Equilibrium Measures for Coupled Map Lattices:
Existence, Uniqueness and Finite-Dimensional Approximations

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Abstract We extend thermodynamic formalism to coupled map lattices of hyperbolic type and prove existence, uniqueness, and mixing properties of equilibrium measures for a class of Hölder continuous potential functions. We also describe finite-dimensional approximations of equilibrium measures. We apply our results to establish existence and uniqueness of SRB-type measures.

Introduction

Coupled map lattices are infinite-dimensional dynamical systems introduced by K. Kaneko [Ka] in 1983 as simple models with essential features of spatio-temporal chaos. These systems usually consist of identical local finite-dimensional subsystems at lattice points each interacting with its neighboring subsystems. Such systems are proven to be useful in studying qualitative properties of spatially extended dynamical systems. They can rather easily be monitored by a computer, and many remarkable results about coupled map lattices were obtained by researchers working in different areas of physics, biology, mathematics, and engineering.

Bunimovich and Sinai initiated rigorous mathematical study of coupled map lattices in [BuSi]. They constructed special SRB-type measures for weakly coupled expanding circle maps (under some additional assumptions that the interaction is of finite range and preserves the unique fixed point of the map). SRB-type measures are invariant under both space and time translations and have strong ergodic properties, for example, mixing. ¿From the physical point of view this is interpreted as evidence of spatio-temporal chaos. In [BK1]–[BK3], Bricmont and Kupiainen extended results of Bunimovich and Sinai to general expanding circle maps. In [KK], Keller and Künzle studied the case when the local subsystems are piecewise smooth interval maps. A detailed survey on this topic can be found in [Bu].

The first attempt to consider coupled map lattices with multidimensional local subsystems of hyperbolic type was made by Pesin and Sinai in [PS]. Assuming that the local subsystem possesses a hyperbolic attractor they constructed conditional distributions for the SRB-type measure on unstable local manifolds. In [J1], [J2], Jiang considered the case when a local subsystem possesses a hyperbolic set and obtained some partial results on the existence and uniqueness of Gibbs distributions. In this paper we extend these results and establish the existence and uniqueness of Gibbs distributions for arbitrary chain of weakly interacting hyperbolic sets.

Our main tool of study is the thermodynamic formalism applied to the lattice spin system of statistical mechanics associated with a given coupled map lattice. We point out that the lattice spin systems corresponding to coupled map lattices are of a special type and have not been studied in the framework of the "classical" statistical mechanics till recently. The study of Gibbs distributions for these special lattice spin systems required new and advanced technique which was developed in [JM] and [BK2], [BK3].

In [JM], the authors considered two-dimensional lattice spin systems. Using polymer expansions of partition functions they found an explicit formula for Gibbs state in terms of potentials and thus, proved existence and uniqueness of Gibbs states for potentials obtained from the corresponding coupled map lattices. They also established continuity of Gibbs states over such potentials. In [BK2], [BK3], the authors considered general multidimensional lattice spin systems. Using expansions of the correlation functions they also established existence and uniqueness of the Gibbs states as well as the mixing property for the same type of potentials. The reader will find a detailed discussion of lattice spin systems and their relation to coupled map lattices in the paper. Appendix contains a description of polymer expansions. This makes the paper more self-contained and can be viewed as an introduction to a highly specialized area of statistical physics.

The paper is divided into five sections. In the first three sections we generalize results of [J1] on the topological structure of coupled map lattices of hyperbolic type. Our main result is that these systems are structurally stable. When the coupling is exponentially weak the conjugacy map allows one to use Markov partitions for the uncoupled map lattice to build up Markov partitions for the coupled map lattice. This leads to a symbolic representation of the lattice system into a lattice spin system of statistical mechanics. In [JM] the authors established uniqueness of Gibbs states and exponential decay of correlations for these lattice spin systems. We use their results as well as results in [BK3] to establish uniqueness and mixing property of equilibrium measures.

In Section 4 we construct "natural" finite-dimensional approximations of equilibrium measures. There are two different types of approximations: one results from considering finite volumes in the lattice and the other one — from considering finite volumes in the lattice spin systems.

In Section 5 we apply our results to establish the existence, uniqueness, and mixing property of SRB-type measures for chains of weakly interacting hyperbolic attractors. We show that these measures are Gibbs states for Hölder continuous functions and we obtain them by describing their finite-dimensional approximations in terms of lattice spin systems.

I. Coupled Map Lattices

1.1. Definition of Coupled Map Lattices.

Let M be a smooth compact Riemannian manifold and f a C^r-map of M, $r \geq 1$. Let also $\mathbb{Z}^d$, $d \geq 1$ be the d-dimensional integer lattice. Set $\mathcal{M} = \otimes_{i \in \mathbb{Z}^d} M_i$, where M_i are copies of M. The space $\mathcal{M}$ admits the structure of an infinite-dimensional Banach manifold with the Finsler metric induced by the Riemannian metric on M, i.e.,

$$\|\bar{v}\| = \sup_{i \in \mathbb{Z}^d} \|v_i\|. \tag{1.1}$$

The distance in $\mathcal{M}$ induced by the Finsler metric is given as follows

$$\rho(\bar{x}, \bar{y}) = \sup_{i \in \mathbb{Z}^d} d(x_i, y_i), \tag{1.2}$$

where $\bar{x} = (x_i)$ and $\bar{y} = (y_i)$ are two points in $\mathcal{M}$ and d is the Riemannian distance on M.

We define the *direct product* map on $\mathcal{M}$ by $F = \otimes_{i \in \mathbb{Z}^d} f_i$, where f_i are copies of f.

Consider a map G on $\mathcal{M}$ which is C^r-close to the identity map *id*. Set $\Phi = F \circ G$. The map G is said to be an *interaction* between points (*space sites*) of the lattice $\mathbb{Z}^d$ and the map Φ is said to be a *perturbation* of F. Iterates of the map Φ generate a $\mathbb{Z}$-action on $\mathcal{M}$ called *time translations.*

We also consider the group action of the lattice $\mathbb{Z}^d$ on $\mathcal{M}$ by *spatial translations* S^k. Namely, for any $k \in \mathbb{Z}^d$ and any $\bar{x} = (x_i) \in \mathcal{M}$, we set $\left(S^k(\bar{x})\right)_i = x_{i+k}$.

The pair of actions (Φ, S) on $\mathcal{M}$ is called a *coupled map lattice* generated by the *local map* f and the interaction G. If G commutes with the spatial translations S^k, i.e., $S^k \circ G = G \circ S^k$, we call G *spatial translation invariant.* In this case the pair (Φ, S) generates a $\mathbb{Z}^{d+1}$-action on $\mathcal{M}$. If $G = id$, the lattice is called *uncoupled.*

One can also define the perturbation in the form $\Phi = G \circ F$. If F is invertible((and in what follows we will always assume this) the study of perturbations of such a form is equivalent to the study of perturbations in the previous form since $G \circ F = F \circ (F^{-1} \circ G \circ F)$ with $F^{-1} \circ G \circ F$ being close to the identity.

1.2. Coupled Map Lattices of Hyperbolic Type.

We consider a special type of coupled map lattice assuming that the local map is hyperbolic. More precisely, let $U \subset M$ be an open set, $f : U \to M$ a C^1-diffeomorphism, and $\Lambda \subset U$ a closed invariant *hyperbolic* set for f. The latter means that the tangent bundle $T_\Lambda M$ over Λ is split into two subbundles: $T_\Lambda M = E^s \bigoplus E^u$, where E^s and E^u are both invariant under the differential Df, and for some $C > 0$ and $0 < \lambda < 1$,

$$\|Df^n v\| \leq C\lambda^n \|v\| \text{ for } n \geq 0, \quad v \in E^s; \tag{1.3}$$

$$\|Df^{-n} w\| \leq C\lambda^n \|w\| \text{ for } n \geq 0, \quad w \in E^u.$$

The hyperbolic set Λ is called *locally maximal* if there exists an open set $U \supset \Lambda$ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(\bar{U})$, where $\bar{U}$ is the closure of U.

For any point x in a hyperbolic set Λ one can construct local stable and unstable manifolds defined by

$$V^s(x) = \{y \in M : d(x,y) \leq \epsilon,\ d(f^n(x), f^n(y)) \to 0,\ n \to +\infty\};$$

$$V^u(x) = \{y \in M : d(x,y) \leq \epsilon,\ d(f^n(x), f^n(y)) \to 0,\ n \to -\infty\}. \tag{1.4}$$

It is known that these submanifolds are as smooth as the map f is.

The definition of hyperbolicity can easily be extended to diffeomorphisms of Banach manifolds. Suppose that H is a C^1-diffeomorphism of an open set U of a Banach manifold $\mathcal{N}$ (endowed with a Finsler metric) and a set $\Delta \subset U$ is invariant under H (note that Δ may not be compact). We say that Δ is *hyperbolic* if the tangent bundle $T_\Delta\mathcal{N}$ over Δ admits a splitting $T_\Delta\mathcal{N} = E^s \oplus E^u$ with the following properties:

1) E^s and E^u are invariant under the differential DH;

2) for any continuous sections v valued in E^s and w valued in E^u we have

$$\|DH^n v\| \leq C\lambda^n\|v\| \quad \text{and} \quad \|DH^{-n}w\| \leq C\lambda^n\|w\|,$$

for some constants $C > 0$ and $0 < \lambda < 1$ independent of v and w;

3) there exists $b > 0$ such that for any z the angle between $E^s(z)$ and $E^u(z)$ is bounded away from zero, i.e.,

$$\inf\{\|\xi - \eta\| : \xi \in E^s(z),\ \eta \in E^u(z)\|,\ \|\xi\| = \|\eta\| = 1\} \geq b. \tag{1.5}$$

Note that in the finite-dimensional case the last condition holds true automatically.

It is easy to see that the map F is hyperbolic in the above sense, i.e., it possesses an infinite-dimensional hyperbolic set

$$\Delta_F = \otimes_{i\in\mathbb{Z}^d}\Lambda_i,$$

where Λ_i is a copy of Λ. Moreover, for each point $\bar{x} = (x_i) \in \Delta_F$ the tangent space $T_{\bar{x}}\mathcal{M}$ admits the splitting $T_{\bar{x}}\mathcal{M} = E^s(\bar{x}) \oplus E^u(\bar{x})$, where

$$E^s(\bar{x}) = \otimes_{i\in\mathbb{Z}^d} E^s(x_i), \quad E^u(\bar{x}) = \otimes_{i\in\mathbb{Z}^d} E^u(x_i). \tag{1.6}$$

Furthermore, for each point $\bar{x} = (x_i) \in \Delta_F$ the local stable and unstable manifolds passing through $\bar{x}$ are

$$V^s_F(\bar{x}) = \otimes_{i\in\mathbb{Z}^d} V^s_i(x_i), \quad V^u_F(\bar{x}) = \otimes_{i\in\mathbb{Z}^d} V^u_i(x_i), \tag{1.7}$$

where $V^s_i(x_i)$ and $V^u_i(x_i)$ are the local stable and unstable manifolds at x_i respectively. If the hyperbolic set Λ is locally maximal, so is Δ_F.

1.3. Short Range Maps.

The goal of this paper is to investigate metric properties of coupled map lattices of hyperbolic type. In the finite-dimensional case one uses *thermodynamic formalism* (see [Bo], [Ru]) to construct invariant measures and then studies ergodicity of hyperbolic maps with respect to these measures. The extension of this formalism to the infinite-dimensional case faces some obstacles. Among them the most crucial one is non-compactness of the hyperbolic set Δ_F. One of the ways to overcome this obstacle is to introduce a new metric on $\mathcal{M}$ with respect to which the space becomes compact. This metric is known as a *metric with weights* and is defined as follows: given $0 < q < 1$ and $\bar{x}, \bar{y} \in \mathcal{M}$, we set

$$\rho_q(\bar{x}, \bar{y}) = \sup_{i\in\mathbb{Z}^d} q^{|i|} d(x_i, y_i), \tag{1.8}$$

where $|i| = |i_1| + |i_2| + \cdots + |i_d|$, $i = (i_1, i_2, \cdots, i_d) \in \mathbb{Z}^d$.

For different $0 < q < 1$ the metrics ρ_q induce the same compact (Tychonov) topology in $\mathcal{M}$.

Although working with ρ_q-metrics gives us some advantages in studying invariant measures for the maps F and Φ it also brings some new problems. For example, the set $\mathcal{M}$ is no longer a differential manifold and the maps F and Φ, while being continuous, need not be differentiable. In particular, the set Δ_F being compact is no longer hyperbolic in the above sense but in some weak one. More precisely, this set is *topologically hyperbolic*, i.e., for every point in Δ_F the local stable and unstable manifolds (1.7) are, in general, only continuous (not smooth).

We will impose a restriction to the class of perturbations we consider to be able to keep track of hyperbolic behavior of trajectories for the perturbation map Φ. More precisely, we consider the special class of perturbations called short range maps. The concept of short range maps was introduced by Bunimovich and Sinai in [BuSi] and was further developed by Pesin and Sinai in [PS] (see also [KK]). We follow their approach.

Let $\mathcal{Y}$ be a subset of $\mathcal{M}$ and $G : \mathcal{Y} \to \mathcal{M}$ a map. We say that G is *short ranged* if G is of the form $G = (G_i)_{i\in\mathbb{Z}^d}$, where $G_i : \mathcal{Y} \to M_i$ satisfy the following condition: for any fixed $k \in \mathbb{Z}^d$ and any points $\bar{x} = (x_j), \bar{y} = (y_j) \in \mathcal{Y}$ with $x_j = y_j$ for all $j \in \mathbb{Z}^d, j \neq k$ we have

$$d(G_i(\bar{x}), G_i(\bar{y})) \leq C\theta^{|i-k|} d(x_k, y_k), \tag{1.9}$$

where C and θ are constants and $C < 0$, $0 < \theta < 1$. We call θ the *decay constant* of G.

If G is spatial translation invariant then G can be shown to be short ranged with a decay constant θ if and only if

$$d(G_0(\bar{x}), G_0(\bar{y})) \leq C\theta^{|k|} d(x_k, y_k) \tag{1.10}$$

for any $\bar{x} = (x_j), \bar{y} = (y_i) \in \mathcal{Y}$ with $x_j = y_j$ for all $j \in \mathbb{Z}$, $j \neq k$.

We list some basic properties of short range maps. The proofs of Propositions 1.1–1.3 can be found in [J1].

Proposition 1.1. *Let G be a C^1-diffeomorphism of an open set $\mathcal{U} \subset \mathcal{M}$ onto its image. Assume that G is short ranged with a decay constant θ. Then*

(1) the differential of G at every point $\bar{x}$, $D_{\bar{x}}G : T_{\bar{x}}\mathcal{M} \to T_{\bar{x}}\mathcal{M}$, is a short range linear map with the same decay constant θ;

(2) the bundle map DG is short ranged with the same decay constant θ.

Moreover, if the map G is continuous with respect to a ρ_q-metric then either of statements (1) or (2) implies that G is short ranged.

Proposition 1.2. *For any $0 < \theta < 1$ there exists $\epsilon > 0$ such that if $G : \mathcal{M} \to \mathcal{M}$ is a short range $C^{1+\alpha}$-diffeomorphism with the decay constant θ and $\mathrm{dist}_{C^1}(G, id) < \epsilon$ then G^{-1} is also a short range map.*

Short range maps are well adopted with the metric structure of $\mathcal{M}$ generated by ρ_q-metrics as the following result shows.

Proposition 1.3. *(1) Let $G : \mathcal{M} \to \mathcal{M}$ be a short range map with a decay constant θ. Then G is Lipschitz continuous as a map from $(\mathcal{M}, \rho_q)$ into itself for any $q > \theta$.*

(2) If G is a Lipschitz continuous map from $(\mathcal{M}, \rho_q)$ to $(\mathcal{M}, \rho_{q_1})$, with some $0 < q_1 < 1$, then G is short ranged with the decay constant $\theta = q$.

(3) For any $\epsilon > 0$ and $0 < \theta < q < 1$ there exist $\delta > 0$ such that if G is a $C^{1+\alpha}$-spatial translation invariant short range map of $\mathcal{M}$ with the decay constant θ and $\mathrm{dist}_{C^1}(G, id) \leq \delta$ then G is Lipschitz continuous in the ρ_q-metric with a Lipschitz constant $L \leq 1 + \epsilon$.

1.4. Structural Stability.

We consider the problem of structural stability of coupled map lattices of hyperbolic type $(\mathcal{M}, F)$. It is well-known that finite-dimensional hyperbolic dynamical systems are structurally stable (see for example, [KH], [Sh]) and so are hyperbolic maps of Banach manifolds which admit a partition of unity (see [Lang]). We stress that $\mathcal{M}$ does not admit a partition of unity and this result can not be applied in the direct way. In order to study structural stability we will exploit the special structure of the system $(\mathcal{M}, F)$ which is the direct product of countably many copies of the *same* finite-dimensional dynamical system (M, f). This enables us to establish structural stability by modifying arguments from the proof in the finite-dimensional case.

