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Lie Algebraic Structure of Anomalously Scaled Fluctuations

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Abstract

We present a general representation theorem for anomalously scaled fluctuations of single-site observables on a square lattice. The set of limiting fluctuation observables is a Lie algebra.

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1 Introduction

The theory of *normal* fluctuations is worked out for general quantum lattice systems in a series of papers by Goderis, Verbeure and Vets [1, 2, 3]. We briefly summarise how a *microscopic* physical system with normal fluctuations yields *macroscopic* fluctuation observables which form an algebra of canonical commutation relations.

Let $\mathcal{B}$ be the C^*–inductive limit of the minimal tensor product algebras $\mathcal{A}_\Lambda = \otimes_{x\in\Lambda}\mathcal{A}_x$, where $\Lambda \in \mathcal{D}(\mathbb{Z}^d)$ is a finite subset of $\mathbb{Z}^d$. The algebras of single–site observables $\mathcal{A}_x$ are copies of a general C^*–algebra with identity. Typical examples are the quantum spin models, where the $\mathcal{A}_x$ are matrix algebras. Let ω be a state on $\mathcal{B}$ which is ergodic for the space translations.

Definition 1.1 *The local fluctuation observable $F_n^0(A)$ of an observable $A = A^*$ in the state ω is defined by*

$$F_n^0(A) \;=\; |\Lambda_n|^{-\frac{1}{2}} \sum_{x\in\Lambda_n} \{\tau_x A - \omega(A)\} \tag{1.1}$$

where the volume Λ_n is the d–dimensional cube $[-n,n]^d \cap \mathbb{Z}^d$, $|\Lambda_n|$ is the cardinality of Λ_n and where τ_x denotes the automorphism of $\mathcal{B}$ representing the space translation over the vector x. ■

If all selfadjoint operators A of a real linear subspace $\mathcal{B}_0$ of $\mathcal{B}$ have normal fluctuations, i.e. if the limits

$$\lim_{n\to\infty} \omega\left(\mathrm{e}^{\mathrm{i}sF_n^0(A)}\right) \;=\; \exp\left[-\frac{s^2}{2}\sum_{x\in\mathbb{Z}^d}\left\{\omega\left(A\tau_x A\right) - \omega\left(A\right)^2\right\}\right]$$

exist and are not trivial, then one can identify $''\lim_n F_n^0(A)''$ with a *Bose field operator* $F^0(A)$ of a macroscopic algebra of canonical commutation relations (CCR). A macroscopic quasi–free state $\tilde\omega$ is induced on this CCR-algebra by the limits $\lim_n \omega\left(\mathrm{e}^{\mathrm{i}sF_n^0(A)}\right) \equiv \tilde\omega\left(\mathrm{e}^{\mathrm{i}sF^0(A)}\right)$. The Bose field operators are called the macroscopic fluctuation observables and they satisfy the commutation relations

$$\left[F^0(A), F^0(B)\right] = \mathrm{i}\sigma(A,B) \equiv \lim_{n\to\infty}\omega\left(\left[F_n^0(A), F_n^0(B)\right]\right).$$

The existence of the sums $\sum_{x\in\mathbb{Z}^d}\left\{\omega\left(A\tau_x A\right)-\omega\left(A\right)^2\right\}$ has been proved for all the observables in the algebra of strictly local observables $\mathcal{A}_L=\bigcup_{\Lambda\in\mathcal{D}(\mathbb{Z}^d)}\mathcal{A}_\Lambda$ under rather weak conditions on the decay of correlations of the state ω (see [3]). Further details can be found in the references.

These results yield interesting applications in physics, among which we mention the contributions to linear response theory and Onsager's relations [4, 5, 6]. Other rigorous results were obtained on the level of the macrodynamics of the fluctuations. One was able to determine discrete points in the spectrum of the macrodynamics in the Schwinger model [7] and the Jellium model [8]. In the Overhauser model, the complete spectrum of the dynamics on the level of the density fluctuations has been determined [9]. Several problems in solid–state and phonon physics have been explained rigorously in the framework of normal fluctuations : e.g. the Peierls–Fröhlich instability and the Kohn anomaly [10], phon(i)on limits and phonon dynamics [11, 9].

Nevertheless, we are faced with the fact that the theory of normal fluctuations only applies to a limited class of physical systems. The presence of long–range correlations in the system causes problems. Therefore, it would be interesting to extend the theory of fluctuations so that it will be applicable to a wider class of physical systems, including systems with long–range correlations.

One speaks about an observable showing abnormal fluctuation behaviour if the variances of the normally scaled fluctuations of that observable become either infinitely large or infinitely small in the infinite volume limit. Both situations can be avoided by the technique of rescaling the local fluctuations *supernormally* and *subnormally* respectively. Both types of anomalous behaviour are known to appear in physical systems. Examples are discussed in the context of several soluble models, showing a phase transition, like the anharmonic crystal [12, 13], the Curie–Weiss model [14] and the Bose gas [15, 16]. In [12, 13] the authors make some observations concerning the commutation relations of these anomalously scaled macroscopic fluctuation observables but the general algebraic structure of the macroscopic abnormally scaled fluctuations – especially in the case that some microscopic observables need to be scaled subnormally – has not been discussed so far, to our knowledge. This is the main subject of this paper.

