THE LYAPUNOV EXPONENTS FOR SCHRÖDINGER OPERATORS WITH SLOWLY OSCILLATING POTENTIALS

Barry Simon and Yunfeng Zhu

Department of Mathematics, 253-37
California Institute of Technology
Pasadena, CA 91125

Abstract. By studying the integrated density of states, we prove the existence of Lyapunov exponents and the Thouless formula for the Schrödinger operator $-d^2/dx^2 + \cos x^\nu$ with $0 < \nu < 1$ on $L^2[0,\infty)$. This yields an explicit formula for these Lyapunov exponents. By applying rank one perturbation theory, we also obtain some spectral consequences.

1. Introduction

Our goal in this paper is to prove Lyapunov behavior and compute a Lyapunov exponent for the one-dimensional half-line Schrödinger operator

$$H_\nu = -\frac{d^2}{dx^2} + \cos x^\nu \qquad x \in [0,\infty) \tag{1.1}$$

with $0 < \nu < 1$.

It is clear that H_ν is regular at 0 and is limit point at infinity. (For the definition of limit point, see [22] or [14].) Therefore, for each $\theta \in [0,\pi)$, H_ν has a unique self-adjoint realization on $L^2[0,\infty)$ with boundary condition at 0 given by

$$u(0)\cos\theta + u'(0)\sin\theta = 0$$

which will be denoted by H_ν^θ.

In the spectral theory of Schrödinger operators, most work has concentrated on the potential $V(x)$, either $V(x) \to 0$ as $|x| \to \infty$ or $V(x)$ is periodic or almost periodic. Such models have been investigated particularly well. Comparatively new are the models with oscillating but not periodic nor almost periodic potentials. Due to recent discoveries of H. Behncke ([2]), W. Kirsch, S.A. Molchanov and L.A. Pastur ([12]) and G. Stolz ([19], [20]), it is clear that some such models may yield very interesting spectrum. As one of his particular examples, Stolz has studied the spectral properties for (1.1) in [20]. Let $\sigma(H)$, $\sigma_{\rm ac}(H)$, $\sigma_{\rm sing}(H)$, $\sigma_{\rm sc}(H)$ and $\sigma_{\rm pp}(H)$ denote the spectrum, absolutely continuous spectrum, singular spectrum, singular continuous spectrum and

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pure point spectrum resp. for H. Then from Stolz's paper, we have known that $\sigma(H_\nu) = [-1, \infty)$, $\sigma_{\text{ac}}(H_\nu) = [1, \infty)$, and $\sigma_{\text{sing}}(H_\nu) = [-1, 1]$. In fact, from an unpublished result of Kirsch and Stolz (see [12]), we have also know that H_ν^θ has pure point spectrum in $[-1, 1]$ for almost all boundary conditions θ.

We already see that this model has some subtle and fascinating spectral properties, especially for $E \in (-1, 1)$. We'll continue working on this model. In particular, we will prove Lyapunov behavior and compute a Lyapunov exponent formula.

We know that the Lyapunov exponent is an important tool in the spectral theory for one-dimensional Schrödinger operators with almost periodic or random potentials. In [17, 16], the rank one perturbation theory shows that Lyapunov behavior can also be used to study Schrödinger operators with deterministic potentials. For almost periodic or random potentials, we have the subadditive ergodic theorem to guarantee the existence of the Lyapunov exponent, but for deterministic potentials, it's often difficult to prove Lyapunov behavior. In this paper, we first study the integrated density of states in detail, then we directly study the existence of the Lyapunov exponent and prove the Thouless formula for a.e. E (with Lebesgue measure).

Now, our formula for $\gamma(E)$, $E \in (-1, 1)$, which we prove off an explicitly given set of measure 0, is strictly positive. It is known (see [7]) that since $(-1, 1) \subset \sigma(H_\nu)$, the complement of $\{ E \mid \gamma(E) \text{ exists and is } > 0 \}$ is a dense G_δ in $[-1, 1]$. By our construction, this dense G_δ has measure zero; indeed, it has Hausdorff dimension zero.

We are unaware of any other explicit (non-random) Schrödinger operators with a computable positive Lyapunov exponent. The explicit formula (3.22) is quasi-classical.

2. The integrated density of states

To prove the Thouless formula, we need to study the integrated density of states, $k(E)$, and the existence of the Lyapunov exponent. Also, we need information on how rapidly $k^{(\ell)}(E)$ converges to $k(E)$ to establish the existence of the Lyapunov exponent. So, we first study the main technical object, the integrated density of states for equation (1.1). We will prove a formula for the integrated density states, and more importantly, we will estimate how fast $k^{(\ell)}(E)$ converges to $k(E)$.

The basic idea to compute the integrated density of states uses the standard Dirichlet-Neumann bracketing technique. Since the potentials in our problem are slowly oscillating, Dirichlet-Neumann bracketing works perfectly.