¿From now on we always assume that the interaction G is short ranged.

Theorem 1.1. *(1) For any $\epsilon > 0$ there exists $0 < \delta < \delta_0$ such that, if $\mathrm{dist}_{C^1}(\Phi, F) \le \delta$, then there is a unique homeomorphism $h : \Delta_F \to \mathcal{M}$ satisfying $\Phi \circ h = h \circ F|_{\Delta_F}$ with $\mathrm{dist}_{C^0}(h, id) \le \epsilon$. In particular, the set $\Delta_\Phi = h(\Delta_F)$ is hyperbolic and locally maximal.*

(2) For any $0 < \theta < 1$ there exists $\delta > 0$ such that if G is a C^2-spatial translation invariant short range map with a decay constant θ and $\mathrm{dist}_{C^1}(G, id) \le \delta$ then the conjugacy map h is Hölder continuous with respect to the metric ρ_q, $0 < q < 1$. Moreover, $h = (h_i(\bar{x}))_{i \in \mathbb{Z}^d}$ satisfies the following property:

$$d(h_0(\bar{x}), h_0(\bar{y})) \le C(\delta) d^{\alpha}(x_k, y_k) \tag{1.11}$$

for every $k \ne 0$ and any $\bar{x}, \bar{y} \in \mathcal{M}$ with $x_i = y_i, i \in \mathbb{Z}^d, i \ne k$, where $0 < \alpha < 1$ and $C(\delta) > 0$ is a constant. Furthermore, $C(\delta) \to 0$ as $\mathrm{dist}_{C^1}(G, id) \to 0$.

Proof. We describe main steps of the proof of Statement 1 recalling those arguments that will be used below (detailed arguments can be found in [J1]). Let $U(\Delta_F)$ be an open neighborhood of Δ_F and $C^0(\Delta_F, U(\Delta_F))$ the space of all continuous maps from Δ_F to $U(\Delta_F)$. Consider the map

$$\mathcal{G} : C^0(\Delta_F, U(\Delta_F)) \to C^0(\Delta_F, \mathcal{M}) \tag{1.12}$$

defined by $\beta \longmapsto \Phi \circ \beta \circ F^{-1}$. We wish to show that $\mathcal{G}$ has a unique fixed point near the identity map. Let $\Gamma^0(\Delta_F, T\mathcal{M})$ be the space of all continuous vector fields on Δ_F. We denote by $\mathcal{I}$ the identity embedding of Δ_F into $\mathcal{M}$, by $B_\gamma(\mathcal{I})$ the ball in $C^0(\Delta_F, U(\Delta_F))$ centered at $\mathcal{I}$ of radius γ, and by $\mathcal{A} : B_\gamma(\mathcal{I}) \to \Gamma^0(\Delta_F, T\mathcal{M})$ the map that is defined as follows:

$$\mathcal{A}\beta(\bar{y}) = (\exp_{y_i}^{-1} \beta_i(\bar{y}))_{i \in \mathbb{Z}}. \tag{1.13}$$

When γ is small $\mathcal{A}$ is a homeomorphism onto the ball $D_\gamma(0)$ in $\Gamma^0(\Delta_F, T\mathcal{M})$ centered at the zero section 0 of radius γ. Set

$$\mathcal{G}' = \mathcal{A} \circ \mathcal{G} \circ \mathcal{A}^{-1} : \ D_\gamma(0) \to \Gamma^0(\Delta_F, T\mathcal{M}). \tag{1.14}$$

If a section $v \in D_\gamma(0)$ is a fixed point of $\mathcal{G}'$ then $\mathcal{A} \circ \mathcal{G} \circ \mathcal{A}^{-1} v = v$ and hence the preimage of v, $\mathcal{A}^{-1} v \in B_\gamma(\mathcal{I})$, is a fixed point of $\mathcal{G}$.

To show that $\mathcal{G}'$ has a fixed point in $D_\gamma(0)$ we want to prove that the following equation has a unique solution v in $D_\gamma(0)$:

$$-((D\mathcal{G}')|_0 - Id)^{-1}(\mathcal{G}'v - (D\mathcal{G}')|_0 v) = v. \tag{1.15}$$

Note that $\Gamma^0(\Delta_F, T\mathcal{M})$ is a Banach space and the map $\mathcal{G}'$ is differentiable in $D_\gamma(0)$. In fact, $D\mathcal{G}'$ is Lipschitz in v since the exponential map and its inverse are both smooth. Since the map G is short ranged, so are the maps $\mathcal{G}'$ and $(D\mathcal{G}')|_0$. Therefore, we can use weak* bases to represent $(D\mathcal{G}')$ in a matrix form. This enables one to readily reproduce the arguments in [KH] (see Lemma 18.1.4) and, exploiting hyperbolicity of F, to show that: 1) the operator $-((D\mathcal{G}')|_0 - Id)^{-1}$ is bounded; 2) the map $\mathcal{K} : D_\gamma(0) \to \Gamma^0(\Delta_F, T\mathcal{M})$ defined by

$$\mathcal{K}v = -((D\mathcal{G}')|_0 - Id)^{-1}(\mathcal{G}'v - (D\mathcal{G}')|_0 v) \tag{1.16}$$

is contracting in a smaller ball $D_{\gamma_0}(0) \subset D_\gamma(0) \subset \Gamma^0(\Delta_F, T\mathcal{M})$; and 3) $\mathcal{K}(D_{\gamma_0}(0)) \subset D_{\gamma_0}(0)$. Thus, $\mathcal{K}$ has a unique fixed point in $D_{\gamma_0}(0)$.

We now proceed with Statement 2 of the theorem. In order to establish (1.11) we need to show that the section v has such a property. Let w be a section satisfying (1.11). Since the map $\mathcal{K}$ is short ranged and sufficiently closed to an uncoupled contracting map it is straightforward to verify that the section $\mathcal{K}w$ also satisfies (1.11).

Since the map G is spatial translation invariant, so is h. The Hölder continuity of h was proved in [J1] by showing that stable and unstable manifolds for Φ vary Hölder continuously in the ρ_q-metric. In Section 5, we describe finite-dimensional approximations for h which can be also used to establish an alternative proof of the Hölder continuity. $\square$

The hyperbolicity of the map $\Phi|_{\Delta_\Phi}$ enables one to establish the following topological properties of this map:

1) the manifolds $V^s_\Phi(h(\bar{x})) = h(V^s_F(\bar{x}))$ and $V^u_\Phi(h(\bar{x})) = h(V^u_F(\bar{x}))$ are stable and unstable manifolds for Φ. They are infinite-dimensional submanifolds of $\mathcal{M}$ and are transversal in the sense that the distance between their tangent bundles is bounded away from 0.

2) stable and unstable manifolds for Φ constitute a local product structure of the set Δ_Φ. This means that there exists a constant δ such that for any $\bar{x}, \bar{y} \in \Delta_\Phi$ with $\rho(\bar{x}, \bar{y}) < \delta$, the intersection $V^s_\Phi(\bar{x}) \cap V^u_\Phi(\bar{y})$ consists of a single point which belongs to Δ_Φ.

Furthermore, in [J1] the author proved the following result.

Theorem 1.2. *If the map $f|_\Lambda$ is topologically mixing then so is the map $\Phi|_{\Delta_\Phi}$.*

Although the space $\mathcal{M}$ equipped with the ρ_q-metric is not a Banach manifold and the maps F and Φ are not differentiable Theorem 1.2 allows one to keep track of the hyperbolic properties of these maps. More precisely, the following statements hold:

1) The local stable and unstable manifolds are Lipschitz continuous with respect to the ρ_q-metric. The map Φ is uniformly contracting on stable manifolds and the map Φ^{-1} is uniformly contracting on unstable manifolds. The contracting coefficients can be estimated from above by $(1+\epsilon)\lambda$ with ϵ arbitrary small.

2) The local stable and unstable manifolds are transversal in the ρ_q-metric in the following sense: for any points $\bar{x}$, $\bar{y} \in V^s_\Phi(\bar{x})$, and $\bar{z} \in V^u_\Phi(\bar{x})$,

$$\rho_q(\bar{x}, \bar{y}) + \rho_q(\bar{x}, \bar{z}) \leq C\rho_q(\bar{y}, \bar{z}), \tag{1.17}$$

where C is a constant depending only on the size of local stable and unstable manifolds and the number q.

The first property was originally proved in [PS] based upon the graph transform technique. The second property was established in [J2]. These properties allows one to say that the map Φ is "topologically hyperbolic".

II. Existence of Equilibrium Measures

Let Ω be a compact metric space and τ a $\mathbb{Z}^{d+1}$-action on Ω induced by $d+1$ commuting homeomorphisms, $d \geq 0$. Let also $\mathcal{U} = \{U_i\}$ and $\mathcal{B} = \{B_i\}$ be covers of Ω. For a finite set $X \subset \mathbb{Z}^{d+1}$ define

$$\mathcal{U}^X = \vee_{x \in X} \tau^{-x} \mathcal{U}. \tag{2.1}$$

Denote by $|X|$ the cardinality of the set X.

The action τ is said to be *expansive* if there exists $\epsilon > 0$ such that for any $\xi, \eta \in \Omega$,

$$d(\tau^x \xi, \tau^x \eta) \leq \epsilon \text{ for all } x \in \mathbb{Z}^{d+1} \text{ implies } \xi = \eta.$$

A Borel measure μ on Ω is said to be τ-*invariant* if μ is invariant under all $d+1$ homeomorphisms. We denote the set of all τ-invariant measures on Ω by $I(\Omega)$.

Let $\mu \in I(\Omega)$ and $\mathcal{U} = \{U_i\}$ be a finite Borel partition of Ω. Define

$$H(\mu, \mathcal{U}) = -\sum_i \mu(U_i) \log \mu(U_i) \tag{2.2}$$

and then set

$$h_\tau(\mu, \mathcal{U}) = \lim_{a_1, \ldots, a_{d+1} \to \infty} \frac{1}{|X(a)|} H(\mu, \mathcal{U}^{X(a)}) = \inf_a \frac{1}{|X(a)|} H(\mu, \mathcal{U}^{X(a)}), \tag{2.3}$$

where $X(a) = \{(i_1 \ldots i_{d+1}) \in \mathbb{Z}^{d+1} : \; a = (a_1 \ldots a_{d+1}),\, a_k > 0,\, |i_k| \leq a_k,\, k = 1, \ldots, d+1\}$. The (measure-theoretic) *entropy* of μ is defined to be

$$h_\tau(\mu) = \sup_{\mathcal{U}} h_\tau(\mu, \mathcal{U}) = \lim_{\operatorname{diam} \mathcal{U} \to 0} h_\tau(\mu, \mathcal{U}), \tag{2.4}$$

where $\operatorname{diam}\mathcal{U} = \max_i(\operatorname{diam} U_i)$.

Let $\mathcal{U}$ be a finite open cover of Ω, φ a continuous function on Ω, and X a finite subset of $\mathbb{Z}^{d+1}$. Define

$$Z_X(\varphi, \mathcal{U}) = \min_{\{B_j\}} \Big\{ \sum_j \exp \Big[\inf_{\xi \in B_j} \sum_{x \in X} \varphi(\tau^x \xi) \Big] \Big\}, \tag{2.5}$$

where the minimum is taken over all subcovers $\{B_j\}$ of $\mathcal{U}^X$. Set

$$P_\tau(\varphi, \mathcal{U}) = \limsup_{a_1, \ldots, a_{d+1} \to \infty} \frac{1}{|X(a)|} \log Z_{X(a)}(\varphi, \mathcal{U}). \tag{2.6}$$

The quantity

$$P_\tau(\varphi) = \lim_{\operatorname{diam}\mathcal{U} \to 0} P_\tau(\varphi, \mathcal{U}) = \sup_{\mathcal{U}} P_\tau(\varphi, \mathcal{U}) \tag{2.7}$$

is called the *topological pressure* of φ (one can show that the limit in (2.7) exists).

For any continuous function φ and any $\nu \in I(\Omega)$ the *variational principle* of statistical mechanics claims that

$$P_\tau(\varphi) = \sup_{\nu \in I(\Omega)} \Big(h_\tau(\nu) + \int \varphi d\nu \Big). \tag{2.8}$$

A measure $\mu \in I(\Omega)$ is called an *equilibrium measure* for φ with respect to a $\mathbb{Z}^{d+1}$-action τ if

$$P_\tau(\varphi) = h_\tau(\mu) + \int \varphi d\mu. \tag{2.9}$$

It is shown in [Ru] that expansiveness of a $\mathbb{Z}^{d+1}$-action implies the upper semi-continuity of the metric entropy $h_\tau(\mu)$ with respect to μ. Therefore, it also implies the existence of equilibrium measures for continuous functions. For uncoupled map lattices one can easily check that the action (F, S) is expansive on Δ_F in the ρ_q-metric. The expansiveness of the action (Φ, S) on Δ_Φ is a direct consequence of the structural stability (see Theorem 1.1). Thus, we have the following result.

Theorem 2.1. *Let $\tau = (\Phi, S)$ be a $\mathbb{Z}^{d+1}-$action on Δ_Φ, where $\Phi = F \circ G$ and G is short ranged spatial translation invariant and sufficiently C^1-close to identity. Then for any $0 < q < 1$ and any continuous function φ on (Δ_Φ, ρ_q) there exists an equilibrium measure μ_φ for φ with respect to τ. The measure μ_φ does not depend on q.*

While this theorem guarantees the existence of equilibrium measures for continuous functions (with respect to ρ_q-metrics), it does not tell us anything about uniqueness and ergodic properties of these measures. One can show that uniqueness of equilibrium measures implies their ergodicity (see [Mañé]) and usually some stronger ergodic properties (mixing, etc.).

Ruelle [Ru] obtained the following general result about uniqueness which is a direct consequence of the convexity of the topological pressure on the Banach space $C^0(\Delta_\Phi)$ of all continuous functions in a ρ_q-metric.

Theorem 2.2. *Assume that the map f is topologically mixing. Then for a residual set of (continuous) functions in $C^0(\Delta_\Phi)$, the corresponding equilibrium measures are unique.*

III. Uniqueness of Equilibrium Measures

Ruelle's theorem does not specify the class of functions for which the uniqueness takes place. In this section we establish uniqueness for Hölder continuous functions with sufficiently small Hölder constant. Our main tool is the thermodynamic formalism applied to symbolic models corresponding to the coupled map lattices.

3.1. Markov Partitions and Symbolic Representations.

One of the main manifestations of Structural Stability Theorem 1.1 is that the conjugacy map h is Hölder continuous in a ρ_q-metric. Therefore, one can see that the uniqueness of an equilibrium measure μ_φ corresponding to a continuous function φ for the perturbed map Φ is equivalent to the uniqueness of an equilibrium measure $\mu_{\varphi \circ h}$ for the unperturbed map F. Thus, we can reduce the study of uniqueness of equilibrium measures to uncoupled map lattices.

We shall assume that f is topologically mixing and the hyperbolic set Λ is locally maximal. For any $\epsilon > 0$ there exists a *Markov partition* of Λ of "size" ϵ. This means that Λ is the union of sets R_i, $i = 1, \dots, m$ satisfying:

1) each set R_i is a "rectangle", i.e., for any $x, y \in R_i$ the intersection of the local stable and unstable manifolds $V^s(x) \cap V^u(y)$ is a single point which lies in R_i;

2) $\mathrm{diam} R_i < \epsilon$ and R_i is the closure of its interior;

3) $R_i \cap R_j = \partial R_i \cap \partial R_j$, where ∂R_i denotes the boundary of R_i;

4) if $x \in R_i$ and $f(x) \in \mathrm{int} R_j$ then $f(V^s(x, R_i)) \subset V^s(f(x), R_j)$; if $x \in R_i$ and $f^{-1}(x) \in \mathrm{int} R_j$ then $f^{-1}(V^u(x, R_i)) \subset V^u(f(x), R_j)$; here $V^s(x, R_i) = V^s(x) \cap R_i$ and $V^u(x, R_i) = V^u(x) \cap R_i$.

The *transfer matrix* $A = (a_{ij})_{1 \le i,j \le m}$ associated with the Markov partition is defined as follows: $a_{ij} = 1$ if $f(\mathrm{int} R_i) \cap \mathrm{int} R_j \neq \emptyset$ and $a_{ij} = 0$ otherwise.