2 Abnormal fluctuations

The main motivation to consider other than normal types of scaling is that not all interesting fluctuations are normal. The theory of normal fluctuations fails in situations where the interactions lead to long–range correlations in the state ω. Typical examples are physical systems at a phase transition. If the phase transition is continuous (or second order), it is common knowledge that quantities such as the heat capacity, or more generally some susceptibilities, diverge in the critical points. This corresponds to the divergence of the variances of normally scaled (i.e. with scaling factor $|\Lambda_n|^{-1/2}$) fluctuations of specific microscopic observables A, i.e. :

$$\omega\left(\left(F_n^0(A)\right)^2\right) \overset{n\to\infty}{\longrightarrow} +\infty.$$

Equivalently : the long–range correlations in the state ω prevent the summability of quantities of the type :

$$\sum_{x\in\mathbb{Z}^d}\left\{\omega(A\tau_x A) - \omega(A)^2\right\}.$$

To make the ideas explained in this section more concrete, we will use the quantum anharmonic crystal, worked out in [12], as a guideline. The displacement operator Q and its powers show such diverging behaviour on the critical line.

The method to deal with these divergent quantities is to rescale the local fluctuations *supernormally*. Supernormal scaling means that one determines a scaling index $\delta_A \in (0, \frac{1}{2})$ for each observable A such that the, supernormally scaled, local fluctuations

$$F_n^{\delta_A}(A) = |\Lambda_n|^{-\delta_A} F_n^0(A) \tag{2.2}$$

yield non–trivial characteristic functions

$$\lim_{n\to\infty}\omega\left(\exp\left[itF_n^{\delta_A}(A)\right]\right) = \phi_A(t).$$

Observe that it is no longer expected that the characteristic functions are Gaussian. The next step is to check whether it is possible to give a meaning to the limits

$$\lim_{n\to\infty} F_n^{\delta_A}(A)$$

as operators — in general unbounded — on some Hilbert space. This will be the topic of a fundamental reconstruction theorem.

When the observable A is associated with the order parameter of the system, one usually describes in the physics literature (see e.g. [17]) the asymptotic form of the connected two–point function with a critical exponent η

$$\omega(A\tau_x A) - \omega(A)^2 \sim O\left(\frac{1}{|x|^{d-2+\eta}}\right) \quad \text{as } |x| \to \infty.$$

The scaling index δ_A is related to the critical exponent η by the straightforward relation

$$\eta = 2 - 2d\delta_A. \tag{2.3}$$

In contrast to the supernormal fluctuations, it was found in [12], that the normally scaled fluctuation of the momentum operator P — i.e. with the usual scaling $|\Lambda_n|^{-\frac{1}{2}}$ — has a trivial characteristic function in the ground state of the quantum anharmonic crystal; i.e.

$$\lim_{n\to\infty} \omega\left(\exp\left[\mathrm{i}tF_n^0(P)\right]\right) = 1.$$

All moments of the local fluctuation of the microscopic momentum observable P converge to zero in the infinite volume limit and consequently the macroscopic fluctuation operator $F^0(P)$ will be the trivial zero operator; in particular the variance converges to zero:

$$\begin{aligned}\omega\left(\left(F_n^0(P)\right)^2\right) &= |\Lambda_n|^{-1} \sum_{x,y\in\Lambda_n} \omega\left(\tau_x P \tau_y P\right) - \omega\left(\tau_x P\right)\omega\left(\tau_y P\right) \\ &\stackrel{n\to\infty}{\longrightarrow} 0.\end{aligned} \tag{2.4}$$

If the right hand side of (2.4) is of asymptotic order $O\left(|\Lambda_n|^{2\delta_P}\right)$ with $\delta_P < 0$, then it is reasonable to expect that the fluctuations live on the specific scale $|\Lambda_n|^{-\frac{1}{2}-\delta_P}$. Instead of working with the normally scaled fluctuation $F_n^0(P)$, one checks whether the *subnormally* scaled local fluctuation

$$F_n^{\delta_P}(P) = |\Lambda_n|^{-\delta_P} F_n^0(P), \tag{2.5}$$

where δ_P is now strictly negative, leads to a non–trivial characteristic function

$$\lim_{n\to\infty} \omega\left(\exp\left[\mathrm{i}tF_n^{\delta_P}(P)\right]\right) = \phi_P(t) \neq 1.$$

Also in this case, one should give a mathematical meaning to the limiting operators :

$$\lim_{n\to\infty} F_n^{\delta_P}(P)$$

3 Algebraic aspects

The focus in this section will be on the algebraic structure of a set of abnormally (super– as well as subnormally) scaled fluctuations. We present the results for single–site observables. Generalisation to strictly local observables is straightforward and does not reveal any new mathematical structures.

First, we mention partial results concerning the algebraic structure of abnormal fluctuations, obtained in the framework of the quantum anharmonic crystal [12]. There it is argued that the commutator of two local fluctuations $F_n^{\delta_A}(A)$ and $F_n^{\delta_B}(B)$ goes to zero in the limit $n \to \infty$ if $\delta_A + \delta_B > 0$ or to the c–number $\omega([A, B])$ if $\delta_A + \delta_B = 0$.