First, let us introduce some notation and definitions. In the following, when we write H_ν, we always mean the Schrödinger operator given by (1.1). Define

$$L = S_\nu(\ell) = (2\pi\ell)^{\frac{1}{\nu}}, \qquad \Omega_\ell = [S_\nu(\ell-1), S_\nu(\ell)], \qquad \text{for } \ell = 1, 2, \cdots.$$

Ω_ℓ is the lth potential well for the potential $V(x) = \cos x^\nu$ $(0 < \nu < 1)$. Let $H_D(\Omega)$, (resp. $H_N(\Omega)$) denote the self-adjoint operator $H_0 + V(x)$ on $L^2(\Omega)$ with Dirichlet (resp. Neumann) boundary conditions, where $H_0 = -\Delta$. When $\Omega = (0, L)$, we use $H_D(L)$, (resp. $H_N(L)$) to denote $H_D(\Omega)$, (resp. $H_N(\Omega)$). In this case, we use $H_{DN}(L)$, (resp. $H_{ND}(L)$) to denote the self-adjoint operator $H_0 + V(x)$ on $L^2(0, L)$ with Dirichlet (resp. Neumann) boundary condition at 0 and Neumann (resp. Dirichlet) boundary condition at L.

Definition. For any self-adjoint operator A, define

$$N(E, A) = \dim P_{(-\infty,E)}(A) = \sum_{E_k < E} 1$$

where $P_\Omega(A)$ is the spectral projection for the operator A, and $\{E_k\}$ are the eigenvalues of A with $E_1 \le E_2 \le E_3 \le \cdots$.

Now, let $H_{\rm bc}(S_\nu(\ell))$ be any self-adjoint realization of H_ν on $L^2(0, S(\ell))$ with some given boundary conditions at 0 and $S_\nu(\ell)$. Let $N_{\rm bc}(E, \ell) = N(E, H_{\rm bc}(S_\nu(\ell)))$.

Definition. Let $N_{\rm bc}(E, \ell)$ be as above, then we define

$$k^{(\ell)}(E) = \frac{1}{S_\nu(\ell)} N_{\rm bc}(E, \ell) \qquad \text{and} \quad k(E) = \lim_{\ell\to\infty} k^{(\ell)}(E).$$

$k(E)$ is called the integrated density of states for (1.1).

We will show that in the above definition, the limit $k(E)$ exists and is independent of the choice of boundary conditions.

By standard Dirichlet-Neumann bracketing (see [15]),

$$\sum_{j=1}^{\ell} N(E, H_D(\Omega_j)) \le N(E, H_D(L)) \le N(E, H_N(L)) \le \sum_{j=1}^{\ell} N(E, H_N(\Omega_j)). \tag{2.1}$$

By explicit construction and counting in boxes, we have

Lemma 2.1. *If we let $N_D(E; a, b)$ (resp. $N_N(E; a, b)$) denote the dimension of the spectral projection $P_{(-\infty,E]}$ for $-\Delta_D$ (resp. $-\Delta_N$) on $L^2(a, b)$. Then for $E < 0$, we have*

$$N_D(E; a, b) = N_N(E; a, b) = 0 \tag{2.2}$$

and for $E \ge 0$, we have

$$\left| N_D(E; a, b) - \frac{\sqrt{E}}{\pi}(b - a) \right| \le 1 \tag{2.3}$$

$$\left| N_N(E; a, b) - \frac{\sqrt{E}}{\pi}(b - a) \right| \le 1. \tag{2.4}$$

First, let us estimate $N(E,H_D(\Omega_j))$ and $N(E,H_N(\Omega_j))$. Let $a_k\in\Omega_j$ and $b_k=a_{k+1}$ such that $\cup_k[a_k,b_k]=\Omega_j$ and $b_k-a_k=j^\alpha$, where $\alpha>0$ (depending on ν) will be determined later. Let $I_k^{(j)}=(a_k,b_k)$ and

$$V_k^D=\sup\{V(x)\ |x\in[a_k,b_k]\},\qquad V_k^N=\inf\{V(x)\ |x\in[a_k,b_k]\}.$$

Define $B_D(I_k^{(j)})=-\Delta_D(I_k^{(j)})+V_k^D$ and $B_N(I_k^{(j)})=-\Delta_N(I_k^{(j)})+V_k^N$, then

$$0\le H_D(I_k^{(j)})\equiv-\Delta_D(I_k^{(j)})+V(x)\le B_D(I_k^{(j)})$$

and

$$0\le B_N(I_k^{(j)})\le-\Delta_N(I_k^{(j)})+V(x)\equiv H_N(I_k^{(j)}).$$

Obviously,

$$N(E,B_D(I_k^{(j)}))\le N(E,H_D(I_k^{(j)})),\qquad N(E,H_N(I_k^{(j)}))\le N(E,B_N(I_k^{(j)}))$$

and by Dirichlet-Neumann bracketing,

$$N(E,H_D(\Omega_j))\ge N(E,H_D(\cup I_k^{(j)}))=\sum_k N(E,H_D(I_k^{(j)}))\ge\sum_k N(E,B_D(I_k^{(j)}))\tag{2.5}$$

and

$$N(E,H_N(\Omega_j))\le N(E,H_N(\cup I_k^{(j)}))=\sum_k N(E,H_N(I_k^{(j)}))\le\sum_k N(E,B_N(I_k^{(j)})).\tag{2.6}$$