Let (Σ_A, σ) be the associated subshift of finite type (where σ denotes the shift). For each $\xi \in \Sigma_A$ the set $\bigcap_{n=-\infty}^{\infty} f^{-n}(R_{\xi(n)})$ contains a single point. The *coding map* $\pi : \Sigma_A \to \Lambda$ defined by $\pi\xi = \bigcap_{n=-\infty}^{\infty} f^{-n}(R_{\xi(n)})$ is a semi-conjugacy between f and σ, i.e., $f \circ \pi = \pi \circ \sigma$.

We consider $\Sigma_A^{\mathbb{Z}^d}$ as a subset of the direct product $\Omega^{\mathbb{Z}^{d+1}}$, where $\Omega = \{1, 2, \ldots, m\}$. Its elements will be denoted by $\bar{\xi} = \bar{\xi}(i,j)_{i \in \mathbb{Z}^d, j \in \mathbb{Z}}$, or sometimes by $\bar{\xi} = \xi_i(j)_{i \in \mathbb{Z}^d, j \in \mathbb{Z}}$. This symbolic space is endowed with the distance

$$\rho_q(\bar{\xi}, \bar{\eta}) = \sup_{(i,j) \in \mathbb{Z}^{d+1}} q^{|i|+|j|} |\bar{\xi}(i,j) - \bar{\eta}(i,j)| \tag{3.1}$$

which is compatible with the product topology. Let σ_t and σ_s be the time and space translations on $\Sigma_A^{\mathbb{Z}^d}$ defined as follows: for $\bar{\xi} = (\xi_i) \in \Sigma_A^{\mathbb{Z}^d}$, $\xi_i = \xi_i(\cdot) \in \Sigma_A$,

$$(\sigma_t^k \bar{\xi})_i(j) = \xi_i(j+k), \quad (\sigma_s^k \bar{\xi})_i = \xi_{i+k}, \; k \in \mathbb{Z}^d. \tag{3.2}$$

We define the coding map $\bar{\pi} = \otimes_{i \in \mathbb{Z}^d} \pi : \Sigma_A^{\mathbb{Z}^d} \to \Delta_F$. It is a semi-conjugacy between the uncoupled map lattice and the symbolic dynamical system, i.e., the following diagram is commutative:

$$\begin{array}{ccc} \Delta_F & \xrightarrow{(F,S)} & \Delta_F \\ \uparrow \bar{\pi} & & \uparrow \bar{\pi} \\ \Sigma_A^{\mathbb{Z}^d} & \xrightarrow{(\sigma_t, \sigma_s)} & \Sigma_A^{\mathbb{Z}^d} \end{array} \tag{3.3}$$

The following statement describes the properties of the map $\bar{\pi}$. Its proof follows from the definitions.

Proposition 3.1. *(1) $\bar{\pi}$ is surjective and Lipschitz continuous with respect to the ρ_q-metric for any $0 < q < 1$.*

(2) $\bar{\pi} \circ \sigma_t = F \circ \bar{\pi}$, $\quad \bar{\pi} \circ \sigma_s = S \circ \bar{\pi}$, i. e., $\quad \bar{\pi} \circ \tau^* = \tau \circ \bar{\pi}$.

(3) $\bar{\pi}$ is injective outside the set $\bigcup_{x \in \mathbb{Z}^{d+1}} \tau^{*x}(\bar{\pi}^{-1}(\mathcal{B}))$, where $\mathcal{B} = \cup_i \partial R_i$ is the boundary of the Markov partition.

3.2. Coupled Map Lattices and Lattice Spin Systems.

The coding map $\bar{\pi}$ enables one to reduce the study of uniqueness and ergodic properties of equilibrium measures corresponding to a (Hölder) continuous function φ on (Δ_F, ρ_q) for the $\mathbb{Z}^{d+1}$-action $\tau^* = (F, S)$ to the study of the same properties of equilibrium measures corresponding to the function $\varphi^* = \varphi \circ \bar{\pi}$ on $\Sigma_A^{\mathbb{Z}^d}$ for the action $\tau = (\sigma_t, \sigma_s)$. In statistical physics the latter is called *the lattice spin system.* We describe the reduction in the following series of results.

Theorem 3.1. *(1) Let φ be a continuous function on Δ_F. Then $P_{\tau^*}(\varphi^*) \geq P_\tau(\varphi)$.*

(2) Let μ^* be a τ^*-invariant measure on $\Sigma_A^{\mathbb{Z}^d}$ and $\mu = \mu^* \circ \bar{\pi}_*^{-1}$. Then $h_\tau(\mu) \le h_{\tau^*}(\mu^*)$.

As in the case of finite-dimensional dynamical systems it is crucial to know that the projection measure $\mu = \mu^* \circ \bar{\pi}_*^{-1}$ of the equilibrium measure μ^* corresponding to the function φ^* is not concentrated on the boundary $\mathcal{B}$ of the Markov partition, i.e., that

$$\mu^*(\bar{\pi}^{-1}(\mathcal{B})) = 0. \tag{3.4}$$

Theorem 3.2. *Let φ be a continuous function on Δ_F. Assume that the condition (3.4) holds for any equilibrium measure μ^* corresponding to $\varphi^* = \varphi \circ \bar{\pi}$. Then,*

(1) $P_{\tau^*}(\varphi^*) = P_\tau(\varphi)$;

(2) the measure $\mu = \mu^* \circ \bar{\pi}_*^{-1}$ is an equilibrium measure corresponding to φ.

(3) if μ_φ is an equilibrium measure for φ on Δ_F, then there exists an equilibrium measure μ^* for $\varphi^* = \varphi \circ \bar{\pi}$ with the property $\mu_\varphi(E) = \mu^*(\bar{\pi}^{-1}(E))$ for any Borel set $E \subset \Delta_F$.

Both Theorems 3.1 and 3.2 follow directly from the definitions (see (2.4) and (2.7)). In the finite-dimensional case Condition (3.4) holds provided the potential function is Hölder continuous. This is due to the fact that the equilibrium measure is unique and hence is ergodic [Ma]. In the infinite-dimensional case the ergodicity of μ^* with respect to time translations is still sufficient for (3.4) to hold.

Theorem 3.3. [J1] *Let μ^* be an equilibrium measure corresponding to a Hölder continuous function on $\Sigma_A^{\mathbb{Z}^d}$. Assume that μ^* is ergodic with respect to the time translation σ_t. Then it satisfies Condition (3.4).*

The proof of this theorem is similar to the proof in the finite-dimensional case (see [Bo]). The boundary $\mathcal{B}$ is a countable union of closed sets invariant under the time shift. By ergodicity the measure of $\mathcal{B}$ is either zero or one. On the other hand, one can show that μ^* takes on positive values on open sets (see below). Therefore, the measure of $\mathcal{B}$ is zero.

Uniqueness of the equilibrium measure implies its ergodicity with respect to the $\mathbb{Z}^{d+1}$-action induced by (F, S). This is weaker than ergodicity with respect to the time translation. In [J1], the author proved directly that for a class of Hölder continuous functions Condition (3.4) holds.

Recall that a function φ on Δ_F is Hölder continuous in the ρ_q-metric if

$$|\varphi(\bar{x}) - \varphi(\bar{y})| \le c\rho_q^\alpha(\bar{x}, \bar{y}),$$

where $\bar{x} = (x_i), \bar{y} = (y_i) \in \Delta_F$. Note that if the function φ is Hölder continuous on Δ_F (in the ρ_q-metric) then the function $\varphi^* = \varphi \cdot \bar{\pi}$ on $\Sigma_A^{\mathbb{Z}^d}$ is also Hölder continuous. The following statement enables one to reduce the study of the uniqueness problem for coupled map lattices to the study of the same problem for lattice spin systems.

Theorem 3.4. [J1] *Let φ be a Hölder continuous function on (Δ_F, ρ_q). Assume in addition that*

$$|\varphi(\bar{x}) - \varphi(\bar{y})| \leq c\rho_q^\alpha(\bar{x}, \bar{y}),$$

where $\bar{x} = (x_i), \bar{y} = (y_i) \in \Delta_F$, $x_0 = y_0$, and c is sufficiently small. Then, $\mu^*(\bar{\pi}^{-1}(\mathcal{B})) = 0$ holds for any equilibrium measure of φ^* on $\Sigma_A^{\mathbb{Z}^d}$. Therefore, for this class of potential functions, the uniqueness of measure μ^* implies the uniqueness of measure μ.

In the next section we shall actually show that the equilibrium measure for φ^* is unique and exponentially mixing for the class of Hölder continuous functions satisfying the condition of Theorem 3.4.

3.3. Gibbs States for Lattice Spin Systems.

We remind the reader the concept of Gibbs states for lattice spin systems of statistical physics.

An element $\bar{\xi} \in \Sigma_A^{\mathbb{Z}^d} \subset \Omega^{\mathbb{Z}^{d+1}}$ is called a *configuration.* For any subset $X \subset \mathbb{Z}^{d+1}$ we set

$$\Omega_X = \{\bar{\eta} \in \Omega^X \ : \ \text{there exists } \bar{\xi} \in \Sigma_A^{\mathbb{Z}^d} \text{ such that } \bar{\eta}(i) = \bar{\xi}(i),\ i \in X\}.$$

The elements of Ω_X will be denoted by $\bar{\xi}_X$, or sometimes by $\bar{\xi}(X)$. One can say that Ω_X consists of restrictions of configurations $\bar{\xi}$ to X.

For each finite subset $X \subset \mathbb{Z}^{d+1}$ define the function $p_X(\bar{\xi})$ on $\Sigma_A^{\mathbb{Z}^d}$ by

$$p_X(\bar{\xi}) = \frac{1}{\sum_{\bar{\eta}, \bar{\eta}(\widehat{X}) = \bar{\xi}(\widehat{X})} \exp\left(\sum_{x \in \mathbb{Z}^{d+1}} \varphi(\tau^x \bar{\eta}) - \varphi(\tau^x \bar{\xi})\right)}, \tag{3.5}$$

where τ^x is the action $(\sigma_t)^i \circ (\sigma_s)^j$, $\widehat{X} = \mathbb{Z}^{d+1} \setminus X$, and $x = (i, j)$, $i \in \mathbb{Z}^d$, $j \in \mathbb{Z}$.

Let φ be a Hölder continuous function on $\Sigma_A^{\mathbb{Z}^d}$. A probability measure μ on $\Sigma_A^{\mathbb{Z}^d}$ is called a *Gibbs state* for φ if for any finite subset $X \subset \mathbb{Z}^{d+1}$,

$$\mu_X(\bar{\xi}(X)) = \int_{\Omega_{\widehat{X}}} p_X(\bar{\xi}) d\mu_{\widehat{X}}, \tag{3.6}$$

where μ_X and $\mu_{\widehat{X}}$ are the probability measures on Ω_X and $\Omega_{\widehat{X}}$ respectively that are induced by natural projections. This equation is known as the *Dobrushin-Ruelle-Lanford equation.*

There is another equivalent way to describe Gibbs states corresponding to Hölder continuous functions on symbolic spaces. Let φ be such a function. For each finite volume

X we define a conditional Gibbs distribution on Ω_X under a given boundary condition $\bar\eta^*$ by

$$\mu_{\bar\eta^*,X}(\bar\xi(X)) = \frac{1}{\sum_{\bar\eta,\bar\eta(\widehat{X})=\bar\eta^*(\widehat{X})} \exp\left(\sum_{x\in\mathbb{Z}^{d+1}} \varphi(\tau^x\bar\eta) - \varphi(\tau^x(\bar\xi(X)+\bar\eta^*(\widehat{X})))\right)}, \tag{3.7}$$

where $\bar\xi(X)+\bar\eta^*(\widehat{X})$ denotes the (admissible) configuration on $X\cup\widehat{X}$ whose restrictions to X and $\widehat{X}$ are $\bar\xi(X)$ and $\bar\eta^*(\widehat{X})$ respectively. The set of all Gibbs states for φ is the convex hull of the thermodynamical limits of the conditional Gibbs distributions.

The relation between translation invariant Gibbs states and equilibrium measures can be stated as follows (see [Ru]).

Theorem 3.5. *If the transfer matrix A is aperiodic then μ is an equilibrium measure for φ if and only if it is a translation invariant Gibbs state for φ.*

In statistical mechanics Gibbs states are usually defined for potentials rather than for functions. We briefly describe this approach.

A *potential* U is a collection of functions defined on the family of all finite configurations, i.e.,

$$U = \{U_X : X \subset \mathbb{Z}^{d+1},\, U_X : \Omega_X \to \mathbb{R}\}.$$

Gibbs states for a potential U is defined as the convex hull of the thermodynamical limits of the conditional Gibbs distributions:

$$\mu_{\bar\eta^*,X}(\bar\xi(X)) = \frac{\exp(\sum_{V\cap X\neq\emptyset} U_V(\bar\xi(X)+\bar\eta^*(\widehat{X}))}{\sum_{\bar\eta,\bar\eta(\widehat{X})=\bar\eta^*(\widehat{X})} \exp(\sum_{V\cap X\neq\emptyset} U_V(\bar\eta))}, \tag{3.8}$$

where $\bar\eta^*$ is a fixed configuration.

We describe potentials corresponding to Hölder continuous functions. Let φ be such a function. We write φ in the form of a series

$$\varphi = \sum_{n=0}^{\infty} \varphi_n. \tag{3.9}$$

Here the value of φ_n depends only on configurations inside the $(d+1)$-dimensional cube Q_n centered at the origin of side $2n\times\cdots\times 2n$. We also set $Q_0=(0,0)$. We define the functions φ_n as follows. Fix a configuration η^* and set

$$\varphi_0(\bar\xi) = \varphi\big(\bar\xi(Q_0)+\bar\eta^*(\widehat{Q}_0)\big). \tag{3.10}$$

Continuing inductively we define

$$\varphi_{n+1(\bar\xi)} = \varphi\big(\bar\xi(Q_{n+1})+\bar\eta^*(\widehat{Q}_{n+1})\big) - \varphi\big(\bar\xi(Q_n)+\bar\eta^*(\widehat{Q}_n)\big), \quad n=1,2,\ldots. \tag{3.11}$$

It is easy to see that $\|\varphi_n\| \to 0$ exponentially fast as $n \to \infty$. We define the potential U_φ associated with the function φ on Q_n by setting

$$U_\varphi(\bar{\xi}(Q_n)) = \varphi_n(\bar{\xi}(Q_n)). \tag{3.12}$$

For other $(d+1)$-dimensional cubes that are translations of Q_n we assign the same value of U_φ. For other finite subsets of $\mathbb{Z}^{d+1}$ we define the potential to be zero. Thus, we obtain a translation invariant potential whose values on finite volumes decrease exponentially when the diameter of the volume grows.

If $\varphi_0 = 0$, the value of the corresponding potential U_φ is bounded by the Hölder constant of the function φ. More generally, let us set

$$\mathcal{F}(\alpha, q, \epsilon) = \{\varphi : \ |\varphi(\bar{\xi}) - \varphi(\bar{\eta})| \le \epsilon \rho_q^\alpha(\bar{\xi}, \bar{\eta})\}, \tag{3.13}$$

$$\|U_{Q_n}\| = \sup_{\xi(Q_n) \in \Omega_{Q_n}} |U_{Q_n}(\bar{\xi}(Q_n))|, \tag{3.14}$$

$$\mathcal{P}(q, \epsilon) = \{U : \sup_{n \ge 1} q^{-n} \|U_{Q_n}\| \le \epsilon\}. \tag{3.15}$$

It is easy to see that, if $\varphi \in \mathcal{F}(\alpha, \epsilon)$, then $U_\varphi \in \mathcal{P}(q^\alpha, \epsilon)$. On the other hand, $U_\varphi \in \mathcal{P}(q, \epsilon)$ implies $\varphi \in \mathcal{F}(1/2, q, \epsilon)$.

The definition of Gibbs states corresponding to potentials is consistent with the one corresponding to functions. More precisely, Gibbs distributions corresponding to a Hölder continuous function φ are exactly the Gibbs distributions corresponding to the potential U_φ.

As we have seen the problem of uniqueness of equilibrium states on symbolic spaces can be reduced to the problem of uniqueness of translation invariant Gibbs states provided the function φ is Hölder continuous. This problem has been extensively studied in statistical physics for a long time. In the one-dimensional case (when $d = 0$) Gibbs states are always unique and are mixing with respect to the shift provided the potential decays exponentially fast as the length of intervals goes to infinity (see [Ru]). In the case of higher dimensional lattice spin systems the well-known Ising model provides an example where the Gibbs states are not unique even for potentials of finite range (see [Sim]). We first describe the two-dimensional Ising model in the context of spin lattice systems.

Example 1: The Ising Model ($d = 1$). Define the potential function φ on Ω by

$$\varphi(\bar{\xi}) = \beta\big(\bar{\xi}(1,0)\bar{\xi}(0,0) + \bar{\xi}(0,0)\bar{\xi}(0,1)\big). \tag{3.16}$$

Then the following statements hold:

(1) $\varphi(\bar{\xi})$ depends only on the values of $\bar{\xi}$ at three lattice points: $(1,0), (0,0)$, *and* $(0,1)$;

(2) There exists $\beta_0 > 0$ such that for $\beta > \beta_0$ Gibbs states corresponding to the function $\varphi(\bar{\xi})$ are not unique.