The nature of the commutator of local fluctuations $F_n^{\delta_A}(A)$ and $F_n^{\delta_B}(B)$ when $\delta_A + \delta_B < 0$ has not been considered in [12]. Precisely this situation turns out to have a rich mathematical structure when studying the algebra of macroscopic fluctuations. We prove that, when $\delta_A + \delta_B < 0$, the commutator $\left[F_n^{\delta_A}(A), F_n^{\delta_B}(B)\right]$ is in fact again the local fluctuation of a microscopic observable. This observation reveals the possibility of obtaining a non–trivial *macroscopic Lie algebra* of macroscopic fluctuations. This structure is a major generalisation of the CCR structure, found so far.

Consider the GNS representation [18] of the physical system $(\mathcal{B}, \omega)$ with ω an ergodic state on $\mathcal{B}$. The GNS representation is specified by a triplet $(\pi, \mathcal{H}, \Omega)$; π is a representation of the C^*–algebra $\mathcal{B}$ into the bounded operators $B(\mathcal{H})$ of the Hilbert space $\mathcal{H}$:

$$\pi : A \in \mathcal{B} \mapsto \pi(A) \in B(\mathcal{H}).$$

and $\Omega \in \mathcal{H}$ is a cyclic vector for the algebra $\pi(\mathcal{B})$ such that :

$$\omega(A) \;=\; (\Omega, \pi(A)\Omega)\,.$$

In the GNS representation of $(\mathcal{B}, \omega)$, we consider in $\pi(\mathcal{B})$ a bounded, skew–symmetric representation on $\mathcal{H}$ of a real Lie algebra $\mathfrak{g}(1)$ generated by m single–site observables $\pi(L_j) \equiv L_j$ multiplied by i. The structure of this Lie algebra is characterised by the following commutation relations :

$$[\mathrm{i}L_j, \mathrm{i}L_k] = \sum_{l=1}^{m} c_{jk}^{l}\,\mathrm{i}L_l. \tag{3.6}$$

The $c^l_{jk} \in \mathbb{R}$ are the structure constants with respect to the basis $\{\mathrm{i}L_j\}_{j=1,\ldots,m}$.

The operator $\sum_{j=1}^m L_j^2$ is bounded and selfadjoint since all L_j are bounded and selfadjoint operators. This implies that the representation of $\mathfrak{g}(1)$ is integrable [19]. Hence, the operators

$$\exp[\mathrm{i}\sum_{j=1}^m \alpha_j L_j] \qquad (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$$

define a Lie group $\mathcal{G}(1)$ of unitary operators on $\mathcal{H}$ such that

$$\exp[\mathrm{i}\sum_{j=1}^m \alpha_j^1 L_j] \exp[\mathrm{i}\sum_{j=1}^m \alpha_j^2 L_j] \;=\; \exp[\mathrm{i}\sum_{j=1}^m \gamma_j^{1,2}(\alpha_1^1, \ldots, \alpha_m^2) L_j]$$

where the coefficients $\gamma_j^{1,2}(\alpha_1^1, \ldots, \alpha_m^2)$ are entire analytic functions of the α^1's and α^2's.

The unitary operators

$$\exp\left[\mathrm{i}\sum_{j=1}^m \alpha_j F_n^{\delta_j}(L_j)\right] = \otimes_{x\in\Lambda_n} \exp\left[\mathrm{i}\sum_{j=1}^m \alpha_j |\Lambda_n|^{-(\frac{1}{2}+\delta_j)} \tau_x\left(L_j - \omega(L_j)\right)\right] \tag{3.7}$$

multiplied with a phase factor $\mathrm{e}^{\mathrm{i}\alpha_0}\mathbb{1}$ define for each $n \in \mathbb{N}$ a regular unitary representation of a Lie group $\mathcal{G}(n)$ on the space $\mathcal{H}$. The associated Lie algebras $\mathfrak{g}(n)$ are given by the $(m+1)$ dimensional real spaces with elements of the type $\alpha_0 \mathrm{i}\mathbb{1} + \sum_{j=1}^m \alpha_j \mathrm{i}F_n^{\delta_j}(L_j)$, $\alpha_j \in \mathbb{R}$.

The purpose of introducing an extra dimension, corresponding to the generator $\mathrm{i}\mathbb{1}$, will be clear from the consideration of the commutator