So, we only need to estimate $N(E,B_N(I_k^{(j)}))$ and $N(E,B_D(I_k^{(j)}))$. But by (2.2) and (2.4),

$$\begin{aligned}N(E,B_N(I_k^{(j)}))&=N_N(E-V_k^N;a_k,b_k)\\&=\begin{cases}\frac{\sqrt{E-V_k^N}}{\pi}(b_k-a_k)+C_0(k), & \text{if } E\ge V_k^N,\\ 0, & \text{if } E<V_k^N\end{cases}\end{aligned}$$

where $|C_0(k)|\le 1$.

Thus, if we use the notation that $[f(x)]_+=\max\{0,f(x)\}$, then we have

$$\left|N(E,B_N(I_k^{(j)}))-\frac{[E-V_k^N]_+^{\frac12}}{\pi}(b_k-a_k)\right|\le 1.\tag{2.7}$$

But

$$\begin{aligned}&\frac1\pi[E-V_k^N]_+^{\frac12}(b_k-a_k)-\frac1\pi\int_{a_k}^{b_k}[E-V(x)]_+^{\frac12}\,dx\\&\qquad=\frac1\pi\int_{a_k}^{b_k}\{[E-V_k^N]_+^{\frac12}-[E-V(x)]_+^{\frac12}\}\,dx\stackrel{\text{def}}{=}J.\end{aligned}\tag{2.8}$$

Since

$$
\begin{aligned}
\big\{[E - V_k^N]_+^{\frac{1}{2}} - &[E - V(x)]_+^{\frac{1}{2}}\big\}^2 \\
&\le \big|[E - V_k^N]_+^{\frac{1}{2}} - [E - V(x)]_+^{\frac{1}{2}}\big|\big\{[E - V_k^N]_+^{\frac{1}{2}} + [E - V(x)]_+^{\frac{1}{2}}\big\} \\
&\le \frac{\nu}{a_k^{1-\nu}}(b_k - a_k) \qquad \text{for } x \in I_k^{(j)}.
\end{aligned}
$$

By Schwartz inequality, we have,

$$
\begin{aligned}
|J| &\le \frac{1}{\pi}(b_k - a_k)^{\frac{1}{2}}\left[\int_{a_k}^{b_k} \{[E - V(x)]_+^{\frac{1}{2}} - [E - V_k^N]_+^{\frac{1}{2}}\}^2\, dx\right]^{\frac{1}{2}} \\
&\le \frac{\sqrt{\nu}}{\pi} a_k^{-\frac{1}{2}(1-\nu)}(b_k - a_k)^{\frac{3}{2}} \\
&\le j^{\frac{3}{2}\alpha - \frac{1}{2}\frac{1-\nu}{\nu}}.
\end{aligned} \tag{2.9}
$$

Therefore, by (2.7)–(2.9), we have

$$
\left|N(E, B_N(I_k^{(j)})) - \frac{1}{\pi}\int_{a_k}^{b_k} [E - V(x)]_+^{\frac{1}{2}}\, dx\right| \le j^{\frac{3}{2}\alpha - \frac{1}{2}\frac{1-\nu}{\nu}} + 1.
$$

Thus, by summing over k and using (2.6), we have

$$
N(E, H_N(\Omega_j)) \le \frac{1}{\pi}\int_{S_\nu(\ell-1)}^{S_\nu(\ell)} [E - V(x)]_+^{\frac{1}{2}}\, dx + C_1 j^{\frac{1}{2}(\alpha + \frac{1-\nu}{\nu})} + C_2 j^{\frac{1-\nu}{\nu} - \alpha} \tag{2.10}
$$

where C_1 and C_2 are independent of j.

Similarly, if we use (2.3) and (2.5) instead of (2.4) and (2.6), then we have

$$
N(E, H_D(\Omega_j)) \ge \frac{1}{\pi}\int_{S_\nu(\ell-1)}^{S_\nu(\ell)} [E - V(x)]_+^{\frac{1}{2}}\, dx - C_1 j^{\frac{1}{2}(\alpha + \frac{1-\nu}{\nu})} - C_2 j^{\frac{1-\nu}{\nu} - \alpha}. \tag{2.11}
$$

Now, by summing over j in (2.10), (2.11) and using (2.1), we have

$$
\begin{aligned}
\frac{1}{\pi}\int_0^{S_\nu(\ell)} [E - V(x)]_+^{\frac{1}{2}}\, dx - C_1 j^{\frac{1}{2}(\alpha + \frac{1-\nu}{\nu}) + 1} - C_2 j^{\frac{1-\nu}{\nu} - \alpha + 1} \le N(E, H_D(L)) \\
\le N(E, H_N(L)) \le \frac{1}{\pi}\int_0^{S_\nu(\ell)} [E - V(x)]_+^{\frac{1}{2}}\, dx + C_1 j^{\frac{1}{2}(\alpha + \frac{1-\nu}{\nu}) + 1} + C_2 j^{\frac{1-\nu}{\nu} - \alpha + 1}.
\end{aligned}
$$