Based upon this Ising model we describe now an example of a coupled map lattice and a Hölder continuous function with non-unique equilibrium measure.

Example 2: Phase Transition For Coupled Map Lattices. Let M be a compact smooth surface and (Λ, f) the Smale horseshoe. One can show that the semi-conjugacy $\bar{\pi}$ between $\mathcal{M} = \otimes_{i\in\mathbb{Z}} M$ and $\{0,1\}^{\mathbb{Z}^2}$ induced by the Markov partition can be chosen as an isometry. Thus, the function $\psi = \varphi \circ \bar{\pi}^{-1}$ is Hölder continuous on Δ_F, where the function φ is chosen as in Example 1. Since the boundary of the Markov partition is empty Condition (3.1) holds. We conclude that there are more than one equilibrium measures for the function ψ.

The following statement provides a general sufficient condition for uniqueness of Gibbs states. Let U be a translation invariant potential on the configuration space $\Omega^{\mathbb{Z}^{d+1}}$, where $\Omega = \{1, 2, \ldots, m\}$.

(1) (Dobrushin's Uniqueness Theorem [D1], [Sim]): Assume that

$$\sum_{X:\,0\in X} (|X| - 1)||U(X)|| < 1. \tag{3.17}$$

Then the Gibbs state for U is unique.

(2) ([Gro], [Sim]): There exist $r > 0$ and $\varepsilon > 0$ such that if

$$\sum_{X:\,0\in X} e^{rd(X)}||U(X)|| \le \varepsilon \tag{3.18}$$

($d(X)$ denotes the diameter of X) then the unique Gibbs state is exponentially mixing with respect to the $\mathbb{Z}^{d+1}$-action on $\Omega^{\mathbb{Z}^{d+1}}$.

The proof of Dobrushin's uniqueness theorem relies strongly upon the direct product structure of the configuration space $\Omega^{\mathbb{Z}^{d+1}}$. This result can not be directly applied to establish uniqueness of Gibbs states for lattice spin systems, which are symbolic representations of coupled map lattices, because the configuration space $\Sigma_A^{\mathbb{Z}^d}$ is, in general, a translation invariant subset of $\Omega^{\mathbb{Z}^{d+1}}$. In [BuSt], the authors constructed examples of strongly irreducible subshifts of finite type for which there are many Gibbs states corresponding to the function $\varphi = 0$. In order to establish uniqueness we will use the special structure of the space $\Sigma_A^{\mathbb{Z}^d}$: it admits subshifts of finite type in the "time" direction and the Bernoulli shift in the "space" direction.

We now present the main result on uniqueness and mixing property of Gibbs states for lattice spin systems which are symbolic representations of coupled map lattices of hyperbolic type. In the two-dimensional case ($d=1$), it was proved by Jiang and Mazel (see [JM]). In the multidimensional case it was established by Bricmont and Kupiainen (see [BK3]).

A potential U_0 on $\Sigma_A^{\mathbb{Z}}$ is called *longitudinal* if it is zero everywhere except for configurations on vertical finite intervals of the lattice. A potential U_0 is said to be *exponentially decreasing* if

$$|U_0(\bar{\xi}(I))| \le Ce^{-\lambda|I|}, \tag{3.19}$$

where $C>0$ and $\lambda>0$ are constants, I is a vertical interval, $|I|$ is its length, and $\bar{\xi}(I)$ is a typical configuration over I. Exponentially deceasing longitudinal potentials correspond to those potential functions whose values depend only on the configuration $\bar{\xi}(0,j)$, $j\in\mathbb{Z}$.

We say that a Gibbs state is exponentially mixing if for every integrable function on the configuration space the $\mathbb{Z}^{d+1}$-correlation functions decay exponentially to zero.

Theorem 3.6 (Uniqueness and Mixing property of Gibbs States). *For any exponentially deceasing longitudinal potential U_0 and every $0<q<1$, there exists $\epsilon>0$ such that the Gibbs state for any potential $U=U_0+U_1$ with $U_1\in\mathcal{P}(q,\epsilon)$ is unique and exponentially mixing.*

Proof. We provide a brief sketch of the proof assuming first that $U_0=0$ and $d=1$. We may assume that the potential is non-negative (otherwise, the non-negative potential $U'(\eta(Q))=U(\eta(Q))+\max_{\eta(Q)}|U(\eta(Q))|$ defines the same family of Gibbs distributions).

We first introduce an *equivalent* potential which is defined on rectangles (i.e., a potential which generates the same Gibbs measures). Consider a square Q and a rectangle P and denote by $b(Q)=(b_1(Q),b_2(Q))$ and $b(P)=(b_1(P),b_2(P))$ the left lowest corners of Q and P, respectively. We define the *rectangular potential* $U(\bar{\eta}(P))$ for all rectangles with $b_2(P)=nL$, $n\in\mathbb{Z}$ of size $l(P)\times Ll(P)$ by

$$U(\bar{\eta}(P))=\sum U(\bar{\eta}(Q)), \tag{3.20}$$

where the sum is taken over all squares Q satisfying the following condition: Q is of size $l(P)\times l(P)$ and $b_1(Q)=b_1(P)$, $b_2(P)\le b_2(Q)<b_2(P)+L$. One can check that both potentials generate the same conditional Gibbs distribution on any finite volume $V\subset\mathbb{Z}^2$.

Let $V\subset\mathbb{Z}^2$ be a finite volume of size $n\times nL$. Fix a boundary condition $\bar{\eta}^*(\widehat{V})$. For any configuration $\bar{\xi}(V)$ such that $\bar{\xi}(V)+\bar{\eta}^*(\widehat{V})$ is a configuration in $\mathbb{Z}^2$ a *conditional Hamiltonian* specified by the potential $U(\bar{\eta}(P))$ is defined as follows

$$H(\bar{\xi}(V)|\bar{\eta}^*(\widehat{V}))=-\sum_{P\cap V\neq\emptyset}U\big(\bar{\eta}(P)|\bar{\xi}(V)+\bar{\eta}^*(\widehat{V})\big). \tag{3.21}$$

The expression $U\big(\bar{\eta}(P)|\bar{\xi}(V)+\bar{\eta}^*(\widehat{V})\big)$ means that the potential $U(\bar{\eta}(P))$ is evaluated under the condition that $\bar{\xi}(V)+\bar{\eta}^*(\widehat{V})$ is fixed. Recall that a conditional Gibbs distribution with the inverse temperature $\beta \geq 0$ is defined by

$$\mu_{V,\bar{\eta}^*}(\bar{\xi}(V)) = \frac{\exp\left(-\beta H(\bar{\xi}(V)|\bar{\eta}^*(\widehat{V}))\right)}{\Xi(V|\bar{\eta}^*(\widehat{V}))}, \tag{3.22}$$

where

$$\Xi(V|\sigma'(\widehat{V})) = \sum_{\bar{\eta}(V)} \exp\left(-\beta H(\bar{\eta}(V)|\bar{\eta}^*(\widehat{V}))\right)$$

is a partition function in the volume V with the boundary condition $\bar{\eta}^*(\widehat{V})$.

Let $B \subset V \subset \mathbb{Z}^2$. We wish to use (3.22) in order to compute the probability $\mu_{V,\bar{\eta}^*}(\bar{\xi}(B))$ of the configuration $\bar{\xi}(B)$ under the boundary condition and to show that it has a limit as $V \to \mathbb{Z}^2$. The latter is the unique Gibbs state for the potential U. Using the Polymer Expansion Theorem (see Appendix) we rewrite the expression (3.22) in the following form:

$$\mu_{V,\bar{\eta}^*}(\bar{\xi}(B)) = N(A)\left[\exp\sum_{P\subseteq B} U(\bar{\xi}(P))\right]\frac{\Xi\left(V\setminus B|\bar{\xi}(B)+\bar{\eta}^*(\widehat{V})\right)}{\Xi\left(V|\bar{\eta}^*(\widehat{V})\right)}$$

$$= N(A)\exp\left[\sum_{P\subseteq B} U(\bar{\eta}(P)) + \sum_{\wp:\wp\cap V\setminus B\neq\emptyset} W(\wp|\bar{\xi}(B)+\bar{\eta}^*(\widehat{V})) - \sum_{\wp:\wp\cap V\neq\emptyset} W(\wp|\bar{\eta}^*(\widehat{V}))\right]$$

$$= N(A)\exp\left[\sum_{P\subseteq B} U(\xi(P)) + \sum_{\wp:\mathrm{dist}(\bar{\wp},B)\leq 1} W(\wp|\xi(B)) - \sum_{\wp:\mathrm{dist}(\bar{\wp},B)\leq 1} W(\wp)\right],$$

where $N(A)$ is a normalizing factor, determined by the transfer matrix A, $W(\wp|\bar{\eta}^*(\widehat{V}))$ and $W(\wp|\bar{\xi}(B)+\bar{\eta}^*(\widehat{V}))$ are the statistical weights for the polymer $\wp$ (see Appendix), and P is a rectangle. By the Abstract Polymer Expansion Theorem (see Appendix) each term in the last sum converges to a limit uniformly in $\mathcal{P}(q,\epsilon)$.

The proof can be easily extended to the case when U_0 is a general exponentially decreasing longitudinal potential (see [JM] for detail). When $d > 1$ the proof is given by Bricmont and Kupiainen in [BK3] by directly obtaining polymer expansions of correlation functions. $\square$

Theorems 3.4 and 3.6 enable us to obtain the following main result about uniqueness and mixing property of equilibrium measures for coupled map lattices.

Theorem 3.7. *Let (Φ, S) be a coupled map lattice and $\varphi = \varphi_0 + \varphi_1$ a function on Δ_Φ, where φ_0 is a Hölder continuous function with a small Hölder constant in the* *metric ρ_q and φ_1 is a Hölder continuous function depending only on the coordinate x_0. Then there exists a unique equilibrium measure μ_φ on Δ_Φ corresponding to φ. This measure is mixing and takes on positive values on open sets. Furthermore, the correlation functions decay exponentially for every Hölder continuous function on Δ_Φ satisfying the above assumptions.*

IV. Finite-Dimensional Approximations

In this section we describe finite-dimensional approximations of equilibrium measures for coupled map lattices. One should distinguish two different types of approximations: by $\mathbb{Z}^{d+1}$-action equilibrium measures and $\mathbb{Z}$-action equilibrium measures. The first come from the corresponding $\mathbb{Z}^{d+1}$-dimension lattice spin system while the second one is a straightforward finite-dimensional approximation of the initial coupled map lattice.

In order to explain some basic ideas concerning finite-dimensional approximations we first consider an uncoupled map lattice $(\mathcal{M}, F)$. Let φ be a Hölder continuous function on $\mathcal{M}$ which depends only on the central coordinate, i.e., $\varphi(\bar{x}) = \psi(x_0)$, where ψ is a Hölder continuous function on M (whose Hölder constant is not necessary small). It is easy to see that the equilibrium measure μ_φ corresponding to φ is unique with respect to the $\mathbb{Z}^{d+1}$-action (F, S) and that $\mu_\varphi = \otimes_{i\in\mathbb{Z}^d}\mu_\psi$, where μ_ψ is the equilibrium measure on $\Lambda \subseteq M$ for ψ with respect to the $\mathbb{Z}$-action generated by f. One can also verify that for any finite set $X \subset \mathbb{Z}^d$ the measure $\mu_X = \otimes_{i\in X}\mu_\psi$ is the unique equilibrium measure on the space $M_X = \otimes_{i\in X} M$ corresponding to the function $\varphi_X = \sum_{i\in X}\varphi(S^i\bar{x})$ with respect to $\mathbb{Z}$-action $F_X = \otimes_{i\in X} f$. Clearly, $\mu_{X_n} \to \mu_\varphi$ in the weak*-topology for any sequence of subset $X_n \to \mathbb{Z}^d$ (i.e., $X_n \subset X_{n+1}$ and $\bigcup_{n\geq 0} X_n = \mathbb{Z}^d$).

It is worth emphasizing that the sequence of the functions φ_{X_n} does not converge to a finite function on $\mathcal{M}$ as $n \to \infty$ while the corresponding $\mathbb{Z}$-action equilibrium measures $\mu_{\varphi_{X_n}}$ approach the $\mathbb{Z}^{d+1}$-action equilibrium measure μ_φ.

On the other hand, one can consider φ as a function on the space M_X provided $0 \in X$. The unique equilibrium measure with respect to the $\mathbb{Z}$-action generated by F_X is $\mu_\psi \times \otimes_{i\in X, i\neq 0}\nu_0$, where ν_0 is the measure of maximal entropy on M.

This simple example illustrates that the $\mathbb{Z}^{d+1}$-action equilibrium measures corresponding to a function φ may not admit approximations by the $\mathbb{Z}$-action equilibrium measures corresponding to the restrictions of φ to finite volumes.

4.1. Continuity of Equilibrium Measures Over Potentials.

In this section we show that equilibrium measures for coupled map lattices depend continuously on their potential functions in the weak*-topology.

Fix $0 < q < 1$ and consider the space of all Hölder continuous functions on Δ_Φ with Hölder exponent $0 < \alpha < 1$ and Hölder constant $\epsilon > 0$ in the metric ρ_q. We denote this space by $\widetilde{\mathcal{F}}(\alpha, q, \epsilon)$. It is endowed with the usual supremum norm $\|\varphi\|$. We also introduce the q^α-norm on this space by

$$\|\varphi\|_{q^\alpha} = \max\{\sup_{n\geq 0} q^{-\alpha n} \sup_{\bar{x},\bar{y}\in\Delta_\Phi} |\varphi(\bar{x}) - \widetilde{\varphi}(\bar{y})|, \|\varphi\|\}, \tag{4.1}$$

where the second supremum is taken over all points $\bar{x}, \bar{y}$ for which $x_i = y_i$ for $|i| \leq n$.

The following statement establishes continuous dependence of equilibrium measures for coupled map lattices for potential functions in $\widetilde{\mathcal{F}}(\alpha, q, \epsilon)$. We provide a proof in the case $d = 1$ using the approach which is based on the polymer expansions. If $d > 1$ the continuous dependence still holds and can be established using methods in [BK3].

Theorem 4.1. *There exists $\epsilon > 0$ such that the unique equilibrium measure μ_φ on Δ_Φ depends continuously (in the weak*-topology) on $\varphi \in \widetilde{\mathcal{F}}(\alpha, q, \epsilon)$ with respect to the norm $\|\cdot\|_{q^\alpha}$, i.e., for $\psi_m \in \widetilde{\mathcal{F}}(\alpha, q, \epsilon)$, $\|\psi_m - \varphi\|_{q^\alpha} \to 0$ implies $\mu_{\psi_m} \to \mu_\varphi$ in the weak*-topology.*

Proof. Observing that $\|\psi_m - \varphi\|_{q^\alpha} \to 0$ implies the convergence of corresponding potentials on the symbolic space we need only to establish the continuity of Gibbs state for the corresponding symbolic representation. For a potential U on $\Sigma_A^{\mathbb{Z}}$ its norm $\|\cdot\|_q$ is defined as

$$\|U\|_q = \sup_{n\geq 0} q^{-n} \|U_{Q_n}(\bar{\xi}_{Q_n})\|, \tag{4.2}$$

where $0 < q < 1$. By theorem 3.6 the Gibbs state is unique when $\|U\|_q$ is sufficiently small. We denote the Gibbs state for U by μ_U. We show that for any cylinder set $E \subset \Sigma_A^{\mathbb{Z}}$, $\mu_U(E)$ depends on U continuously in a neighborhood of the zero potential in the set $\mathcal{P}(q, 1) = \{U : \|U\|_q \leq 1\}$.

For this purpose we use the explicit expression of $\mu_U(E)$ in terms of the potential U provided by the polymer expansion theorem (see Appendix).

For non-negative potential $U \in \mathcal{P}(q, \epsilon)$, $U(\eta(Q)) \geq 0$ and any finite set $B \subset \mathbb{Z}^2$ we have the unique Gibbs state:

$$\mu_U(\bar{\xi}(B)) = N(A) \exp\left[\sum_{P\subseteq B} U(\bar{\xi}(P)) + \sum_{\wp:\mathrm{dist}(\bar{\wp},B)\leq 1} W(\wp|\bar{\xi}(B)) - \sum_{\wp:\mathrm{dist}(\bar{\wp},B)\leq 1} W(\wp)\right], \tag{4.3}$$

where $N(A)$ is a normalizing factor, which is determined by the transfer matrix A, $W(\wp)$ and $W(\wp|\bar{\xi}(B)$ are the statistical weights for the polymer $\wp$ (see Appendix), and P is a rectangle. By the Abstract Polymer Expansion Theorem (see Appendix) all three terms in the above sum converge uniformly in $\mathcal{P}(q,\epsilon)$ and the statistical weight $W(\wp)$ depends continuously on $U(\eta(P))$ with respect to the norm $\|\cdot\|_q$. This implies that μ_U depends on $U \geq 0$ weakly continuously.