$$\begin{aligned}
&\left[\mathrm{i}F_n^{\delta_j}(L_j), \mathrm{i}F_n^{\delta_k}(L_k)\right] \\
&= |\Lambda_n|^{-(1+\delta_j+\delta_k)} \sum_{x\in\Lambda_n} \tau_x\left[\mathrm{i}L_j, \mathrm{i}L_k\right] \\
&= |\Lambda_n|^{-(1+\delta_j+\delta_k)} \sum_{x\in\Lambda_n} \left\{\tau_x\left[\mathrm{i}L_j, \mathrm{i}L_k\right] - \omega\left(\left[\mathrm{i}L_j, \mathrm{i}L_k\right]\right)\right\} \\
&\quad + |\Lambda_n|^{-(\delta_j+\delta_k)} \omega\left(\left[\mathrm{i}L_j, \mathrm{i}L_k\right]\right)\mathbb{1} \\
&= \sum_{l=1}^m c_{jk}^l |\Lambda_n|^{-\left(\frac{1}{2}+\delta_j+\delta_k-\delta_l\right)} |\Lambda_n|^{-(\frac{1}{2}+\delta_l)} \sum_{x\in\Lambda_n} \left\{\tau_x \mathrm{i}L^l - \omega(\mathrm{i}L^l)\right\} \\
&\quad + |\Lambda_n|^{-(\delta_j+\delta_k)} \omega\left(\left[\mathrm{i}L_j, \mathrm{i}L_k\right]\right)\mathbb{1} \\
&= \sum_{l=1}^m c_{jk}^l |\Lambda_n|^{-\left(\frac{1}{2}+\delta_j+\delta_k-\delta_l\right)} \mathrm{i}F_n^{\delta_l}(L^l) \\
&\quad - \mathrm{i}|\Lambda_n|^{-(\delta_j+\delta_k)} \omega\left(\left[\mathrm{i}L_j, \mathrm{i}L_k\right]\right)\mathrm{i}\mathbb{1}
\end{aligned}$$

This simple calculation reveals the (volume dependent) structure constants of $\mathfrak{g}(n)$ with respect to the basis $\mathrm{i}\mathbb{1}$, $\{\mathrm{i}F_n^{\delta_j}(L_j)\}_{j=1,\ldots,m}$. They are given by the expressions :

$$c_{jk}^l(n) \;=\; c_{jk}^l |\Lambda_n|^{-\left(\frac{1}{2}+\delta_j+\delta_k-\delta_l\right)} \quad \text{for } l, j, k \in \{1, \ldots m\} \tag{3.8}$$

$$\begin{aligned} c^0_{jk}(n) &= -\mathrm{i}|\Lambda_n|^{-(\delta_j+\delta_k)}\omega\left([\mathrm{i}L_j, \mathrm{i}L_k]\right) \quad \text{for } j,k \in \{1,\ldots .m\} & (3.9)\\ c^l_{0k}(n) &= 0. & (3.10) \end{aligned}$$

Assume now that it is possible to determine for each L_j, $j = 1, \ldots, m$, a scaling index $\delta_j < \frac{1}{2}$ such that

$$\lim_{n\to\infty} \omega\left(\exp\left[\mathrm{i}\sum_{j=1}^{m} \alpha_j F_n^{\delta_j}(L_j)\right]\right) = \phi(\alpha_1, \ldots, \alpha_m) \tag{3.11}$$

defines a non–trivial analytic[4] function ϕ of the real variables $\alpha_1, \ldots, \alpha_m$. In particular, we assume the existence of the limits

$$\lim_{n\to\infty} \omega\left(\exp[\mathrm{i}\alpha_j F_n^{\delta_j}(L_j)]\right) = \phi_j(\alpha_j) \tag{3.12}$$

as non–trivial analytic, (*not* necessarily Gaussian) characteristic functions for every $j = 1, \ldots, m$.

The next step in the determination of the macroscopic fluctuation observables, is the analysis of the limits

$$\begin{aligned} &\lim_{n\to\infty} \omega\left(\exp\left[\mathrm{i}\sum_{j=1}^{m} \alpha^1_j F_n^{\delta_j}(L_j)\right] \exp\left[\mathrm{i}\sum_{j=1}^{m} \alpha^2_j F_n^{\delta_j}(L_j)\right]\right) \\ &= \lim_{n\to\infty}\left\{\omega\left(\exp\left[\mathrm{i}\sum_{j=1}^{m} \gamma^{1,2}_j(n) F_n^{\delta_j}(L_j)\right]\right) \mathrm{e}^{\mathrm{i}\gamma^{1,2}_0(n)}\right\}. \end{aligned} \tag{3.13}$$

where for $j = 1, \ldots, m$

$$\gamma^{1,2}_j(n) = |\Lambda_n|^{\frac{1}{2}+\delta_j}\gamma^{1,2}_j\left(\frac{\alpha^1_1}{|\Lambda_n|^{\frac{1}{2}+\delta_1}}, \ldots, \frac{\alpha^2_m}{|\Lambda_n|^{\frac{1}{2}+\delta_m}}\right) \tag{3.14}$$

as is easily verified with the help of formula (3.7).

By our assumption (3.11) about the existence of a smooth distribution $\phi(\alpha_1, \ldots, \alpha_m)$, the limits (3.13) will be well–defined if and only if the limits $n \to \infty$ of the numbers $\gamma^{1,2}_0(n)$ and $\gamma^{1,2}_j(n)$ are defined. This property is directly related to the convergence of the structure constants $c^l_{jk}(n)$ as $n \to \infty$.

[4] the scaling indices were introduced in such a way that ϕ should only be C^2 which corresponds to the existence of the variance of the macroscopic fluctuation observable. But it is logical to require that also the higher moments exist and that the analyticity of the central limit is assured [20].