So, if we take $\alpha = \frac{1}{3}\frac{1-\nu}{\nu}$, then we have

$$
\begin{aligned}
\frac{1}{\pi}\int_0^{S(\ell)} [E - V(x)]_+^{\frac{1}{2}}\, dx - C\ell^{\frac{2}{3}\frac{1-\nu}{\nu} + 1} \le N(E, H_D(L)) \\
\le N(E, H_N(L)) \le \frac{1}{\pi}\int_0^{S(\ell)} [E - V(x)]_+^{\frac{1}{2}}\, dx + C\ell^{\frac{2}{3}\frac{1-\nu}{\nu} + 1}
\end{aligned} \tag{2.12}
$$

where $C = C_1 + C_2$.

Also, we have the following estimation

$$\begin{aligned}\frac{1}{S_\nu(\ell)}\int_0^{S_\nu(\ell)}[E-V(x)]_+^{\frac12}\,dx &= \frac{1}{\nu(2\pi\ell)^{\frac1\nu}}\int_0^{2\pi\ell} y^{\frac{1-\nu}{\nu}}[E-\cos y]_+^{\frac12}\,dy \qquad (x^\nu = y)\\ &= \frac{1}{\nu(2\pi\ell)^2}\sum_{k=1}^{\ell}\int_{-2\pi}^{0}(z+2k\pi)^{\frac{1-\nu}{\nu}}[E-\cos z]_+^{\frac12}\,dz \qquad (y = z+2k\pi)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[E-\cos x]_+^{\frac12}\,dx + O(\ell^{-1}).\end{aligned}$$

Thus, if we denote

$$k_N^{(\ell)}(E) \equiv \frac{1}{S(\ell)}N(E, H_N(L)),$$

then by the above estimations and (2.12), we have

$$\left|k_N^{(\ell)}(E) - \frac{1}{2\pi^2}\int_{-\pi}^{\pi}[E-\cos x]_+^{\frac12}\,dx\right| = O(\ell^{-\frac13\frac{1-\nu}{\nu}}) + O(\ell^{-1}). \tag{2.13}$$

Since variations of boundary condition are rank one perturbations (see [16]),

$$|N(E, H_N(L)) - N(E, H_{bc}(L))| \le 2 \tag{2.14}$$

where $H_{\mathrm{bc}}(L)$ is defined by any other self-adjoint boundary condition.

Thus, by (2.13) and (2.14), we have proved the following

Theorem 2.1. *The integrated density of states for the Schrödinger operator* (*2.1*) *exists, which is independent of the boundary conditions, and is given by*

$$k(E) = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}[E-\cos x]_+^{\frac12}\,dx.$$

Moreover, we have the following estimation

$$|k^{(\ell)}(E) - k(E)| = O(\ell^{-\kappa(\nu)}) \tag{2.15}$$

where

$$\kappa(\nu) = \min\left\{\frac13\frac{(1-\nu)}{\nu}, 1\right\}. \tag{2.16}$$

3. The Thouless Formula and Lyapunov Exponent

Now, we begin to study the Lyapunov exponent by first proving the Thouless formula which relates the Lyapunov exponent to the integrated density of states. In [1], the Thouless formula is proved for almost periodic potentials and random potentials. To prove the Thouless formula in our case, we can closely follow the proof given in [1] for Schrödinger operators. However, we will prove the existence of the Lyapunov exponent by using the information on how fast $k^{\ell}(E)$ converges to $k(E)$ which is given in Theorem 2.1.

First, we define the transfer matrix for the Schrödinger operator (1.1) as follows. Let $u(x,a,E)$, $v(x,a,E)$ $(x \geq 0, a \geq 0)$ solve the equation $-\phi'' + (V(x) - E)\phi = 0$ with the boundary conditions given by $u(a) = 0$, $u'(a) = 1$; $v(a) = 1$, $v'(a) = 0$. Then the transfer matrix is defined by

$$T_{a,x}(E) = \begin{pmatrix} v(x,a,E) & u(x,a,E) \\ \frac{\partial v(x,a,E)}{\partial x} & \frac{\partial u(x,a,E)}{\partial x} \end{pmatrix}. \tag{3.1}$$

In particular, when $a = 0$, we use $T_x(E)$ to denote $T_{0,x}(E)$.

Definition. For a given E, if $\gamma(E) = \lim_{x\to\infty} x^{-1} \ln \|T_x(E)\|$ exists, then we say that for the energy E, H has Lyapunov behavior, and $\gamma(E)$ is called the Lyapunov exponent.