To show that μ_U depends on U continuously for all $U \in \mathcal{P}(q,\epsilon/4)$ let us consider the potential U_ϵ defined as $U(\bar{\xi}(Q_n)) = \epsilon q^n$. Then, for any $U \in \mathcal{P}(q,\epsilon/4)$ we have that

$$U + U_{\epsilon/4} \geq 0, \quad U + U_{\epsilon/4} \in \mathcal{P}(q, 1/2\epsilon).$$

Note that given Q_n, U_ϵ is a constant potential on Q_n. Therefore, Gibbs distributions for U and $U + U_{\epsilon/4}$ coincide and hence,

$$\mu_U = \mu_{U+U_{\epsilon/4}}. \tag{4.4}$$

This implies the desired result. $\square$

4.2. Finite-Dimensional $\mathbb{Z}^{d+1}$-Approximations.

We describe finite-dimensional $\mathbb{Z}^{d+1}$-approximations of equilibrium measures for coupled map lattices.

Let $\varphi \in \widetilde{\mathcal{F}}(\alpha, q, \epsilon)$ be a Hölder continuous function on Δ_Φ. Fix a point $\bar{x}^* = (x_i^*)$ which we call the *boundary condition.* Given a finite volume $V \subset \mathbb{Z}^d$ consider the function on Δ_Φ

$$\varphi_{n,\bar{x}^*}(\bar{x}) = \varphi(\bar{x}|_V, \bar{x}^*|_{\widehat{V}}). \tag{4.5}$$

One can see that

$$\|\varphi_{n,\bar{x}^*} - \varphi\|_{q_1^\alpha} \to 0 \tag{4.6}$$

as $n \to \infty$ for any q_1 with $0 < q < q_1$. The following result is an immediate corollary of Theorem 4.1.

Theorem 4.2. $\mu_{\varphi_{n,\bar{x}^*}} \overset{weak^*}{\longrightarrow} \mu_\varphi$ *independently of the boundary condition* $\bar{x}^*$ (recall that $\mu_{\varphi_{n,\bar{x}^*}}$ is the unique equilibrium measure corresponding to the function $\varphi_{n,\bar{x}^*}$ and μ_φ is the unique equilibrium measure corresponding to the function φ).

4.3. Finite-Dimensional $\mathbb{Z}$-Approximations I: Uncoupled Map Lattices.

We describe some "natural" finite-dimensional approximations of equilibrium measures for coupled map lattices by $\mathbb{Z}$-action equilibrium measures. We first consider an uncoupled map lattice (F, S) in the space $(\mathcal{M}, \rho_q)$.

For every volume $V \subset \mathbb{Z}^d$ we set $\mathcal{M}_V = \otimes_{i\in V} M_i$, $F_V = \otimes_{i\in V} f_i$, and $\Delta_{F,V} = \otimes_{i\in V}\Lambda_i$. One can see that $\mathcal{M}_V$ is a smooth finite-dimensional manifold, F_V is a C^r-diffeomorphism of $\mathcal{M}_V$, and $\Delta_{F,V}$ is a locally maximal hyperbolic set for F_V.

Fix a point $\bar{x}^* = (x_i^*) \in \Delta_F$ (*the boundary condition*) and consider a Hölder continuous function $\varphi \in \widetilde{\mathcal{F}}(\alpha, q, \epsilon)$ on Δ_F. Define the function $\psi_{V,\bar{x}^*}$ on $\Delta_{F,V}$ by

$$\psi_{V,\bar{x}^*}(x) = \sum_{i \in \Delta_{F,V}} \varphi(S^i(x, x^*|_{\widehat{\Delta_{F,V}}})). \tag{4.7}$$

Consider the $\mathbb{Z}$-action equilibrium measure ν_V corresponding to the function $\psi_{V,\bar{x}^*}$. We can view these measures as being supported on $\mathcal{M}$. Let also μ_φ be the $\mathbb{Z}^{d+1}$-action equilibrium measure corresponding to φ. This measure is concentrated on Δ_F and thus can be viewed as being supported on $\mathcal{M}$.

Theorem 4.3. *There exists $c_0 > 0$ such that if $0 < \epsilon \le c_0$ then μ_φ is the limit (in the weak*-topology) of equilibrium measures ν_V as $V \to \mathbb{Z}^{d+1}$.*

Proof. We consider only the case $d = 1$. For $d > 1$ the proof is the same. It is sufficient to prove the convergence of the measures $\nu_V^* = \nu_V \bar{\pi}$ to the measure $\mu^* = \mu_{\varphi^*}$ on the symbolic space $\otimes_{\mathbb{Z}} \Sigma_A$ as $V \to \mathbb{Z}^2$.

Let us fix a configuration $\bar{\eta}^*$ on $\mathbb{Z}^2$. Given $n > 0$ and $m > 0$, consider the rectangle $V_{nm} = \{x = (i,j) \in \mathbb{Z}^2 : |i| \le n, |j| \le m\}$ and define the Gibbs distribution on V_{nm} as follows: for any configuration $\bar{\xi}(V_{nm})$ we set

$$\mu_{nm}(\bar{\xi}(V_{nm})) = \frac{\exp \sum_{x \in V_{nm}} \varphi\big(\tau^x(\bar{\xi}(V_{nm}) + \bar{\eta}^*(\widehat{V}_{nm})\big)}{\sum_{\bar{\eta}(V_{nm})} \exp \sum_{x \in V_{nm}} \varphi\big(\tau^x(\bar{\eta}(V_{nm}) + \bar{\eta}^*(\widehat{V}_{nm})\big)}, \tag{4.8}$$

where $\bar{\xi}(V_{nm})$ is a configuration on V_{nm}. By the definition of a Gibbs state and the uniqueness of μ^* the measure μ^* is the limit of measures μ_{nm}, i.e., for any finite volume $V \subset \mathbb{Z}^2$,

$$\mu^*(\xi(V)) = \lim_{V_{nm} \to \mathbb{Z}^2} \mu_{nm}(\xi(V)),$$

where V_{nm} converges to $\mathbb{Z}^2$ in the sense of van Hove, i.e., for any fixed $a \in \mathbb{Z}^2$

$$\lim_{n \to \infty} \frac{|\tau^a\big(V_{nm(n)}\big) \setminus V_{nm(n)}|}{|V_{nm(n)}|} = 0.$$

We observe that for each $n > 0$, there exists the limit $\nu_n^* = \lim_{m \to \infty} \mu_{nm}$ which is the $\mathbb{Z}$-action Gibbs state for the function $\psi^*_{V_n, \eta^*}$ on $V_n = \otimes_{i=-n}^n \Sigma_A$. Thus, for each fixed n there exists $m(n)$ such that

$$|\mu_{nm(n)}(\bar{\xi}(V)) - \nu_n(\bar{\xi}(V))| \le \frac{1}{n}$$

for every $V \subset V_{nm}$. Notice that $V_{nm(n)} \to \mathbb{Z}^2$ in the sense of van Hove. This implies that $\lim_{n\to\infty} \nu_n = \lim_{n\to\infty} \mu_{nm(n)} = \mu_\varphi$. $\square$

4.4. Finite-Dimensional $\mathbb{Z}$-Approximations II: Coupled Map Lattices.

We consider a coupled map lattice (Φ, S) in the space $(\mathcal{M}, \rho_q)$ and define its finite-dimensional approximations as follows.

Fix a point $\bar{x}^* \in \Delta_\Phi$ (*the boundary condition*). For any finite volume $V \subset \mathbb{Z}^d$ consider the map on M_V

$$\big(\Phi_V(x)\big)_i = \big(\Phi((x, x^*|_{\widehat{V}})\big)_i, \tag{4.9}$$

where $()_i$ denotes the coordinate at the lattice site i. One can see that if the perturbation is sufficiently small then Φ_V is a diffeomorphism of M_V. It can be written as $\Phi_V = G_V \circ F_V$, where G_V is the restriction of G to M_V:

$$G_V(x) = G(F_{\widehat{V}}(x^*|_{\widehat{V}}), x). \tag{4.10}$$

Since the diffeomorphism Φ_V is closed to the diffeomorphism F_V by the structural stability theorem it possesses a locally maximal hyperbolic set which we denote by $\Delta_{\Phi,V}$. Moreover, there exists a conjugacy homeomorphism $h_V : \Delta_{F,V} \to \Delta_{\Phi,V}$ which is close to identity.

The maps Φ_V and h_V provide finite-dimensional approximations for the infinite-dimensional maps Φ and h respectively. In order to describe this in a more explicit way we introduce the following maps:

$$\tilde{\Phi}_V(\bar{x}) = (\Phi_V(\bar{x}|_V),\, F_{\widehat{V}}(\bar{x}|_{\widehat{V}})), \quad \tilde{h}_V(\bar{x}) = (h_V(\bar{x}|_V),\, id_{\widehat{V}}(\bar{x}|_{\widehat{V}})).$$

We denote by d_q^0 and d_q^1 the C^0 and respectively C^1 distances in the space of diffeomorphisms induced by the ρ_q-metric. We also use $d(0, \partial V)$ to denote the shortest distance from the origin of the lattice to the boundary of the set V.

Theorem 4.4. *There exist constants $C > 0$ and $\beta > 0$ such that for any $V \subset V' \subset \mathbb{Z}^d$,*
(1) $d_q^1(\Phi_V, \Phi_{V'}) \le Ce^{-\beta d(0,\partial V)}$ and $\Phi_V \to \Phi$.
(2) $d_q^0(h_V, h_{V'}) \le Ce^{-\beta d(0,\partial V)}$ and $h_V \to h$.

Proof. The first statement is obvious since Φ is short ranged. The proof of the second statement is based upon arguments in the proof of structural stability theorem (see Theorem 1.1). We recall that the conjugacy map h is determined by a unique fixed point for a contracting map $\mathcal{K}$ acting in a ball $D_\gamma(0)$ of the Banach space $\Gamma^0(\Delta_F, T\mathcal{M})$ of all continuous vector fields on Δ_F (see (1.16)).

In order to obtain the conjugacy map h_V one needs to find a unique fixed point for a contracting map $\mathcal{K}_V$ acting in $D_\gamma(0)$ by the formula similar to (1.16):

$$\mathcal{K}_V v = -((D\mathcal{G}'_V)|_0 - Id)^{-1}(\mathcal{G}'_V v - (D\mathcal{G}'_V)|_0 v),$$

where $\mathcal{G}'_V = \mathcal{A} \circ \mathcal{G}_V \circ \mathcal{A}^{-1}$ (see (1.14)) and $\mathcal{G}'_V \beta = \tilde{\Phi}_V \circ \beta \circ F^{-1}$. One can show that the contracting constant of $\mathcal{F}_V$ is uniform over V and that $\mathcal{F}_V$ converges exponentially fast to $\mathcal{F}$. Therefore, the corresponding fixed point h_V converges exponentially fast to h. $\Box$

For a Hölder continuous function $\varphi \in \widetilde{\mathcal{F}}(\alpha, q, \epsilon)$ on Δ_Φ consider the function $\tilde{\varphi} = \varphi \circ h$ on Δ_F, where $h : \Delta_F \to \Delta_\Phi$ is a conjugacy homeomorphism. Let $\tilde{\nu}_V$ be the $\mathbb{Z}$-action equilibrium measure on $\Delta_{F,V}$ corresponding to the function $\tilde{\psi}_{V,\bar{x}^*}$ which is determined by (4.7) with respect to the function $\tilde{\varphi}$. Finally, we define the measure $\nu_V = (h_V^{-1})^* \circ \tilde{\nu}_V$ on $\Delta_{\Phi,V}$. It also can be considered as a measure on $\mathcal{M}$. As a direct consequence of Theorem 4.3 we conclude that *if ϵ is sufficiently small then the measure μ_φ is the limit (in the weak*-topology) of the measures ν_V as $V \to \mathbb{Z}^d$.*

V. Uniqueness and Mixing Properties of Sinai–Ruelle–Bowen (SRB)-Measures

In this section we discuss existence, uniqueness, and mixing properties of SRB-type measures for coupled map lattices. The first construction of these measures appeared in [BuSi]. In [BK2], Bricmont and Kupiainen constructed these measures for general expanding circle maps. Their approach is based upon the study of the Perron–Frobenius operator. In [PS], Pesin and Sinai developed another method for constructing SRB-type measures for coupled map lattices assuming that the local map possesses a hyperbolic attractor.

In this section we develop a new approach and obtain stronger results under more general assumptions.

Let f be a C^r-diffeomorphism of a compact finite-dimensional manifold M possessing a hyperbolic attractor Λ. The latter means that Λ is a hyperbolic set and there exists an open neighborhood U of Λ such that $f(\bar{U}) \subset U$. In particular, $\Lambda = \cap_{n>0} f^n(U)$ and is a locally maximal invariant set. We assume that the map f is topologically mixing. Then an SRB-measure μ on Λ is unique and is characterized as follows:

1) the restriction of μ on the unstable manifolds is absolutely continuous with respect to the Lebesgue measure;

2) for any continuous function g and almost all $x \in U$ with respect to the Lebesgue measure in U,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(f^k x) = \int g d\mu; \tag{5.1}$$

3) μ is the unique equilibrium measure corresponding to the Hölder continuous function $\varphi^u(x) = -\log |\mathrm{Jac}^u f(x)|$, where $\mathrm{Jac}^u f(x)$ denotes the Jacobian of f at x along the unstable subspace.

In the infinite-dimensional case we construct a measure on Δ_Φ which has similar properties. This is an SRB-type measure for the coupled map lattice. Our construction is based upon symbolic representations of the finite-dimensional approximations of the lattice constructed in the previous section.

Let $V \in \mathbb{Z}^d$ be a finite volume. Consider the diffeomorphisms F_V and Φ_V. Since Φ_V is close to F_V it has a hyperbolic attractor $\Delta_{\Phi,V}$.

Since we assume that the map f is topologically mixing then so are the maps F, Φ, F_V, and Φ_V. Therefore, the map Φ_V possesses the unique SRB-measure μ_V that is supported on $\Delta_{\Phi,V}$. This measure is the unique equilibrium measure corresponding to the Hölder continuous function $\varphi_V^u(x) = -\log|\mathrm{Jac}^u\,\Phi_V(x)|$, where $\mathrm{Jac}^u\,\Phi_V(x)$ is the Jacobian of the map Φ_V at x along the unstable subspace. We can consider the measure μ_V to be supported on the compact space $(\mathcal{M}, \rho_q)$. Our main result is the following.

Theorem 5.1. *The SRB-measures μ_V weak* converge to a measure on $\mathcal{M}$ which is an equilibrium measure corresponding to a Hölder continuous function on $\mathcal{M}$ and is mixing. Furthermore, the correlation functions decay exponentially for every continuous function on $\mathcal{M}$ satisfying the assumptions of Theorem 3.1.*

Remarks.

(1) It is clear that for an uncoupled map lattice the SRB measures μ_V converge to the measure $\otimes_{i\in\mathbb{Z}^d}\mu_f$ which is the equilibrium measure for the potential function $-\log|Jac^u f(x)|$. The potential function for the SRB measure in Theorem 5.1 is a small perturbation of $-\log|Jac^u f(x)|$. Its precise description is given by (5.15).

(2) We follow the approach suggested in [BK2], [BK3]. We thank J. Bricmont who suggested to use the formula (5.8) to expand the Jacobian.

(3) To avoid some technical obstacles we assume that f is an Anosov map. In this case $\Delta_{\Phi_V} = \Delta_{F_V} = \mathcal{M}$.

(4) There is another approach to prove existence of SRB-type measures suggested in [PS]. It is based on a rather detailed analysis of conditional measures generated on the unstable manifolds. One can show that these conditional measures determine the SRB measure in the unique way.

Proof of the theorem. Let $\pi_V = \otimes_{i\in V}\pi_i$ be the semi-conjugacy map between the symbolic dynamical system $(\sigma_t, \otimes_{i\in V}\Sigma_A)$ and $(F_V, \mathcal{M}_V)$ (here π_i are copies of the coding map π). Define the measure ν_V on $\Sigma_A^V = \otimes_{i\in V}\Sigma_A$ by the following relation $\mu_V = (h_V\pi_V)^*\nu_V$. It is easy to see that the following statement holds.

Lemma 1. *The measures μ_V weak* converge to a measure on $\mathcal{M}$ if the measures ν_V weak* converge to a measure on $\Sigma_A^{\mathbb{Z}^d}$ as $V \to \mathbb{Z}^d$.*

The desired result is now a consequence of Lemma 1 and the following lemma.