Note that when all scaling indices are zero, the theory of normal fluctuations for single–site observables is recovered. For the case of normal fluctuations, the structure constants reduce to the numbers

$$
\begin{aligned}
c^l_{jk}(n) &= c^l_{jk}|\Lambda_n|^{-\frac{1}{2}} \quad \text{for } l,j,k \in \{1,\ldots,m\} \\
c^0_{jk}(n) &= -\mathrm{i}\omega\left([\mathrm{i}L_j, \mathrm{i}L_k]\right) \quad \text{for } j,k \in \{1,\ldots,m\} \\
c^l_{0k}(n) &= 0.
\end{aligned}
$$

It is obvious that one obtains the CCR structure, as in [3], in the limit $n \to \infty$, because the $c^0_{jk}(n)$ are volume independent numbers and the $c^l_{jk}(n)$ converge to zero for $l \in \{1,\ldots,m\}$.

Consider now again the general case (3.8)–(3.10)in which some (or all) scaling indices differ from zero and the structure constants $c^l_{jk}(n)$ for values of j and k for which the corresponding scaling indices have $\delta_j + \delta_k > 0$. An immediate consequence of the strict positivity of the sum of the scaling indices is that the structure constants $c^0_{jk}(n) = -\mathrm{i}|\Lambda_n|^{-(\delta_j+\delta_k)}\omega\left([\mathrm{i}L_j, \mathrm{i}L_k]\right)$ converge to zero as $n \to \infty$. The ergodicity of the state ω implies moreover that the other structure constants $c^l_{jk}(n)$ also converge to zero because the scaling index δ_l cannot exceed $+\frac{1}{2}$ and hence the exponent of the factor $|\Lambda_n|^{-(\frac{1}{2}+\delta_j+\delta_k-\delta_l)}$ in (3.8) is strictly negative as well. This means that for a couple (j,k) for which $\delta_j + \delta_k > 0$, all structure constants $c^l_{jk}(n)$, where $l = 0,\ldots,m$, converge to zero. This indicates that the macroscopic fluctuation operators associated to L_j and L_k, will commute. In physical terms, one speaks about classically behaving fluctuations if all scaling indices are positive.

If $\delta_j + \delta_k = 0$, the ergodicity of the state ω still implies the convergence of the $c^l_{jk}(n)$ to zero for all $l \neq 0$. On the other hand, it is trivial that in this case the $c^0_{jk}(n)$ are the volume independent numbers $-\mathrm{i}\omega\left([\mathrm{i}L_j, \mathrm{i}L_k]\right)$. This indicates that the systems of macroscopic fluctuation operators of a pair L_j and L_k behave again as Bose fields.

The most delicate situation is when j and k are such that $\delta_j + \delta_k < 0$: the exponent $-(\delta_j + \delta_k)$ of $|\Lambda_n|$ in the $c^0_{jk}(n) = -\mathrm{i}|\Lambda_n|^{-(\delta_j+\delta_k)}\omega\left([\mathrm{i}L_j, \mathrm{i}L_k]\right)$ is now strictly positive. This is the new situation. We prove

Lemma 3.1 *If, for an ergodic state ω, the fluctuations of L_j and L_k exist with scaling indices δ_j and δ_k; i.e. assumption (3.11) is satisfied, such that $\delta_j + \delta_k < 0$, then*

$\omega\left([L_j, L_k]\right) = 0$

Proof : Using Schwarz inequality

$$\left|\omega\left(\left[F_n^{\delta_j}(L_j), F_n^{\delta_k}(L_k)\right]\right)\right|^2 \leq 4\,\omega\left(\left(F_n^{\delta_j}(L_j)\right)^2\right)\omega\left(\left(F_n^{\delta_k}(L_k)\right)^2\right). \tag{3.15}$$

The right hand side of (3.15) is bounded uniformly in n by a constant on the basis of the definition of δ_j and δ_k and of the assumption (3.11).

On the other hand :

$$\begin{aligned}
\omega\left(\left[F_n^{\delta_j}(L_j), F_n^{\delta_k}(L_k)\right]\right) &= |\Lambda_n|^{-(\frac{1}{2}+\delta_j+\frac{1}{2}+\delta_k)} \sum_{x,y\in\Lambda_n} \omega\left([\tau_x L_j, \tau_y L_k]\right) \\
&= |\Lambda_n|^{-(1+\delta_j+\delta_k)} \sum_{x\in\Lambda_n} \omega\left(\tau_x\,[L_j, L_k]\right) \\
&= |\Lambda_n|^{-(\delta_j+\delta_k)}\omega\left([L_j, L_k]\right)
\end{aligned}$$

Hence, when $n\to\infty$, the bound (3.15) implies that $\omega\left([L_j, L_k]\right) = 0$ when $\delta_j + \delta_k < 0$. ■

It follows trivially from this lemma that for each $n \in \mathbb{N}$, the structure constants $c^0_{jk}(n) = 0$ when $\delta_j + \delta_k < 0$.

Concerning the other structure constants $c^l_{jk}(n) = c^l_{jk}|\Lambda_n|^{-(\frac{1}{2}+\delta_j+\delta_k-\delta_l)}$ for $l = 1, \ldots, m$, one sees immediately that we are in trouble if for some $l \in \{1, \ldots, m\}$ the exponent $\frac{1}{2} + \delta_j + \delta_k - \delta_l < 0$ while $c^l_{jk} \neq 0$. If this situation occurs, the structure constants $c^l_{jk}(n)$ diverge. We are not able to proceed with the analysis if there exists a value of l such that $c^l_{jk} \neq 0$ and $\frac{1}{2} + \delta_j + \delta_k - \delta_l < 0$.