To give the Thouless formula, we need to define the resonance set first. In Section 2, we defined the operators $H_D(L)$, $H_N(L)$, $H_{DN}(L)$ and $H_{ND}(L)$. Now, let $\{E_k(\ell, D)\}$, $\{E_k(\ell, N)\}$, $\{E_k(\ell, DN)\}$, and $\{E_k(\ell, ND)\}$ be the corresponding eigenvalues.

Definition. For each given $\nu \in (0,1)$, let ϵ_ν be a fixed small number such that $\epsilon_\nu < \kappa(\nu)$, where $\kappa(\nu)$ is defined by (2.16). Then the resonance set, R_ν, for the operator H_ν is defined by

$$R_\nu = R_D \cup R_N \cup R_{DN} \cup R_{ND} \tag{3.2}$$

where

$$R_D = \bigcup_{d=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_k \{\, E \in [-d,d] \mid |E - E_k(n,D)| < \exp(-n^{\kappa(\nu)-\epsilon_\nu}) \,\}. \tag{3.3}$$

R_N, R_{DN} and R_{ND} are defined by replacing $\{E_k(\ell, D)\}$ in (3.3) by $\{E_k(\ell, N)\}$,$\{E_k(\ell, DN)\}$ and $\{E_k(\ell, ND)\}$ resp.

Remark. We conjecture that instead of (3.2) and (3.3), the resonance set in $[-1,1]$ can be defined by

$$R_\nu = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_k \{\, E \in [-1,1] \mid |E - E_k^{(n)}| < \exp(-n^{\min\{\frac{1-\nu}{2\nu}, \frac{1}{2}\}}) \,\}$$

where $\{E_k^{(n)}\}$ are the eigenvalues of $H_\nu = H_0 + V(x)$ on the nth potential well, $[(2n\pi - 2\pi)^{\frac{1}{\nu}}, (2n\pi)^{\frac{1}{\nu}}]$, with Dirichlet boundary conditions. We believe that this is the reasonable definition of the resonance set. However, in our proof of the Thouless formula, we need to use the resonance set defined by (3.2) and (3.3).

From the definition, it is easy to show that

Theorem 3.1. *Let R_ν be the resonance set for H_ν which is defined by (3.2) and (3.3) and let* $\dim_H$ *denote the Hausdorff dimension. Then*

$$|R_\nu| = \dim_H R_\nu = 0.$$

Now, we are ready to prove one of our main results.

Theorem 3.2 (Thouless formula). *Let H_ν be the Schrödinger operator given by (1.1). Let $\gamma_0(E) = [\max(0, -E)]^{\frac{1}{2}}$ and $k_0(E) = \pi^{-1}[\max(0, E)]^{\frac{1}{2}}$. Then for any $E \notin R_\nu$, where R_ν is defined by (3.2) and (3.3), we have*

$$\gamma(E) = \gamma_0(E) + \int_{-\infty}^{\infty} \ln|E - E'|\, d(k - k_0)(E') \tag{3.4}$$

where $\gamma(E)$ is the Lyapunov exponent for H_ν, and $k(E)$ is the integrated density of states for H_ν.

We prove this theorem by proving the following series of lemmas. The first three lemmas are already given in [1], so we will not give a proof for these results here.

Lemma 3.3 [1]. *For a.e. E,*

$$\lim_{\ell \to \infty} \ell^{-1} \ln|u_0(\ell, E)| = \gamma_0(E) \tag{3.5}$$

the limit being through the integers.

Lemma 3.4 [1]. *Let $E_k(\ell)$ be the eigenvalue of H_ν on $L^2[0, S_\nu(\ell)]$ with vanishing boundary conditions, and let $E_k^{(0)}(\ell) = (\pi k / S_\nu(\ell))^2$ be the corresponding eigenvalue of H_0. Then*

$$|E_k(\ell) - E_k^{(0)}| \leq \|V\|_\infty = 1. \tag{3.6}$$

Lemma 3.5 [1]. *For fixed ℓ, we have that*

$$\frac{u(S_\nu(\ell), E)}{u_0(S_\nu(\ell), E)} = \prod_{k=1}^{\infty} \left[\frac{E - E_k(\ell)}{E - E_k^{(0)}(\ell)} \right]. \tag{3.7}$$

¿From [1], we also know that

$$\begin{aligned} &\lim_{M \to \infty} \left[\int_{k(E') \leq M} \ln|E - E'|\, dk(E') - \int_{k_0(E') \leq M} \ln|E - E'|\, dk_0(E') \right] \\ &\quad = \int_{-\infty}^{\infty} \ln|E - E'|\, d(k - k_0)(E'). \end{aligned} \tag{3.8}$$

Lemma 3.6. *For $E \notin R_D$, we have*

$$\lim_{\ell\to\infty} \frac{1}{S_\nu(\ell)} \ln \prod_{k=1}^{\infty} \left| \frac{E - E_k(\ell)}{E - E_k^{(0)}(\ell)} \right| = \int_{-\infty}^{\infty} \ln|E - E'| d(k - k_0)(E'). \tag{3.9}$$

Proof. For a given $E \notin R_D$, without loss of generality, we can also suppose that $E \notin R_D^{(0)}$, where $R_D^{(0)}$ is the corresponding resonance set for H_0 with Dirichlet boundary condition. From now on, we always suppose that E is fixed and $E \notin R_D \cup R_D^{(0)}$.

