the value of F, where and $L = (-\omega\{r(\exp(2(r - r_0))) - 1\})$

$$F = (1 - \exp(-2(r - r_0)))^{\frac{2\omega+1}{2}} \times \exp\left(\int L(r)\, dr\right) \tag{15}$$

From this, the equations for ρ, p_t and p_r can be calculated from (5-7) – the values are given by (16-18), where $A = \exp(-2(r - r_0))$ and $B = \exp(2(r - r_0))$.

$$\rho = A(1 - A)^{\omega+\frac{1}{2}}\frac{1 - 2r}{r^2} \times \exp\int (L(r))\, dr \tag{16}$$

$$p_t = \omega\left(\frac{1 - 2r}{r^2}\right)(1 - A)^{\omega+\frac{1}{2}} \times \exp(\int L(r)\, dr) \tag{17}$$

$$p_r = -\rho\left[\frac{1}{1 - 2r}\left(1 + \frac{r^2(r(1 + 2\omega) - \omega)}{r \times B - 1}\right) + \frac{r^2 \times B}{1 - 2r}\right. \\ \left(\frac{(1 + 2\omega)^2 - 2(1 + 2\omega)B}{(B - 1)^2} + \frac{\omega^2 + \omega((2r + 1) \times B - 1)}{r^2(B - 1)^2} + \right. \\ \frac{2\omega(1 + 2\omega)}{r(B - 1)^2} \tag{18}$$

We can find T and $\Box F$ and get the field equations. Note that the Ricci scalar is $R = \frac{2b'(r)}{r^2}$, which for our shape function (13) gives $R = \frac{2(1-2r)\exp(2(r-r_0))}{r^2}$. We then see that $f(R)$ is of the form

$$f(R) = \frac{F}{2}\Bigg[\frac{2(1-2r)\times B}{r^2} + 2(1-A)\left(\frac{(1+2\omega)^2 - 2(1+2\omega\times B)}{(B-1)^2} + \frac{\omega^2 + \omega((2r+1)\times B - 1)}{r^2(B-1)^2} - \frac{2(1+2\omega)}{r(B-1)^2}\right) + \frac{r(1+2\omega)-\omega}{r(B-1)}\left(\frac{(-r+2(B-1))}{r\times B}\right) - \frac{1-2r}{r^2}(1-2\omega)\times A + \frac{1}{r^2\times B} + A\frac{(1+2\omega)r-\omega}{r(B-1)} \tag{19}$$

In this case, we found the function $f(R)$ from the shape function and by imposing the condition that the model satisfies the differential equation (14). We will now analyse a model with a different shape function and investigate the energy conditions for such a model.

B. Setting $b(r) = r_0^n r^{(1-n)}$

In the previous case, we derived the function $f(R)$ for the case of a shape function of the form (13). Our procedure was to derive ρ, p_t and p_r after finding F. We then derive the trace of $T_{\mu\nu}$ and $\Box F$, and use the field equations in $f(R)$ gravity to find out the function $f(R)$. We will now consider the case of the shape function of the form [17]

$$b(r) = r_0^n r^{(1-n)} \tag{20}$$

We again require the function F to satisfy the differential equation (14). In this case, F is determined to be of the form

$$F = \Theta(1 - r_0^n r^{-n})^{\frac{n+2\omega(n-1)}{2n}} \tag{21}$$

Where Θ is a constant. The field equations can be shown to give the following values of ρ, p_t and p_r:

$$\rho = \Theta(1-n)r_0^n r^{-(n+2)}(1 - r_0^n r^{-n})^{\frac{n+2\omega(n-1)}{2n}} \tag{22}$$

$$p_t = \Theta\times\omega r_0^n r^{-(n-2)}(1-n)(1 - r_0^n r^{-n})^{\frac{n+2\omega(n-1)}{2n}} \tag{23}$$

$$p_r = \frac{\Theta}{2}\frac{(1 - r_0^n r^{-n})^{\frac{-n+2\omega(n-1)}{2n}}}{r^{2(n+1)}}[r^n\{2 - (n+1)(n+2\omega(n-1))\} + r_0^n\{(n+2\omega(n-1))(n+\omega(n-1)+1) - 2\} \tag{24}$$