However when $\frac{1}{2}+\delta_j+\delta_k-\delta_l \geq 0$ for the values of $l \in \{1, \ldots, m\}$ for which $c^l_{jk} \neq 0$, the structure constants $c^l_{jk}(n)$ either converge to zero (if the inequality is strict) or they are equal to c^l_{jk} (if there is equality). The latter case can be interpreted as a phenomenon of *scale invariance* in the physical system $(\mathcal{B}, \omega)$. Moreover it has the interesting consequence that new algebraic structures for the macroscopic fluctuations will be obtained which are more complex than the CCR–structure.

The discussion of the limiting structure constants $\lim_n c^l_{jk}(n)$ and the continuity properties of the characteristic function ϕ in (3.11) guarantee that the limits (3.13)

$$\lim_{n\to\infty}\left\{\omega\left(\exp\left[\mathrm{i}\sum_{j=1}^m \gamma_j^{1,2}(n) F_n^{\delta_j}(L_j)\right]\right)\mathrm{e}^{\mathrm{i}\gamma_0^{1,2}(n)}\right\}$$

$$= \mathrm{e}^{\mathrm{i}\lim_n \gamma_0^{1,2}(n)} \phi\left(\lim_n \gamma_1^{1,2}(n), \ldots, \lim_n \gamma_m^{1,2}(n)\right)$$

exist.

These observations indicate the possibility of having a Lie algebraic structure for the limiting fluctuation observables.

It is explained in [21] how to associate a C^*–algebra to a topological group $\mathcal{G}$ and how complex functions of the positive type on the group define states on the C^*–algebra. These notions are essential in order to formulate the *reconstruction theorem*, in which the limiting fluctuation observables are characterised as operators on a Hilbert space.

Denote by $W(\mathcal{G})$ the complex vector space generated by the functions

$$W(g) : \mathcal{G} \to \mathbb{C} : h \mapsto W(g)(h) = \begin{cases} 0 & if \quad g \neq h \\ 1 & if \quad g = h. \end{cases}$$

The group–product of $\mathcal{G}$ defines a product on $W(\mathcal{G})$ by

$$W(g_1)W(g_2) = W(g_1 g_2)$$

and gives $W(\mathcal{G})$ the structure of an algebra with unit $W(e)$ where e is the neutral element of the group $\mathcal{G}$.

$W(\mathcal{G})$ is a *–algebra for the involution

$$W(g) \mapsto W(g)^* = W(g^{-1})$$

In order to equip $W(\mathcal{G})$ with the C^*–structure, we use the construction from [22, 23]. The function

$$\begin{aligned} ||.||_1 \quad : \quad & W(\mathcal{G}) \to \mathrm{I\!R}^+ \\ & \sum_{i=1}^{n} \lambda_i W(g_i) \mapsto \sum_{i=1}^{n} |\lambda_i| \quad \forall \lambda_i \in \mathbb{C}, g_i \in \mathcal{G} \end{aligned}$$

is a *–norm and the completion $\overline{W(\mathcal{G})}^1$ is a *–Banach algebra with unity. The norm $||.||_1$ is not a C^*–norm, however.

Lemma 3.2 *Every positive linear form on $W(\mathcal{G})$ is continuous w.r.t. $||.||_1$–topology and hence extends continuously to a positive linear form on $\overline{W(\mathcal{G})}^1$.*

Proof : By Schwarz inequality, one has for all positive linear forms ρ

$$|\rho(W(g))| = |\rho(W(g)W(e))| \leq \rho\left(W(g^{-1})W(g)\right)^{\frac{1}{2}} \rho(\mathbb{1})^{\frac{1}{2}} = \rho(\mathbb{1})$$

Hence for all $\sum_i \lambda_i W(g_i)$,

$$|\rho(\sum_i \lambda_i W(g_i))| \leq \sum_i |\lambda_i| \rho(\mathbb{1}) = ||\sum_i \lambda_i W(g_i)||_1 \rho(\mathbb{1})$$

and ρ is continuous. ∎

Define now on $\overline{W(\mathcal{G})}^1$ the following norm :

$$||A|| = \sup_{\rho \in \mathcal{S}} \sqrt{\rho(A^* A)} \quad \forall A \in \overline{W(\mathcal{G})}^1$$

where $\mathcal{S}$ stands for the set of states on $\overline{W(\mathcal{G})}^1$. This norm is called the *minimal regular norm* [24]. One can check that $||.||$ is a C^*–norm. By construction, it is immediate that $||A|| \leq ||A||_1$. The completion $\overline{W(\mathcal{G})}$ of $\overline{W(\mathcal{G})}^1$ with respect to this minimal regular norm is called the C^*–algebra on the Lie group $\mathcal{G}$.