For each fixed E, we can choose $M(\ell)$ such that $M(\ell) \to \infty$ as $\ell \to \infty$ and $a_i(\ell) > E + 1$ $(i = 0, 1)$, where

$$a_0(\ell) = \sup\{E' \mid k_0^{(\ell)}(E') \le M(\ell)\}, \qquad a_1(\ell) = \sup\{E' \mid k^{(\ell)}(E') \le M(\ell)\}.$$

For convenience, we define

$$f_\ell(E) = \frac{1}{S_\nu(\ell)} \ln \prod_{k=1}^{\infty} \left| \frac{E - E_k(\ell)}{E - E_k^{(0)}(\ell)} \right|, \qquad f(E) = \int_{-\infty}^{\infty} \ln|E - E'|\, d(k - k_0)(E').$$

Then we have

$$\begin{aligned}
|f_\ell(E) - f(E)| = &\Bigg| \frac{1}{S_\nu(\ell)} \ln \prod_{k \le M(\ell)S_\nu(\ell)} \big|[E - E_k(\ell)]/[E - E_k^{(0)}(\ell)]\big| \\
&+ \frac{1}{S_\nu(\ell)} \ln \prod_{k > M(\ell)S_\nu(\ell)} \big|[E - E_k(\ell)]/[E - E_k^{(0)}(\ell)]\big| - f(E) \Bigg| \\
\le &\left| \int_{-\infty}^{a_1(\ell)} \ln|E - E'|\, d(k^{(\ell)} - k)(E') - \int_{-\infty}^{a_0(\ell)} \ln|E - E'|\, d(k_0^{(\ell)} - k_0)(E') \right| \\
&+ \left| \int_{a_1(\ell)}^{\infty} \ln|E - E'|\, dk(E') - \int_{a_0(\ell)}^{\infty} \ln|E - E'|\, dk_0(E') \right| \\
&+ \left| \frac{1}{S_\nu(\ell)} \ln \prod_{k > M(\ell)S_\nu(\ell)} \big|[E - E_k(\ell)]/[E - E_k^{(0)}(\ell)]\big| \right|.
\end{aligned} \tag{3.10}$$

By (3.8), we have

$$\lim_{\ell\to\infty} \left| \int_{a_1(\ell)}^{\infty} \ln|E - E'|\, dk(E') - \int_{a_0(\ell)}^{\infty} \ln|E - E'|\, dk_0(E') \right| = 0. \tag{3.11}$$

Since $E_k^{(0)}(\ell) = \big(\pi k/S_\nu(\ell)\big)^2$, by using lemma 3.4, we have

$$\begin{aligned}
\ln \prod_{k > M(\ell)S_\nu(\ell)} \left| \frac{E - E_k(\ell)}{E - E_k^{(0)}(\ell)} \right| &= \sum_{k > M(\ell)S_\nu(\ell)} \ln \left| 1 + \frac{E_k(\ell) - E_k^{(0)}}{E_k^{(0)}(\ell) - E} \right| \\
&\le \sum_{k > M(\ell)S_\nu(\ell)} S_\nu^2(\ell)/[\pi^2 k^2 - S_\nu^2(\ell)E] \\
&\le S_\nu(\ell) \int_{M(\ell)}^{\infty} \frac{dx}{\pi^2 x^2 - E}.
\end{aligned}$$

Therefore,

$$\left|\frac{1}{S_\nu(\ell)}\ln\prod_{k>M(\ell)S_\nu(\ell)}\left|[E-E_k(\ell)]/[E-E_k^{(0)}(\ell)]\right|\right| = O\left(\frac{1}{M(\ell)}\right). \tag{3.12}$$

So, it remains to estimate

$$J_\ell \equiv \left|\int_{-\infty}^{a_1(\ell)} \ln|E-E'|\, d(k^{(\ell)}-k)(E') - \int_{-\infty}^{a_0(\ell)} \ln|E-E'|\, d(k_0^{(\ell)}-k_0)(E')\right|.$$

We define

$$I_\ell(E) = \left[E-\delta_\ell, E+\delta_\ell\right], \qquad \delta_\ell = \frac{1}{3}\exp(-\ell^{\kappa(\nu)-\epsilon_\nu}) \tag{3.13}$$

where $\kappa(\nu)$ is defined by (2.16) and ϵ_ν is given in definition of the resonance set.