Lemma 2. *The measures ν_V weak* converge to a measure on the $(d+1)$-dimensional* *lattice spin system $\Sigma_A^{\mathbb{Z}^d}$ which is the unique Gibbs state for a Hölder continuous function. It is also exponentially mixing with respect to the $\mathbb{Z}^{d+1}$-action of the lattice.*

Proof of the lemma. Note that the measure ν_V is the unique Gibbs state for the Hölder continuous function

$$\varphi_V(\xi_V) = -\log \mathrm{Jac}^u \Phi_V(h_V \pi_V(\xi_V)) \tag{5.2}$$

on Σ_A^V. We express the Jacobian $\mathrm{Jac}^u\Phi_V(x_V)$, $x_V \in M_V$ as a product

$$\mathrm{Jac}^u\Phi_V(x_V) = \det(D\Phi_V | W^u_{\Phi_V}(x_V)) = \det(I + A_V(x_V))\Big(\prod_{i\in V} \mathrm{Jac}^u f(x_i)\Big), \tag{5.3}$$

where I is the identity matrix and A_V is a matrix whose entries are submatrices satisfying some special properties which we specify later.

Let $E^u_{\Phi_V}(x_V)$ be the unstable subspace at x_V for the map Φ_V. One can see that $E^u_{\Phi_V}(x_V)$ is close to the direct product $\otimes_{i\in V} E^u_f(x_i)$. We choose a basis $\{\mathbf{u}_i(x_i), \mathbf{s}_i(x_i), i \in V\}$ in the space

$$\otimes_{i\in V} T_{x_i} M = \big(\otimes_{i\in V} E^u_f(x_i)\big) \otimes \big(\otimes_{i\in V} E^s_f(x_i)\big)$$

such that $\mathbf{u}_i(x_i)$ and $\mathbf{s}_i(x_i)$ are bases in $E^u_f(x_i)$ and $E^s_f(x_i)$ respectively, and we assume that they depend Hölder continuously on the base point x_V. The derivative $D\Phi_V(x_V)$ can now be written as follows:

$$D\Phi_V(x_V) = \begin{pmatrix} (D^u f(x_i)) & 0 \\ 0 & (D^s f(x_i)) \end{pmatrix} \left(I + \begin{pmatrix} \mathbf{a}^{uu}_{ij}(x_V) & \mathbf{a}^{us}_{ij}(x_V) \\ \mathbf{a}^{su}_{ij}(x_V) & \mathbf{a}^{ss}_{ij}(x_V) \end{pmatrix} \right). \tag{5.4}$$

where we arrange the elements of the basis $\{\mathbf{u}_i(x_i), \mathbf{s}_i(x_i), i \in V\}$ in an arbitrary linear order, $\mathbf{u}_i$ first, followed by $\mathbf{s}_i$. Since Φ is C^1-close to F and is short ranged the submatrices $(\mathbf{a}^*_{ij}(x_V))$ satisfy the following conditions (we use * to denote one of the symbols uu, us, su, or ss):

(1) $\|(\mathbf{a}^*_{ij}(x_V))\| \le \epsilon e^{-\beta|i-j|}$, where $|i-j|$ is the distance between the lattice sites i and j and constants $\epsilon > 0$ and $\beta > 0$ are independent of the volume V as well as of the base point x_V;

(2) each submatrix $\mathbf{a}^*_{ij}(x_V)$ depends Hölder continuously on x_V:

$$\|\mathbf{a}^*_{ij}(x_V) - \mathbf{a}^*_{ij}(y_V)\| \le \epsilon e^{-\beta|i-k|} d^\delta(x_k, y_k), \tag{5.5}$$

where $x_V = (x_i)$ and $y_V = (y_i)$ are such that $x_i = y_i$ for $i \ne k$ (recall that d is the Riemannian distance on M).

The constant $\epsilon > 0$ can be chosen arbitrarily small as the C^1-distance between Φ and F goes to zero. The constant δ is independent of the volume V and the base point x_V.

Using the graph transform technique one can identify the unstable subspace $E^u_{\Phi_V}(x_V)$ with the graph of a linear map $H_{x_V} : \otimes_{i\in V} E^u_f(x_i) \to \otimes_{i\in V} E^s_f(x_i)$, i.e.,

$$E^u_{\Phi_V}(x_V) = (\otimes_{i\in V} E^u_f(x_i),\, H_{x_V} \otimes_{i\in V} E^u_f(x_i)). \tag{5.6}$$

The linear map H_{x_V} has a unique matrix representation $(\mathbf{c}^{us}_{ij})$ in the basis $\{\mathbf{u_i}(x_i),\, \mathbf{s}_i(x_i)\}$

$$H_{x_V}\mathbf{u}_i(x_i) = \sum_j \mathbf{c}^{us}_{ij}\mathbf{s}_j(x_j), \tag{5.7}$$

where each submatrix $\mathbf{c}^{us}_{ij}$ satisfies conditions similar to Conditions (1) and (2):

(3) $\|\mathbf{c}^{us}_{ij}\| \le \epsilon e^{-\beta|i-j|}$;

(4) $\|\mathbf{c}^{us}_{ij}(x_V) - \mathbf{c}^{us}_{ij}(y_V)\| \le \epsilon e^{-\beta|i-k|} d^\delta(x_k, y_k)$, where $x_V = (x_i)$ and $y_V = (y_i)$ are such that $x_i = y_i$ for $i \ne k$.

In order to prove Condition (3) one can use the graph transform technique in the form described in [JLP] and combine it with the fact that the linear map H_{x_V} is short ranged. Condition (4) follows from the fact that distributions $E^u_{\Phi_V}(x_V)$, $\otimes_{i\in V} E^u_f(x_i)$, and $\otimes_{i\in V} E^s_f(x_i)$ depend Hölder continuously over the base point x_V.

Moreover, the entries $\mathbf{c}^{us}_{ij}$ satisfy the following crucial condition which allows one to pass from a finite volume to a bigger one:

(5) $\|\mathbf{c}^{us}_{ij}(x_V) - \mathbf{c}^{us}_{ij}(y_{V'})\| \le \epsilon e^{-\beta d(i,\partial V)}$ for any finite volume $V \subset V'$ and any point $y_{V'}$ satisfying $y_{V'}|_V = x_V$.

In order to prove (5), we apply graph transform technique to the map $\Phi_{V'}$ on $M_{V'}$ with the ρ_q-metric restricted to $M_{V'}$. Note that the ρ_q-distance between $\Phi_{V'}$ and $\Phi_V \otimes F_{V'\setminus V}$ is proportional to $\epsilon e^{-\beta d(V)}$. Therefore, using results in [PS] we obtain that the ρ_q-distance between subspaces $E^{u,s}_{\Phi'_V}(x'_V)$ and $E^{u,s}_{\Phi_V}(x_V) \otimes_{i\in V'\setminus V} E^{u,s}_f(y_i)$ is also proportional to $\epsilon e^{-\beta d(V)}$. Hence, so is the ρ_q-distance between linear operators $H_{x_{V'}}$ and H_{x_V}. This implies (5).

We choose $\{\tilde{\mathbf{u}}_i\} = \{\mathbf{u}_i + H\mathbf{u_i}\} = \{\mathbf{u}_i + \sum_j \mathbf{c}^{us}_{ij}\mathbf{s}_j\}$ as a basis in $E^u_{\Phi_V}(x_V)$ and we write the derivative $D\Phi|E^u_{\Phi_V}(x_V)$ in the new basis $\{\tilde{\mathbf{u}}_i, \mathbf{s}_i, i \in V\}$ into the following matrix form:

$$D\Phi|E^u_{\Phi_V}(x_V) = (D^u f(x_i))(I + \mathbf{a}^{uu}_{ij}(x_V)) + (\mathbf{a}^{us}_{ij}(x_V))(\mathbf{c}^{us}_{ij}(x_V)).$$

The latter expression can be rewritten in the form

$$(D^u f(x_i))(I + (\mathbf{a}_{ij}(x_V))),$$

where $A_V(x_v) = (\mathbf{a}_{ij}(x_V))$ is the matrix whose submatrix entries $\mathbf{a}_{ij}(x_V)$ satisfy the following conditions (which follow immediately from (1)–(5)):

(6) $\|\mathbf{a}_{ij}\| \le \epsilon e^{-\beta|i-j|}$;

(7) $\|\mathbf{a}_{ij}(x_V) - \mathbf{a}_{ij}(y_V)\| \le \epsilon e^{-\beta|i-k|} d^\delta(x_k, y_k)$, where $x_V = (x_i)$ and $y_V = (y_i)$ are such that $x_i = y_i$ for $i \ne k$.

(8) $\|\mathbf{a}_{ij}(x_V) - \mathbf{a}_{ij}(y_{V'})\| \le \epsilon e^{-\beta d(i, \partial V)}$ for any $V \subset V'$.

Next, we apply the well-known formula:

$$\det(\exp(B)) = \exp(\operatorname{trace}(B)).$$

In our case, $\exp(B) = I + A_V(x_V)$ and hence,

$$\det(I + A_V) = \exp(\operatorname{trace}(\ln(I + A_V)) = \exp(-\sum_{i \in V} w_{Vi}), \tag{5.8}$$

where

$$w_{Vi}(x_V) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \operatorname{trace}(\mathbf{a}_{ii}^n(x_V)) \tag{5.9}$$

and $\mathbf{a}_{ii}^n(x_V)$ are submatrices on the main diagonal of $(A_V)^n$.

Sublemma. The functions $w_{Vi}(x_V)$ satisfy:

(1) $|w_{Vi}(x_V)| \le C\epsilon$;

(2) $|w_{Vi}(x_V) - w_{Vi}(y_V)| \le C\epsilon \exp(-\frac{\beta}{2}|i-k|) d^\delta(x_k, y_k)$, where $x_V = (x_i)$ and $y_V = (y_i)$ are such that $x_i = y_i$ for $i \ne k$;

(3) if $V \subset V'$ then $|w_{Vi}(x) - w_{V'i}(y)| \le C\epsilon \exp(-\frac{\beta}{2} d(i, \partial V))$;

(4) there exists the limit $\varphi_i = \lim_{V \to \mathbb{Z}^d} w_{Vi}(x)$ which is translation invariant in the following sense: $\varphi_i(\bar{x}) = \varphi_0(\sigma_s^i \bar{x})$. Moreover, φ_0 is Hölder continuous with Hölder constant which goes to zero as $\epsilon \to 0$.

Proof of the sublemma. The proof is a straightforward calculation. We first show the following inequality

$$\|\mathbf{a}_{ij}^n\| \le (C\epsilon)^n e^{-\tilde{\beta}|i-j|}, \tag{5.10}$$

where $\tilde{\beta}$ is a number smaller than β and $C = C(\tilde{\beta})$ is a constant.

We use the induction. For $n = 2$ we have

$$\begin{aligned} \|\mathbf{a}_{ij}^2\| = \|\sum_{l \in V} \mathbf{a}_{il}\mathbf{a}_{lj}\| &\le \sum_{l \in V} \epsilon^2 \exp(-\beta(|i-l| + |l-j|)) \\ &\le \sum_{l \in V} \epsilon^2 \exp(-\tilde{\beta}(|i-l| + |l-j|) - (\beta - \tilde{\beta})|l-j|) \\ &\le \epsilon^2 e^{-\tilde{\beta}|i-j|} \sum_{l \in V} \exp(-(\beta - \tilde{\beta})|l-j|) \le C\epsilon^2 e^{-\tilde{\beta}|i-j|}, \end{aligned} \tag{5.11}$$

where $C = C(\tilde{\beta}) = \sum_{l \in \mathbb{Z}^d} \exp(-(\beta - \tilde{\beta})|l|)$.

Let us assume that $\|\mathbf{a}_{ij}^{n-1}\| \leq C^{n-2}\epsilon^{n-1}\exp(-\tilde{\beta}|i-j|)$. Then

$$\|\mathbf{a}_{ij}^{n}\| = \|\sum_{l\in V}\mathbf{a}_{il}^{n-1}\mathbf{a}_{lj}\| \leq \sum_{l\in V} C^{n-2}\epsilon^{n}\exp(-\tilde{\beta}(|i-l|+|l-j|)-(\beta-\tilde{\beta})|l-j|)$$

$$\leq C^{n-1}\epsilon^{n}\exp(-\tilde{\beta}|i-j|). \tag{5.12}$$

Therefore, Statement 1 follows directly from the definition of w_{Vi}.

To prove Statement 2 we need only to show the following inequality:

$$\|\mathbf{a}_{ij}^{n}(x_V)-\mathbf{a}_{ij}^{n}(y_V)\| \leq (C\epsilon)^n e^{-\frac{\beta}{2}|i-k|}d^{\delta}(x_k,y_k),$$

where $x_V=(x_i)$ and $y_V=(y_i)$ are such that $x_i=y_i$ for $i\neq k$. We again use the induction. For $n=2$,

$$\|\mathbf{a}_{ij}^{2}(x_V)-\mathbf{a}_{ij}^{2}(y_V)\| = \sum_{l\in V}\mathbf{a}_{il}(x_V)\mathbf{a}_{lj}(x_V)-\mathbf{a}_{il}(y_V)\mathbf{a}_{lj}(y_V)$$

$$=\sum_{l\in V}\mathbf{a}_{il}(x_V)[\mathbf{a}_{lj}(x_V)-\mathbf{a}_{lj}(y_V)]+\mathbf{a}_{lj}(y_V)[\mathbf{a}_{il}(x_V)-\mathbf{a}_{il}(y_V)]$$

$$\leq \sum_{l\in V}\epsilon^2[\exp(-\beta(|l-k|+|i-l|))+\exp(-\beta(|l-j|+|i-k|))]d^{\delta}(x_k,y_k)$$

$$\leq C\epsilon^2\exp(-\frac{\beta}{2}|i-k|)d^{\delta}(x_k,y_k), \tag{5.13}$$

where $C=2\sum_{l\in\mathbb{Z}^d}\exp(-\frac{\beta}{2}|l|)$.

For $n>2$ we argue similarly using Statement (1):

$$\|\mathbf{a}_{ij}^{n}(x_V)-\mathbf{a}_{ij}^{n}(y_V)\| = \sum_{l\in V}\mathbf{a}_{il}^{n-1}(x_V)\mathbf{a}_{lj}(x_V)-\mathbf{a}_{il}^{n-1}(y_V)\mathbf{a}_{lj}(y_V)$$

$$=\sum_{l\in V}\mathbf{a}_{il}^{n-1}(x_V)[\mathbf{a}_{lj}(x_V)-\mathbf{a}_{lj}(y_V)]+\mathbf{a}_{lj}(y_V)[\mathbf{a}_{il}^{n-1}(x_V)-\mathbf{a}_{il}^{n-1}(y_V)]$$

$$\leq \sum_{l\in V}(C\epsilon)^{n-1}\epsilon\exp(-\frac{\beta}{2}|i-l|-\beta|l-k|-\beta|l-j|-\frac{\beta}{2}|i-k|)d^{\delta}(x_k,y_k)$$

$$\leq (C\epsilon)^n\exp(-\frac{\beta}{2}|i-k|)d^{\delta}(x_k,y_k). \tag{5.14}$$

Statement 3 follows from Condition (8) while Statement 4 is a consequence of Statements 2 and 3 and our assumption that the map Φ is spatial translation invariant. $\square$

We proceed with the proof of the theorem. Let V be a d-dimensional cube centered at the origin. Choose any finite volume $V_0 \subset V$ and numbers $0 < m < n$. We have that

$$\nu_V(\xi_{(V_0,m)}) = \lim_{n\to\infty} \nu_V(\xi_{(V_0,m)}|\eta^*_{\widehat{(V,n)}}).$$

In order to obtain the desired result we shall show that the one-dimensional Gibbs distributions $\nu_V(\xi_{(V,n)}|\eta^*_{\widehat{(V,n)}})$ has a unique thermodynamic limit as $V \to \mathbb{Z}^{d+1}$ and $n \to \infty$. This thermodynamic limit is precisely the unique $d+1$-Gibbs state for the potential function

$$\varphi(\bar{\xi}) = (\varphi_0 - \log \mathrm{Jac}^u f)(h\bar{\pi}(\bar{\xi})) \tag{5.15}$$

on $\Sigma_A^{\mathbb{Z}^d}$.