Each state η on this algebra determines a *generating functional*, which is a complex function of the positive type on the group [21] :

$$g \in \mathcal{G} \mapsto \eta\left(W(g)\right).$$

Conversely, a continuous complex function ϕ on the group $\mathcal{G}$ defines a unique state η of the C^*–algebra $\overline{W(\mathcal{G})}$ by the identification $\phi(g) = \eta\left(W(g)\right)$ if and only if

$$\sum_{i,j=1}^{N} \lambda_i \overline{\lambda_j} \phi(g_i g_j^{-1}) \geq 0 \quad \text{for } \lambda_i \in \mathbb{C} \text{ and } g_i \in \mathcal{G}. \tag{3.16}$$

The mathematical characterisation of the limiting fluctuation operators can now be continued. If the limits $\lim_n \gamma_0^{1,2}(n)$ and $\lim_n \gamma_j^{1,2}(n)$ (3.14) exist, then it is possible to consider the following Lie group structure on $\mathbb{R}^{m+1}$: the group product is defined by

$$\left(\alpha_0^1, \ldots, \alpha_m^1\right)\left(\alpha_0^2, \ldots, \alpha_m^2\right) = \left(\lim_n \gamma_0^{1,2}(n), \ldots, \lim_n \gamma_m^{1,2}(n)\right).$$

where the functions $\gamma_j^{1,2}(n)$ are defined by formula (3.14). We denote this (connected) group by $\mathcal{G}(\infty)$. It is easily verified that the neutral element e of $\mathcal{G}(\infty)$ is :

$$e = (0, \ldots, 0)$$

and the inverse of an element $(\alpha_0, \ldots, \alpha_m)$ is given by

$$(\alpha_0, \ldots, \alpha_m)^{-1} \;=\; (-\alpha_0, \ldots, -\alpha_m)$$

Finally, we construct the associated C^*–algebra $\overline{W(\mathcal{G}(\infty))}$ as above.

The characterisation of the macroscopic fluctuations is a result of the following theorem.

Theorem 3.1 (Reconstruction theorem) *For each $(\alpha_0, \ldots, \alpha_m) \in \mathbb{R}^{(m+1)}$, the limits*

$$\lim_{n\to\infty} \omega\left(\mathrm{e}^{\mathrm{i}\sum_{j=1}^m \alpha_j F_n^{\delta_j}(L_j)}\right) \;=\; \phi(\alpha_1, \ldots, \alpha_m)$$

define a state $\tilde{\omega}$ on the C^*–algebra $\overline{W(\mathcal{G}(\infty))}$ with generating functional

$$\begin{aligned}\tilde{\phi}(\alpha_0, \alpha_1, \ldots, \alpha_m) &= \tilde{\omega}\left(W(\alpha_0, \ldots, \alpha_m)\right) \\ &= \mathrm{e}^{\mathrm{i}\alpha_0}\phi(\alpha_1, \ldots, \alpha_m)\end{aligned}$$

<u>*Proof*</u> : In order to prove that $\tilde{\omega}$ is a state of the C^*–algebra $\overline{W(\mathcal{G}(\infty))}$, we have to check the condition (3.16) for $\tilde{\phi}$.

The positivity condition of the generating functional $\tilde{\phi}$ is easily checked : for arbitrary $\lambda_i \in \mathbb{C}$ and $(\alpha_0^i, \ldots, \alpha_m^i) \in \mathcal{G}(\infty)$, one has

$$\begin{aligned}&\sum_{i,j=1}^N \lambda_i\overline{\lambda_j}\tilde{\phi}((\alpha_0^i, \ldots, \alpha_m^i)(\alpha_0^j, \ldots, \alpha_m^j)^{-1}) \\ &\quad= \sum_{i,j=1}^N \lambda_i\overline{\lambda_j}\tilde{\phi}\left(\lim_n \gamma_0^{i,j}(n), \ldots, \lim_n \gamma_m^{i,j}(n)\right)\end{aligned}$$

The $\gamma_l^{i,j}(n)$ are defined in (3.14). The upperindices i,j indicate that the $\gamma_l^{i,j}(n)$ are the coordinates of the element $(\alpha_0^i, \ldots, \alpha_m^i)(\alpha_0^j, \ldots, \alpha_m^j)^{-1}$. By (3.13), the last expression equals :

$$\begin{aligned}&= \sum_{i,j=1}^N \lambda_i\overline{\lambda_j}\mathrm{e}^{\mathrm{i}\alpha_0^i}\mathrm{e}^{-\mathrm{i}\alpha_0^j}\lim_n \omega\left(\mathrm{e}^{\mathrm{i}\sum_{j=1}^m \alpha_j F_n^{\delta_j}(L_j)}\mathrm{e}^{-\mathrm{i}\sum_{j=1}^m \alpha_j F_n^{\delta_j}(L_j)}\right) \\ &= \lim_n \omega\left(\left(\sum_{i=1}^N \lambda_i \mathrm{e}^{\mathrm{i}\alpha_0^i}\mathrm{e}^{\mathrm{i}\sum_{j=1}^m \alpha_j F_n^{\delta_j}(L_j)}\right)\left(\sum_{i=1}^N \lambda_i \mathrm{e}^{\mathrm{i}\alpha_0^i}\mathrm{e}^{\mathrm{i}\sum_{j=1}^m \alpha_j F_n^{\delta_j}(L_j)}\right)^*\right) \\ &\geq 0\end{aligned}$$

by the positivity of the state ω. This proves the theorem. ∎

Consider now the GNS representation [18] of the C^* system $\left(\overline{W(\mathcal{G}(\infty))}, \tilde{\omega}\right)$, given by the triplet $\left(\tilde{\pi}, \tilde{\mathcal{H}}, \tilde{\Omega}\right)$.