Since $E \notin R_D \cup R_D^{(0)}$, there are no eigenvalues of $H_D(L)$ and $H_{0D}(L)$ on the interval $I_\ell(E)$ which is defined by (3.13). Thus, $k^{(\ell)}(E)$, $k_0^{(\ell)}(E)$ are constant on the interval $I_\ell(E)$. Also, we notice that

$$\left|\int_{I_\ell(E)} \ln|E-E'|\, dk(E')\right| \le C_E[|I_\ell(E)|]^{\frac{1}{2}} \tag{3.14}$$

where C_E is a constant for a given E. So, we have

$$\begin{aligned} J_\ell = &\left|\int_{(-\infty,a_1(\ell)]\setminus I_\ell(E)} \ln|E-E'|\, d(k^{(\ell)}-k)(E') + \int_{I_\ell(E)} \ln|E-E'|\, d(k^{(\ell)}-k)(E')\right.\\ &\left.- \int_{(-\infty,a_0(\ell)]\setminus I_\ell(E)} \ln|E-E'|\, d(k_0^{(\ell)}-k_0)(E') - \int_{I_\ell(E)} \ln|E-E'|\, d(k_0^{(\ell)}-k_0)(E')\right|\\ \le &\left|\int_{(-\infty,a_1(\ell)]\setminus I_\ell(E)} \ln|E-E'|\, d(k^{(\ell)}-k)(E')\right| + \left|\int_{I_\ell(E)} \ln|E-E'|\, dk(E')\right|\\ &+ \left|\int_{(-\infty,a_0(\ell)]\setminus I_\ell(E)} \ln|E-E'|\, d(k_0^{(\ell)}-k_0)(E')\right| + \left|\int_{I_\ell(E)} \ln|E-E'|\, dk_0(E')\right|. \end{aligned} \tag{3.15}$$

By (3.14), we know that

$$\lim_{\ell\to\infty}\int_{I_\ell(E)} \ln|E-E'|\, dk(E') = 0. \tag{3.16}$$

Similarly,

$$\lim_{\ell\to\infty}\int_{I_\ell(E)} \ln|E-E'|\, dk_0(E') = 0. \tag{3.17}$$

Using integration by parts, we have

$$\begin{aligned} &\left|\int_{(-\infty,a_1(\ell)]\setminus I_\ell(E)} \ln|E-E'|\, d(k^{(\ell)}-k)(E')\right|\\ &\quad\le (k^{(\ell)}-k)(a_1(\ell))\ln|E-a_1(\ell)| + \{(k^{(\ell)}-k)(E+\delta_\ell) - (k^{(\ell)}-k)(E-\delta_\ell)\}\ln\delta_\ell\\ &\qquad+ \left|\int_{(-\infty,a_1(\ell)]\setminus I_\ell(E)} \frac{(k^{(\ell)}-k)(E')}{E'-E}\, dE'\right|. \end{aligned}$$

By theorem 2.1 and (3.13), we know that

$$\begin{aligned}
&\lim_{\ell\to\infty}(k^{(\ell)}-k)(a_1(\ell))\ln|E-a_1(\ell)|=0\\
&\lim_{\ell\to\infty}\{(k^{(\ell)}-k)(E+\delta_\ell)-(k^{(\ell)}-k)(E-\delta_\ell)\}\ln\delta_\ell=0
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_{(-\infty,a_1(\ell)]\setminus I_\ell(E)}\frac{(k^{(\ell)}-k)(E')}{E'-E}\,dE'\right| &\le C_1\ell^{-\kappa(\nu)}\left|\int_{(-\infty,a_1(\ell)]\setminus I_\ell(E)}\frac{1}{E'-E}\,dE'\right|\\
&\le \ell^{-\kappa(\nu)}\{C_2\ln\delta_\ell+C_3\ln|a_1(\ell)-E|\}\\
&\to 0 \qquad \text{as } \ell\to\infty.
\end{aligned}$$

Thus,

$$\lim_{\ell\to\infty}\left|\int_{(-\infty,a_1(\ell)]\setminus I_\ell(E)}\ln|E-E'|\,d(k^{(\ell)}-k)(E')\right|=0. \tag{3.18}$$

Similarly,

$$\lim_{\ell\to\infty}\left|\int_{(-\infty,a_0(\ell)]\setminus I_\ell(E)}\ln|E-E'|\,d(k_0^{(\ell)}-k_0)(E')\right|. \tag{3.19}$$

So, by (3.15)–(3.19),

$$\lim_{\ell\to\infty}J_\ell=0. \tag{3.20}$$

Now, by (3.10)–(3.12) and (3.20), we have proved that $\lim_{\ell\to\infty}|f_\ell(E)-f(E)|=0$. Therefore, Lemma 3.6 is proved. □

Now, by combining the results of Lemma 3.5 and Lemma 3.6, we have proved the following result.