We can now derive T and $\Box F$ and get the field equations for this theory. The Ricci scalar in this case is of the form $R = 2(1-n)r_0^n r^{-(n+2)}$. We then can find $f(R)$ to be of the following form (where R_0 is R for $b(r)|_{r=r_0}$):

$$f(R) = \frac{\Theta\times R}{2}\left(1 - \left(\frac{R}{R_0}\right)^{\frac{-n+2\omega(n-1)}{2n}}\right)\times\Bigg[\left(\frac{R}{R_0}\right)^{\frac{n}{n+2}}\left(-\frac{3(1+\omega)}{2}\left(\frac{n+2\omega(n-1)}{2}\right) - \frac{1}{4(1-n)}(5\omega n - 2\omega - 3n - 2\omega^2 - n^2 + 4\omega^2 n + 3\omega n^2 - 2\omega^2 n^2 + 4) + \left(\frac{3}{2}\frac{n+2\omega(n-1)}{2} - \frac{1}{4(1-n)(n^2(2\omega+1) + 2\omega - 4\omega n + 3n - 4)}\right) \tag{25}$$

Let us consider the throat radius$r_0 = 0.1$ and let $\omega = 0.5$. We have the following results by setting $n = 0.59$:

- Null energy condition (NEC): The NEC term in the radial component is negative for certain regions of the wormhole geometry, implying a violation of NEC in the radial component, while NEC in the tangential component is always preserved.

- Strong energy condition (SEC): The SEC term is negative for certain values of r/r_0, therefore implying a violation of SEC.

- Dominant energy condition (DEC): DEC in the radial component is negative, while DEC in the tangential component is positive, implying a preservation of DEC in the tangential component.

For different values of n we would observe different forms of energy conditions. For isntance, minor changes in n would yield different natures of the energy conditions around Θ and ω.

The nature of the energy conditions of this model can be summarized in the following table:

	NEC	SEC	DEC
Radial	Violated	Violated	Violated
Tangential	Preserved	Violated	Preserved

Table-5: Violation of energy conditions for the shape function (20) and $f(R)$ of the form (25).

Therefore, for a shape function $b(r)$ of the form (20), we see that the nature of the energy conditions is as shown in Table-5, where we see that NEC in tangential component is preserved along with DEC in the tangential component.

In this section, we reviewed the process of deriving $f(R)$ from $b(r)$ and have reviewed the energy conditions for a shape function. It is interesting to note that we can easily find out the embedded topology of the wormhole solution. We can do this by considering a hypersurface of the geometry $\mathcal{H}$ with a constant $t = const.$ and setting $\theta = \pi/2$. For this, we would have the following:

$$ds^2_{\mathcal{H}} = dr^2 \left(1 - \frac{b(r)}{r}\right)^{-1} + r^2 d\phi^2 (1)^2 \qquad (26)$$

This would be equivalent to the cylindrical geometry

$$ds^2_{\mathcal{H}} = \left(1 + \left(\frac{dz}{dr}\right)^2\right) dr^2 + r^2 d\phi^2 (1)^2 \qquad (27)$$

From this, we can find out the integral for $z(r)$, using which we can find out the embedding diagram for the wormhole.

When working with exotic matter in wormhole solutions, it becomes important to ask how much exotic matter is present in the solution. For instance, the *averaged null energy condition* (ANEC) is given by the following line integral, where γ is a null curve:

$$\int_\gamma T_{\mu\nu} x^\mu x^\nu \; d\lambda \qquad (28)$$

Where x^μ is a null vector, and for γ being a null geodesic, we would have λ as the affine parameter. In this case, we can clearly not say anything about the amount of exotic matter in the model, since we would require an integral over V instead. For this, we consider definite volume integrals of ρ and p_r, called the *volume integral quantifier*. Regions for a given wormhole solution where these quantifiers are negative indicate the regions where exotic matter is present. This is to say that we find out the amount of matter in the solution for which ANEC is violated, which allows us to find the amount of exotic matter present. In [15] it was shown that in the general relativistic case, for a given shape function, ANEC can be made to be violated with an infinitesimal amount of exotic matter.

CONCLUSION

Wormholes are an interesting arena of research, particularly for understanding the physical implications of modified theories of gravity. Two characteristic features of wormhole solutions are the shape function and the energy conditions and how they are violated. The shape function for a wormhole solution determines the topological structure of the solution, while it also affects the way the energy conditions are influenced. The framework of the theory also affects the energy conditions and the physical nature of the model. For instance, GR affects all the energy conditions for any given wormhole solution, since the throat of the wormhole requires exotic matter to be open. This "anomaly" can be changed by introducing modified theories of gravity, particularly those in the curvature sector (i.e. those modified theories of gravity derived by introducing curvature terms rather than matter terms) such as $f(R)$ gravity and Lovelock gravity. In this review, we considered a few of the many possible $f(R)$ theories with viable shape functions (i.e. that satisfy the physical conditions imposed in section I) and reviewed the nature of the energy conditions for such models. In section IV, we found out the function $f(R)$ from the shape function and investigated the energy conditions for a model.

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