In analogy with the philosophy of [1, 2, 3] we call the representation algebra $\tilde{\pi}\left(\overline{W(\mathcal{G}(\infty))}\right)$ *the algebra of abnormally scaled macroscopic fluctuations* for the system $(\mathcal{B}, \mathfrak{g}(1), \omega)$.

¿From the assumption (3.11) that the function ϕ is analytic, it is trivial that the functions

$$t \in \mathbb{R} \mapsto \tilde{\phi}\left((\alpha_0^2, \ldots, \alpha_m^2)(t\alpha_0^1, \ldots, t\alpha_m^1)(\alpha_0^3, \ldots, \alpha_m^3)\right) \tag{3.17}$$

are continuous for each $(\alpha_0^1, \ldots, \alpha_m^1)$, $(\alpha_0^2, \ldots, \alpha_m^2)$, $(\alpha_0^3, \ldots, \alpha_m^3) \in \mathbb{R}^{m+1}$. The construction of the GNS representation [18], guarantees that there exists an element

$$A = \sum_{i=1}^{n} \lambda_i W(\alpha_0^i, \ldots, \alpha_m^i) \in W\left(\mathcal{G}(\infty)\right)$$

such that

$$||\psi_A - \mu|| \leq \frac{\epsilon}{4}$$

for any element $\mu \in \tilde{\mathcal{H}}$ and any $\epsilon \in \mathbb{R}^+$. The vector ψ_A is the canonical image of A into the representation space $\tilde{\mathcal{H}}$ of $\tilde{\pi}$. Hence

$$\begin{aligned}
&||\tilde{\pi}\left(W(t\alpha_0^1, \ldots, t\alpha_m^1)\right)\mu - \mu|| \\
&\leq \; 2||\mu - \psi_A|| + ||\tilde{\pi}\left(W(t\alpha_0^1, \ldots, t\alpha_m^1)\right)\psi_A - \psi_A|| \\
&\leq \; \frac{\epsilon}{2} + 2\left(\tilde{\omega}(A^*A) - \mathrm{Re}\tilde{\omega}(A^*W(t\alpha^1, \ldots, t\alpha_m^1)A)\right) \\
&\leq \; \epsilon
\end{aligned}$$

for t sufficiently small by continuity of the functions (3.17). This observation implies that the maps

$$t \in \mathbb{R} \mapsto \tilde{\pi}\left(W(t\alpha_0, \ldots, t\alpha_m)\right)$$

define for each $(\alpha_0, \ldots, \alpha_m) \in \mathbb{R}^{m+1}$ a strongly continuous group of isometries, due to the definition of the group multiplication (3.14).

Due to Stone's theorem, there exists a selfadjoint linear operator $F(\alpha_0, \ldots, \alpha_m)$ on $\tilde{\mathcal{H}}$, for all $(\alpha_0, \ldots, \alpha_m) \in \mathbb{R}^{m+1}$, such that

$$\tilde{\pi}\left(W(t\alpha_0, \ldots, t\alpha_m)\right) \;=\; \mathrm{e}^{\mathrm{i}tF(\alpha_0, \ldots, \alpha_m)}.$$

By the assumption of analyticity of ϕ, it is trivial that the map

$$F : \mathbb{R}^{m+1} \to \mathcal{L}(\tilde{\mathcal{H}}) : (\alpha_0, \ldots, \alpha_m) \mapsto F(\alpha_0, \ldots, \alpha_m),$$

where $\mathcal{L}(\tilde{\mathcal{H}})$ denotes the set of all linear operators on $\tilde{\mathcal{H}}$, defines a skew-symmetric representation of a Lie algebra $\mathfrak{g}(\infty)$ on the Hilbert space $\tilde{\mathcal{H}}$.

The elements $F(0, \ldots, 1, 0, \ldots, 0) \equiv \tilde{L}_j$ define a basis of this representation of the Lie group $\mathfrak{g}(\infty)$ on $\tilde{\mathcal{H}}$ and by the reconstruction theorem 3.1, one can make the identification

$$\text{"}\lim_{n\to\infty}\text{"} F_n^{\delta_j}(L_j) \;=\; \tilde{L}_j$$

for $j = 1, \ldots, m$ and $\mathbb{1} \equiv \tilde{L}_0$. The "$\lim_{n\to\infty}$" is the central limit, which is assumed to hold in the theorem.

The structure constants w.r.t. the basis $\left\{ \mathrm{i}\tilde{L}_j \right\}_{j=0,\ldots,m}$ are obviously given by the limits $\lim_{n\to\infty} c_{jk}^l(n)$.

The ground state of the quantum harmonic crystal is a nice illustration of the phenomenon that the macroscopic abnormally scaled fluctuations of some observables have a non–trivial Lie algebraic structure. The details about this model will be published in a separate paper [25].

Finally, the problem which we have treated here is closely related to the theory of contractions of Lie algebra representations, at least from the mathematical point of view. The concept of contraction of Lie algebra representations was introduced in [26, 27] and is a subject of active interest, particularly in connection with the theory of quantum groups [28, 29].

We feel that our approach has the advantage of characterising unambiguously the limiting Hilbertspace for the representation of the limiting Lie algebra in contrast to the methods used in [30, 31, 32].

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