For $E\notin R_D$, then we have that

$$\lim_{\ell\to\infty}\frac{1}{S_\nu(\ell)}\ln\left|\frac{u(S_\nu(\ell),E)}{u_0(S_\nu(\ell),E)}\right|=\int_{-\infty}^{\infty}\ln|E-E'|d(k-k_0)(E').$$

By using Lemma 3.3, we obtain the following control on the limit

$$\lim_{\ell\to\infty}\frac{1}{S_\nu(\ell)}\ln|u(S_\nu(\ell),E)|=\gamma_0(E)+\int_{-\infty}^{\infty}\ln|E-E'|\,d(k-k_0)(E').$$

By using different boundary conditions, we can obtain similar control of $\frac{1}{S_\nu(\ell)}\ln\big|v(S_\nu(\ell),E)\big|$, $\frac{1}{S_\nu(\ell)}\ln\big|\frac{\partial u(S_\nu(\ell),E)}{\partial x}\big|$ and $\frac{1}{S_\nu(\ell)}\ln\big|\frac{\partial v(S_\nu(\ell),E)}{\partial x}\big|$. Therefore, we obtain control of $\frac{1}{S(\ell)}\ln\|T_{S(\ell)}(E)\|$, namely

Lemma 3.7. *For $E \notin R_\nu$, where R_ν is the resonance set defined by (3.2) and (3.3), then*

$$\lim_{\ell\to\infty} \frac{1}{S(\ell)} \ln \|T_{S(\ell)}(E)\| = \gamma_0(E) + \int_{-\infty}^{\infty} \ln|E-E'|\, d(k-k_0)(E') \tag{3.21}$$

where $\|\cdot\|$ denotes the matrix norm, and $T_x(E)$ is defined by (3.1).

Now, Theorem 3.2 follows from Lemma 3.7 and definition of the Lyapunov exponent.

Next, we want to compute an explicit formula for the Lyapunov exponent by using the Thouless formula and the formula for integrated density of states. First, (3.4) asserts that $\pi k + i\gamma$ is the boundary value of an analytic function in the upper half plane. Let $F(z) = \pi k(z) + i\gamma(z)$ for $\operatorname{Im} z \geq 0$, and define $\tilde{F}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt{z-\cos x}\, dx$ with branch cut from -1 to ∞ along the real axis. Then $\tilde{F}(z)$ is analytic for $\operatorname{Im} z > 0$ and by Theorem 2.1, $\operatorname{Re}\tilde{F}(z) \to \pi k(E)$ as $z \to E$ ($\operatorname{Im} z > 0$, $E \in \mathbb{R}$). Therefore,

$$\gamma(E) = \lim_{\operatorname{Im} z>0, z\to E} \operatorname{Im}\tilde{F}(z) + C$$

where C is a real constant. That is,

$$\gamma(E) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos x - E]_+^{\frac{1}{2}}\, dx + C.$$

Notice that for $E > 1$, $\gamma(E) = 0$ and the integral in the right-hand side is also zero, so $C = 0$. Therefore, we have

Theorem 3.3. *For all $E \notin R_\nu$, where R_ν is defined by (3.2) and (3.3), the operator H_ν in (1.1) has Lyapunov behavior with the Lyapunov exponent given by*

$$\gamma(E) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos x - E]_+^{\frac{1}{2}}\, dx \tag{3.22}$$

where $[f(x)]_+ = \max\{0, f(x)\}$.

Remarks. 1. In fact, there is no mystery for this beautiful Lyapunov exponent formula if we use the WKB (see [8],[9]) heuristic argument. However, it's not easy to justify the WKB solutions.

2. Note that while R_ν is ν-dependent, the right-hand side of (3.22) is ν-independent!

4. Some Spectral Consequences

We have already proved that for a.e. $E \in [-1,1]$, H_ν has positive Lyapunov exponent. By simply applying the Kotani argument (see [13]) or rank one spectral theory (see [7, 16, 17]), we can get dense pure point spectrum on $(-1,1)$ for almost all boundary conditions. Also, we can show that the eigenfunctions are exponentially decaying. The result on pure point spectrum is an unpublished result by Kirsch and Stolz which is stated in [12] by Kirsch, Molchanov and Pastur, and the result on exponentially decaying is proved by Stolz in [21]. Now, we can give an explicit decaying rate of eigenfunctions.

Theorem 4.1. *Let H_ν^θ be the operator H_ν given by (1.1) with the θ boundary condition at 0, $u(0)\cos\theta + u'(0)\sin\theta = 0$. Then for a.e. $\theta \in [0,\pi)$ (with respect to Lebesgue measure), H_ν^θ has dense pure point spectrum on $(-1,1)$, and the eigenfunctions of H_ν^θ to all eigenvalues $E \in (-1,1)$ decay like $e^{-\gamma(E)x}$ at ∞ for almost every θ, where $\gamma(E)$ is given by (3.22).*

Next, as we have shown that the resonance set has Hausdorff dimension zero, by applying rank one perturbation theory, we get a new result on singular continuous spectral.

Theorem 4.2. *Let H_ν^θ be the operator H_ν given by (1.1) with the boundary condition at 0 given by $u(0)\cos\theta + u'(0)\sin\theta = 0$ for $\theta \in [0,\pi)$. Then for $\theta \neq \frac{\pi}{2}$, the singular continuous part, $(d\mu_\theta)_{\rm sc}$, of the spectral measure $d\mu_\theta$ for H_ν^θ is supported on a Hausdorff dimension zero set.*

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