Note that the function φ is the sum of two functions, $\varphi = \tilde{\varphi}_0 + \varphi_1$, where $\tilde{\varphi}_0 = \varphi_0 - \log \mathrm{Jac}^u f \circ h \circ \bar{\pi} + \log \mathrm{Jac}^u f \circ \bar{\pi}$ and $\varphi_1 = -\log \mathrm{Jac}^u f \circ \bar{\pi}$. By Statements 1, 2, and 4 of Sublemma and Theorem 1.1 the function $\tilde{\varphi}_0$ is Hölder continuous with a small Hölder constant in the metric ρ_q provided ϵ is sufficiently small. The function φ_1 is also Hölder continuous and depends only on the coordinate ξ_0. Therefore, by Theorem 3.7 the Gibbs state corresponding to this function is unique.

Since the measure ν_V is the unique Gibbs state for the Hölder continuous function $\varphi_V(\xi_V)$ on Σ_A^V (see (5.2)) it satisfies the following equation [Ru]: given a configuration $\eta^* \in \Sigma_A^{\mathbb{Z}^d}$,

$$\nu_V(\xi_{(V,n)}|\eta^*_{\widehat{(V,n)}}) = \frac{\exp \sum_{k\in\mathbb{Z}} \varphi_V(\sigma_t^k(\xi_{(V,n)} + \eta^*_{\widehat{(V,n)}}))}{\sum_{\eta_{(V,n)}} \exp \sum_{k\in\mathbb{Z}} \varphi_V(\sigma_t^k(\eta_{(V,n)} + \eta^*_{\widehat{(V,n)}})}, \tag{5.16}$$

where $\xi_{(V,n)}$ is a configuration over the finite volume $(V,n) = V \times [-n,n] \subset \mathbb{Z}^{d+1}$, and $\eta^*_{\widehat{(V,n)}}$ is the restriction of the configuration η^* to $\widehat{(V,n)} = \mathbb{Z}^{d+1} \backslash V \times [-n,n]$.

Using (5.3) and (5.8) we rewrite (5.16) in the following way

$$\nu_V(\xi_{(V,n)}|\eta^*_{\widehat{(V,n)}}) = \frac{\exp \sum_{k\in\mathbb{Z}} \varphi_V(h_V\pi_V\sigma_t^k(\xi_{(V,n)} + \eta^*_{\widehat{(V,n)}}))}{\sum_{\eta_{(V,n)}} \exp \sum_{k\in\mathbb{Z}} \varphi_V(h_V\pi_V\sigma_t^k(\eta_{(V,n)} + \eta^*_{\widehat{(V,n)}}))}$$

$$= \frac{\exp \sum_{k\in\mathbb{Z}} \sum_{i\in V} (w_{Vi} - \log \mathrm{Jac}^u f)(h_V\pi_V\sigma_t^k(\xi_{(V,n)} + \eta^*_{\widehat{(V,n)}}))}{\sum_{\eta_{(V,n)}} \exp \sum_{k\in\mathbb{Z}} \sum_{i\in V} (w_{Vi} - \log \mathrm{Jac}^u f)(h_V\pi_V\sigma_t^k(\eta_{(V,n)} + \eta^*_{\widehat{(V,n)}}))}.$$

The rest of the proof is split into the following steps.

Step 1: We wish to rewrite the last expression for the conditional distributions $\nu_V(\xi_{(V,n)}|\eta^*_{\widehat{(V,n)}})$ in terms of potentials (see Section 3). The potential U corresponding to the function $(\varphi_0 - \log \mathrm{Jac}^u f)(h\bar{\pi})$ can be constructed using (3.9)–(3.12).

Given a finite volume V and $i \in V$, consider the function $(w_{Vi} - \log \mathrm{Jac}^u f)(h_V \pi_V)$. In order to construct the potential U^{Vi} corresponding to this function we again follow the procedure described in Section 3 and use $(w_{Vi} - \log \mathrm{Jac}^u f)(h_V \pi_V)$ for each $\mathbb{Z}^{d+1}$-cube centered at $(i,k) \in V \times \mathbb{Z}$. Not that the resulting potential is invariant under time translations but may not be invariant under spatial translations.

Step 2: We now rewrite the distributions $\nu_V(\xi_{(V,n)}|\eta^*_{\widehat{(V,n)}})$ in terms of potentials U^{Vi}:

$$\nu_V(\xi_{(V,n)}|\eta^*_{\widehat{(V,n)}}) = \frac{\exp \sum_{Q\cap(V,n)\neq\emptyset} U_Q^{Vi}(\xi_{(V,n)} + \eta^*_{\widehat{(V,n)}})}{\sum_{\eta_{(V,n)}} \exp \sum_{Q\cap(V,n)\neq\emptyset} U_Q^{Vi}(\eta_{(V,n)} + \eta^*_{\widehat{(V,n)}})}. \tag{5.17}$$

Step 3: By Statement 3 of Sublemma $w_{Vi} \to \varphi_i = \varphi_0(\sigma_s^i)$ exponentially fast. Using the fact that $h_V \to h$ exponentially fast in the ρ_q-metric (see Theorem 4.4) we obtain that for any $\mathbb{Z}^{d+1}$-cube Q centered at $(i,k) \in V \times \mathbb{Z}$

$$|U^{Vi}(\xi(Q)) - U(\xi(Q))| \leq C\epsilon e^{-\beta d(i,\partial V)}. \tag{5.18}$$

By Statement 2 of Sublemma both potentials $U^{Vi}|_Q$ and $U|_Q$ go to zero exponentially fast as the side length of Q increases.

Step 4: Take a larger volume $(V',n') \subset \mathbb{Z}^{d+1}$ such that $(V,n) \subset (V'/2, n'/2)$ where $V'/2$ is the d-dimensional cube centered at the origin of the side length equal to 1/2 of the side length of V. We follow the approach elaborated by Ruelle in [Ru] (see Section 1.7). (For the reader's convenience we provide the correspondence between Ruelle's notations and ours: $M = (V',n')$, $\Lambda = (V,n)$, $X = Q$, and $\Phi = U^{Vi}, U$).

We first decompose the numerator of (5.17) (for volume (V',n')) into two terms.

$$\exp \sum_{Q\cap(V',n')\neq\emptyset} U_Q^{Vi}(\xi_{(V,n)} + \eta^*_{\widehat{(V,n)}}) = \exp\left(H_{(V,n)}(\xi_{(V,n)}) + B_{(V',n')}(\xi_{(V',n')})\right),$$

where the *main* term $H_{(V,n)}(\xi_{(V,n)})$, the Hamiltonian in volume (V,n), is given by

$$H_{(V,n)}(\xi_{(V,n)}) = \sum_{Q\subset(V,n)} U_Q(\xi_{(V,n)} + \eta^*_{\widehat{(V,n)}})$$

while the *boundary* term is given as follows:

$$B_{(V',n')}(\xi_{(V',n')}) = \sum_{Q\cap(V',n')\neq\emptyset} U_Q^{V'i}(\xi_{(V',n')} + \eta^*_{\widehat{(V',n')}}) - U_Q(\xi_{(V',n')} + \eta^*_{\widehat{(V',n')}})$$

$$+\sum_{\substack{Q\cap(V',n')\neq\emptyset\\ Q\cap\widehat{(V',n')}\neq\emptyset}} U_Q(\xi_{(V',n')}+\eta^*_{\widehat{M}}).$$

By (5.17) and results in [Ru] (see Section 1.6) we now only need to verify that the boundary term satisfies the conditions stated in section 1.7 of [Ru].

We first split $B_{(V',n')}(\xi_{(V',n')})$ into two terms $B_{(V',n')}(\xi_{(V',n')}) = B'(\eta)+B''(\xi_{(V,n)}+\eta)$, where $\xi_{(V',n')} = \xi_{(V,n)}+\eta$ and $B'(\eta)$ collects the terms depending only on $\eta \in \Omega_{(V',n')\backslash(V,n)}$, i.e.,

$$B'(\eta) = \sum_{\substack{X\cap(V',n')\neq\emptyset\\ Q\cap(V,n)=\emptyset}} \Big(U_Q^{V'i}(\xi_{(V',n')}+\eta^*_{\widehat{(V',n')}}) - U_Q(\xi_{(V',n')}+\eta^*_{\widehat{(V',n')}})\Big)$$

$$+\sum_Q{}^{*} U_Q(\xi_{(V',n')}+\eta^*_{\widehat{(V',n')}})$$

while the second term is given as follows:

$$B''(\xi_\Lambda+\eta) = \sum_{Q\cap(V,n)\neq\emptyset} \Big(U_Q^{V'i}(\xi_{(V',n')}+\eta^*_{\widehat{(V',n')}}) - U_Q(\xi_{(V',n')}+\eta^*_{\widehat{(V',n')}})\Big)$$

$$+\sum_Q{}^{**} U_Q(\xi_{(V',n')}+\eta^*_{\widehat{(V',n')}}).$$

Here $\sum_Q^*$ runs over $\{Q : Q\cap(V',n')\neq\emptyset, Q\cap\widehat{(V',n')}\neq\emptyset, Q\cap\Lambda=\emptyset\}$ and $\sum_Q^{**}$ runs over $\{Q : Q\cap(V',n')\neq\emptyset, Q\cap\widehat{(V',n')}\neq\emptyset, Q\cap\Lambda\neq\emptyset\}$.

According to [Ru] in order to show that the thermodynamic limit of $\nu_V(\xi_{(V,n)}|\eta^*_{\widehat{(V,n)}})$ goes to a $\mathbb{Z}^{d+1}$-Gibbs state of U, we only need to check that for any fixed (V,n), $B''(\xi_{(V,n)}+\eta)$ as a function of $\eta\in\Omega_{(V',n')\backslash(V,n)}$ goes to zero uniformly in $\Omega_{(V',n')\backslash(V,n)}$ as $(V',n')\to\mathbb{Z}^{d+1}$. The second sum in B'', $\sum_Q^{**}$, goes to zero uniformly since the potential U decays exponentially. The first sum in B'' can be further decomposed into two sums. Let $(i(Q),k(Q))\in\mathbb{Z}^{d+1}$ denote the center of Q. We may assume that (V',n') is a $\mathbb{Z}^{d+1}$-cube with equal sides. Then,

$$\sum_{Q\cap(V,n)\neq\emptyset} U_Q^{V'i}(\xi_{(V',n')}+\eta^*_{\widehat{(V',n')}}) - U_Q(\xi_{(V',n')}+\eta^*_{\widehat{(V',n')}})$$

$$= (\sum_{\substack{i(Q)\in(V',n')/2\\ Q\cap(V,n)\neq\emptyset}} + \sum_{\substack{i(Q)\notin(V',n')/2\\ Q\cap(V,n)\neq\emptyset}}) U_Q^{V'i}(\xi_{(V',n')}+\eta^*_{\widehat{(V',n')}}) - U_Q(\xi_{(V',n')}+\eta^*_{\widehat{(V',n')}}).$$

By (5.18) we have

$$|\sum_{\substack{i(Q)\in(V',n')/2\\ Q\cap(V,n)\neq\emptyset}} U_Q^{V'i}(\xi_{(V',n')}+\eta^*_{\widehat{(V',n')}}) - U_Q(\xi_{(V',n')}+\eta^*_{\widehat{(V',n')}})|$$

$$\leq C'\varepsilon|(V,n)||(V',n')/2|e^{-\beta d((V',n'))},$$

where $|(V,n)|$ and $|(V',n')/2|$ are the cardinalities of the corresponding sets and $d((V',n'))$ is the side length of (V',n'). The sum

$$\sum_{\substack{i(Q)\notin(V',n')/2\\ Q\cap(V,n)\neq\emptyset}} U_Q^{V'i}(\xi_{(V',n')}+\eta^*_{\widehat{(V',n')}})-U_Q(\xi_{(V',n')}+\eta^*_{\widehat{(V',n')}})$$

also goes to zero uniformly as $d((V',n'))\to\infty$ since both potentials $U^{V'i}$ and U go to zero exponentially fast as $d((V',n'))\to\infty$.

This completes the proof of the theorem. □

Appendix: Spin Lattice Systems

1. Abstract Polymer Expansion Theorem

Consider a finite or countable set Θ. Its elements are called (abstract) *contours* and denoted by θ,θ', etc. Fix some reflexive and symmetric relation on $\Theta\times\Theta$. A pair $\theta,\theta'\in\Theta\times\Theta$ is called incompatible ($\theta\not\sim\theta'$) if it belongs to the given relation. Otherwise, this pair is called compatible ($\theta\sim\theta'$). A collection $\{\theta_j\}$ is called a *compatible collection of contours* if any two of its elements are compatible. A *statistical weight* is a complex function on the set of contours. For any finite subset $\Lambda\subseteq\Theta$ an *abstract partition function* is defined as

$$Z(\Lambda)=\sum_{\{\theta_j\}\subseteq\Lambda}\prod_j w(\theta_j), \qquad (A.1.1)$$

where the sum is extended to all compatible collections of contours $\theta_i\in\Lambda$. The empty collection is compatible by definition and it is included in $Z(\Lambda)$ with statistical weight 1.

A *polymer* $\wp=[\theta_i^{\alpha_i}]$ is an (unordered) finite collection of different contours $\theta_i\in\Theta$ with positive integer multiplicity α_i. For every pair $\theta',\ \theta''\in\wp$ there exists a sequence $\theta'=\theta_{i_1},\ \theta_{i_2},\dots,\theta_{i_s}=\theta''\in\wp$ with $\theta_{i_j}\not\sim\theta_{i_{j+1}},\ j=1,2,\dots,s-1$. The notation $\wp\subseteq\Lambda$ means that $\theta_i\in\Lambda$ for every $\theta_i\in\wp$.

With every polymer $\wp$ we associate an (abstract) graph $\Gamma(\wp)$ which consists of $\sum_i\alpha_i$ vertices labeled by the contours from $\wp$ and edges joining every two vertices labeled by incompatible contours. It follows from the definition of $\Gamma(\wp)$ that it is connected and we denote by $r(\wp)$ the quantity

$$r(\wp)=\prod_i(\alpha_i!)^{-1}\sum_{\Gamma'\subset\Gamma(\wp)}(-1)^{|\Gamma'|}, \qquad (A.1.2)$$

where the sum is taken over all connected subgraphs Γ' of $\Gamma(\wp)$ containing all of $\sum_i \alpha_i$ vertices and $|\Gamma'|$ denotes the number of edges in Γ'. For any $\theta \in \wp$ we denote by $\alpha(\theta, \wp)$ the multiplicity of θ in the polymer $\wp$.

The polymer expansion theorem below is a modification of results of [Se] and [KP] proven in [MSu] (see also [D2] for close results).

Abstract Polymer Expansion Theorem *(1) Suppose that there exists a function $a(\theta): \ \Theta \mapsto \mathbb{R}^+$ such that for any contour θ*

$$\sum_{\theta':\ \theta' \not\sim \theta} |w(\theta')| e^{a(\theta')} \leq a(\theta). \tag{A.1.3}$$

Then, for any finite Λ,

$$\log Z(\Lambda) = \sum_{\wp \subseteq \Lambda} w(\wp), \tag{A.1.4}$$

where the statistical weight of a polymer $\wp = [\theta_i^{\alpha_i}]$ equals to

$$w(\wp) = r(\wp) \prod_i w(\theta_i)^{\alpha_i}. \tag{A.1.5}$$

Moreover, the series (A.1 .4) converges absolutely in view of the estimate

$$\sum_{\wp:\ \wp \ni \theta} \alpha(\theta, \wp) |w(\wp)| \leq |w(\theta)| e^{a(\theta)}, \tag{A.1.6}$$

which holds true for any contour θ.

2. Gibbs States

Let $S = \{1, 2, \cdots, p\}$ and A be a $p \times p$ transfer matrix with entries a_{ij} equal to either 0 or 1. Assume that A is transitive, i.e., there is a constant n_0 such that every entry of A^{n_0} is positive. For any volume $V \subseteq \mathbb{Z}^2$ a *configuration* in V is an element $\eta(V)$ of S^V with the value $\eta_x(V)$ at point $x = (i, j) \in V$. A configuration η is called admissible if $a_{\eta_{x_1} \eta_{x_2}} = 1$ for any pair $x_1 = (i, j), x_2 = (i, j + 1) \in V$. For the family of configurations $\eta(V_i)$ in mutually disjoint volumes V_i we denote by $\sum_i \eta(V_i)$ the corresponding configuration in $\cup_i V_i$ provided such a configuration exists (i.e., is admissible). When $V = \mathbb{Z}^2$ we have the configuration space $\Sigma_A^{\mathbb{Z}} = \bigotimes_{\mathbb{Z}} \Sigma_A$, where Σ_A is the subshift generated by the matrix A.

Let Q be a square in $\mathbb{Z}^2$ and $l(Q)$ its side length. Consider a potential U satisfying

$$0 \leq U(\eta(Q)) \leq \exp\ [-l(Q)]. \tag{A.2.1),}$$

for every square $Q \subset \mathbb{Z}^2$.

Take a finite volume V and fix a configuration η' over $\widehat{V} = \mathbb{Z}^2 \setminus V$. The configuration $\eta'(\widehat{V})$ is called a boundary condition.

A Gibbs distribution over V under the boundary condition $\eta'(\widehat{V})$ is defined by

$$\mu_{V,\eta'}(\eta(V)) = \frac{\exp\left[-\beta H(\eta(V)|\eta'(\widehat{V}))\right]}{\Xi(V|\eta'(\widehat{V}))} . \qquad (A.2.2)$$

Here $\beta > 0$ is called the *inverse temperature*, $\eta(V)$ is a configuration over V such that $\eta(V) + \eta'(\widehat{V})$ is also a configuration in $\mathbb{Z}^2$,

$$H(\eta(V)|\eta'(\widehat{V})) = -\sum_{Q \subseteq V} U(\eta(Q)) - \sum_{Q\cap V \neq \emptyset,\ Q \cap \widehat{V} \neq \emptyset} U\big(\eta(Q\cap V) + \eta'(Q \cap \widehat{V})\big) \qquad (A.2.3)$$

is the conditional Hamiltonian, and the denominator in (A.2.2) is the partition function in the volume V with the boundary condition $\eta'(\widehat{V})$:

$$\Xi(V|\eta'(\widehat{V})) = \sum_{\eta(V)} \exp\left[-\beta H(\eta(V)|\eta'(\widehat{V}))\right] . \qquad (A.2.4).$$

3. Contour Representation of Partition Functions

We shall show that the partition function $\Xi(V|\eta'(\widehat{V}))$ can be represented in the form of an abstract partition function (A.1.1). It has a polymer expansion (A.1.4) if β is sufficiently small. We shall describe the terms in (A.1.1) in our specific context.

We first introduce a new potential which are equivalent to the original one (A.2.1)-(A.2.4). This means that the new potential defines the same Gibbs distributions over any finite volume under a fixed boundary condition.

Let $b(Q)$ be the leftmost lower corner of Q. Take an integer $L \geq n_0$ and consider a rectangle P of size $n(P) \times Ln(P)$ such that its leftmost lower corner $b(P) = (b_1(P), b_2(P))$ has $b_2(P) = rL$, where r and $n(P)$ are integers. We say that the square Q with $b(Q) = (b_1(Q), b_2(Q))$ is associated with the rectangle P if $b_1(Q) = b_1(P)$, $L[b_2(Q)/L] = b_2(P)$, $l(Q) = n(P)$ and hence $Q \subseteq P$ (here $[\,\cdot\,]$ denotes the integer part). For any rectangle P we define

$$U(\eta(P)) = \sum_{Q} U(\eta(Q)), \qquad (A.3.1)$$

where the sum is taken over all squares Q associated with the rectangle P. Clearly,

$$0 \leq U(\eta(P)) \leq L\exp\ [-n(P)] \qquad (A.3.2)$$

and absorbing L in β one can assume that the potential is defined on rectangles P (instead of squares Q) and satisfies

$$0 \le U(\eta(P)) \le \exp\ [-n(P)]\,. \tag{A.3.3}$$

Set $\partial^I V = \{x \in V \mid \text{dist }(x, \widehat{V}) = 1\}$, $\partial^E V = \{x \in \widehat{V} \mid \text{dist }(x, V) = 1\}$. We call $\partial^I V$ and $\partial^E V$ an *internal* and an *external* boundaries of V respectively. Observe that every finite volume V can be uniquely partitioned into vertical segments V_n with each segment being a connected component of the intersection of V and some vertical line. We denote by $a(V_n)$ and $b(V_n)$ the points of $\partial^E V$ adjacent to V_n from above and from below, respectively. The collection of such elements will be denoted by $a(V)$ and $b(V)$. In addition, we restrict our considerations to the volumes with

$$L[a(V_n)/L] = a(V_n) \text{ and } \ L[b(V_n) + 1/L] - 1 = b(V_n). \tag{A.3.4}$$

As we still allow arbitrary boundary conditions it is sufficient to prove the uniqueness of the limiting Gibbs state when the limit is taken over volumes of the special shape described above.

3.1 Definition of contours

A *precontour* $\gamma = \{P_j\}$ is a family of rectangles which satisfy the following conditions:
(1) $\bar{\gamma} = \cup_j P_j$ is a connected subset of $\mathbb{Z}^2$;
(2) every P_j contains a point which does not belong to any other rectangle of γ.

Consider a finite family of rectangles $\Gamma = \{P_i\}$ such that $\bar{\Gamma} = \cup_i P_i$ is a connected subset of $\mathbb{Z}^2$. This family of rectangles $\gamma(\Gamma)$ will be a precontour by our definition. It is called the *precontour of* Γ. We describe an algorithm which produces a unique minimal covering $\gamma(\Gamma)$ of $\bar{\Gamma}$.

(i) Fix the *leftmost lower* point in $\bar{\Gamma}$. Among all rectangles of Γ that begins at this point choose the rectangle P_{i_1} with the maximal linear size $n(P_{i_1})$ and include it in $\gamma(\Gamma)$.

(ii) Suppose that the rectangles $P_{i_1}, \ldots, P_{i_k}$ are already selected to $\gamma(\Gamma)$ during the previous steps of the algorithm. Fix the *leftmost lower* point $x \in \bar{\Gamma} \setminus (\cup_{j=1}^k P_{i_j})$. Consider all rectangles of Γ covering x. Among them choose the rectangles with the maximal right upper corner (here maximal means *rightmost upper*). From this family of rectangles include in $\gamma(\Gamma)$ the rectangle $P_{i_{k+1}}$ which has the maximal linear size.

(iii) Repeat step (ii) until $\bar{\Gamma}$ will be totally covered, i.e. $\bar{\Gamma} = \cup_j P_{i_j}$.

We say that a rectangle P is *compatible* with precontour $\gamma = \{P_j\}$ and denote it by $P \prec \gamma$ if for $\Gamma = \{P_j\} \cup \{P\}$ one has $\gamma(\Gamma) = \gamma$. Obviously, any $P \prec \gamma$ belongs to $\bar{\gamma}$ and any P embedded into some $P_j \in \gamma$ is compatible with γ. It is also clear that some of the rectangles $P \subseteq \bar{\gamma}$ can be incompatible with γ.

A collection of precontours $\{\gamma_i\}$ is called a compatible if for any $\gamma_{i_1}, \gamma_{i_2} \in \{\gamma_i\}$ either dist $(\bar{\gamma}_{i_1}, \bar{\gamma}_{i_2}) > 1$ or $\bar{\gamma}_{i_1} \subseteq \bar{\gamma}_{i_2} \setminus \partial^I \bar{\gamma}_{i_2}$. For $V \subset \mathbb{Z}^2$, the inclusion $\Gamma \subset V$ means that every rectangle of Γ is contained in V. Furthermore, $\Gamma \cap V \neq \emptyset$ mean that $P \cap V \neq \emptyset$ for every $P \subset \Gamma$. A collection of precontours $\{\Gamma_i\} \cap V \neq \emptyset$ if $\Gamma_i \cap V \neq \emptyset$ for each i.

A *contour* is a triple $\Omega = \big(\{\gamma_i\},\ \{\tau_j\},\ \eta\big)$, where

(i) either $\{\gamma_i\} \cap V \neq \emptyset$ is a compatible collection of precontours or $\{\Gamma_i\}$ is an empty set;

(ii) $\{\tau_j\} \subseteq V \setminus (\cup_i \partial^I \bar{\gamma}_i)$ is a collection of mutually disjoint finite vertical segments with $a(\tau_j),\ b(\tau_j) \in \cup_i(\partial^I \bar{\gamma}_i \cap V) \cup \partial^E V$;

(iii) η is a configuration in $\cup_i(\partial^I \bar{\gamma}_i \cap V)$;

(iv) either $\{\gamma_i\}$ is non empty and for every τ_j at least one of its ends ($a(\tau_j)$ or $b(\tau_j)$) belongs to $\cup_i(\partial^I \bar{\gamma}_i \cap V)$ or $\{\gamma_i\}$ is empty and $\{\tau_j\}$ consists of a single segment τ with $a(\tau),\ b(\tau) \in \partial^E V$;

(v) for every pair $\gamma_{i'}$ and $\gamma_{i''}$ there exists a sequence $\gamma_{i'} = \gamma_{i_1}, \tau_{j_1}, \ldots, \gamma_{i_s}, \tau_{j_s}, \gamma_{i_{s+1}} = \gamma_{i''}$ such that for any $1 \leq k \leq s$ either $a(\tau_{j_k}) \in \partial^I \bar{\gamma}_{i_k}$ and $b(\tau_{j_k}) \in \partial^I \bar{\gamma}_{i_{k+1}}$ or $b(\tau_{j_k}) \in \partial^I \bar{\gamma}_{i_k}$ and $a(\tau_{j_k}) \in \partial^I \bar{\gamma}_{i_{k+1}}$.

The contour clearly depends on V. In the special case when $V = \mathbb{Z}^2$ we obtain so called *free* contours.

Given a contour $\Omega = \big(\{\gamma_i\},\ \{\tau_j\},\ \eta\big)$, we set $\bar{\Omega}^\tau = \cup_j \tau_j$, $\bar{\Omega}^\gamma = \cup_i \bar{\gamma}_i$, $\bar{\Omega} = \bar{\Omega}^\tau \cup \bar{\Omega}^\gamma$, $\tilde{\Omega} = \bar{\Omega}^\tau \cup (\cup_i \partial^I \bar{\gamma}_i)$.

A collection $\{\Omega_l\}$ is *compatible* if for any Ω_{l_1} and Ω_{l_2} one has $\tilde{\Omega}_{l_1} \cap \tilde{\Omega}_{l_2} = \emptyset$ and the total collection $\{\gamma_i(\Omega_{l_1}),\ \gamma_i(\Omega_{l_2})\}$ is a compatible collection of precontours.

A contour Ω belongs to the volume V if the corresponding precontours $\gamma_i \subseteq V$ and $\bar{\Omega} \subseteq V$. A contour Ω has non empty intersection with the volume V if $\{\gamma_i\} \cap V \neq \emptyset$ and $\bar{\Omega}^\tau \subseteq V$.

3.2 Definition of statistical weight for contours

1. Since β is small exp $[\beta U(\eta(P))]-1$ is small. We denote this difference by $U(\beta, \eta(P))$ for any rectangle P and any configuration $\eta(P)$.

2. We partition the finite volume V into vertical segments V_n and denote the distance between $a(V_n)$ and $b(V_n)$ by $||V_n|| = |V_n| + 1$. The number of configurations in V with the boundary condition $\eta'(\widehat{V})$ can be calculated as

$$N(V|\eta'(\partial^E V)) = \prod_n N\left(V_n|\eta'_{a(V_n)}, \eta'_{b(V_n)}\right), \tag{A.3.5}$$

where $N\left(V_n|\eta'_{a(V_n)}, \eta'_{b(V_n)}\right)$ is the matrix entry of $A^{||V_n||}$ indexed by $\eta'_{a(V_n)}, \eta'_{b(V_n)}$. By Perron-Frobenius theorem both matrices A and its adjoint A^* have a unique maximal eigenvalue $\lambda > 1$ and the corresponding eigenvectors $\mathbf{e}$ and $\mathbf{e}^*$ with positive components e_η and e^*_η. We normalize $\mathbf{e}$ and $\mathbf{e}^*$ in such a way that $\sum_\eta e_\eta e^*_\eta = 1$. Using the Jordan normal form for matrix A, one can show that

$$N\left(V_n|\eta'_{a(V_n)},\eta'_{b(V_n)}\right) = e_{\eta'_{a(V_n)}} e^*_{\eta'_{b(V_n)}} \lambda^{||V_n||} \left(1 + F\left(V_n|\eta'_{a(V_n)},\eta'_{b(V_n)}\right)\right), \qquad (A.3.6)$$

where for some $0 < \rho(A) < 1$ and $\nu(A) > 0$

$$\left|F\left(V_n|\eta'_{a(V_n)},\eta'_{b(V_n)}\right)\right| \leq \nu(A)\rho(A)^{||V_n||}. \qquad (A.3.7)$$

We define

$$L(V) = \lambda^{-\sum_n ||V_n||},$$

$$E(\eta(\partial^E V)) = \left(\prod_n e_{\eta_{a(V_n)}}\right)^{-1} \left(\prod_n e^*_{\eta_{b(V_n)}}\right)^{-1},$$

$$E^*(\eta(\partial^E V)) = \left(\prod_n e^*_{\eta_{a(V_n)}}\right)^{-1} \left(\prod_n e_{\eta_{b(V_n)}}\right)^{-1}. \qquad (A.3.8)$$

Similarly, we define $E(\eta(\partial^I V))$ and $E^*(\eta(\partial^I V))$ by using the top and bottom elements of V_n instead of $a(V_n)$ and $b(V_n)$.

3. Given a precontour γ and a fixed configuration $\eta(\partial^I\bar{\gamma}\cap V)$, we define a *precontour partition function* by

$$\Xi\big(\gamma,\eta(\partial^I\bar{\gamma}\cap V)\big|\eta'(\widehat{V})\big) = L\big((\bar{\gamma}\setminus\partial^I\bar{\gamma})\cap V\big)E^*\big(\eta(\partial^I\bar{\gamma}\cap V)\big)^{-1}E\big(\eta'(\partial^E V\cap\bar{\gamma})\big)$$

$$\times \sum_{\eta\left((\bar{\gamma}\setminus\partial^I\bar{\gamma})\cap V\right)} \prod_{P\in\gamma} U\big(\beta,\eta(P\cap V)+\eta'(P\cap\widehat{V})\big) \prod_{P\prec\gamma}\Big(1+U\big(\beta,\eta(P\cap V)+\eta'(P\cap\widehat{V})\big)\Big). \qquad (A.3.10)$$

Set

$$\Xi^*(V|\eta'(\partial^E V)) = L(V)E(\eta'(\partial^E V))\sum_{\eta(V)}\prod_{P:\,P\subseteq V}\Big(1+U\big(\beta,\eta(P)\big)\Big).$$

The *statistical weight of precontour* is defined by

$$W\big(\gamma,\eta(\partial^I\bar{\gamma}\cap V)\big|\eta'(\widehat{V})\big) = \frac{\Xi\big(\gamma,\eta(\partial^I\bar{\gamma}\cap V)\big|\eta'(\widehat{V})\big)}{\Xi^*\big((\bar{\gamma}\cap V)\setminus\partial^I\bar{\gamma}\big|\eta(\partial^I\bar{\gamma}\cap V)+\eta'(\partial^E V\cap\bar{\gamma})\big)}. \qquad (A.3.11)$$

4. For any contour $\Omega = \big(\{\gamma_i\},\ \{\tau_j\},\ \eta\big)$, the *statistical weight* is

$$W(\Omega|\eta'(\widehat{V})) = \prod_i W\big(\gamma_i,\eta(\partial^I\bar{\gamma}_i\cap V)\big|\eta'(\widehat{V})\big)\prod_j F(\tau_j|\eta''_{a(\tau_j)},\eta''_{b(\tau_j)}), \qquad (A.3.29)$$

where $\eta'' = \eta'\big(\partial^E V \setminus \big(\cup_i \bar{\gamma}_i\big)\big) + \sum_i \eta\big(\partial^I \bar{\gamma}_i \cap V\big)$.

4. Polymer Expansion Theorem. (see [JM]) *Suppose that $U(\eta(P))$ is a potential which is defined on rectangles of size $n(P) \times Ln(P)$. Assume that U satisfies (A.3.3). Then there exists a constant $\beta_0 > 0$ such that for any $0 < \beta \leq \beta_0$, any finite volume V satisfying (A.3.4), and arbitrary boundary condition $\eta'(\widehat{V})$, the following equation holds:*

$$L(V)E(\eta'(\partial^E V))\Xi(V|\eta'(\widehat{V})) = \sum_{\{\Omega_j\}\cap V \neq \emptyset} \prod_j W(\Omega_j|\eta'(\widehat{V})), \tag{A.4.1}$$

where the partition function $\Xi(V|\eta'(\widehat{V}))$ on the left-hand side is defined by (A.2.1)–(A.2.4) with $U(\eta(P))$ replacing $U(\eta(Q))$ and the right-hand side is the abstract partition function over contours defined in the previous sections. Thus, the partition function has the polymer expansion

$$L(V)E(\eta'(\partial^E V))\Xi(V|\eta'(\widehat{V})) = \exp\big(\sum_{\wp\cap\Lambda\neq\emptyset} w(\wp)\big),$$

where the statistical weight $w(\wp)$ is defined in (A.1.5)

For a polymer $\wp = [\Omega_i^{\alpha_i}]$, $\bar{\wp} = \cup_i \bar{\Omega}_i$. This notation is used in (4.26). We note that the infinite sum on the right-hand side is convergent uniformly for all potentials satisfying (A.3.3) and $\beta \leq \beta_0$.

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