Canonical Quantization on a Doubly Connected Space and the Aharonov-Bohm Phase

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Abstract

We consider the canonical quantization (Schrödinger representation) on a doubly connected space $\Omega_R \equiv \mathbb{R}^2 \setminus \{(x\,,\,y)\,|\,x^2+y^2 \le R^2\}$ $(R>0)$, and find the Aharonov-Bohm phase in Weyl's canonical commutation relations.

1 Introduction

We would like to construct quantum mechanics on Riemann surfaces. This paper is a way in our attempt. There are, indeed, many insightful works, for instance, by Ruijsenaars [21], Helffer [12], Exner [6], and Albeverio, Breźniak and Dabrowski [3], et al. for the Schrödinger operator on non-Euclidean spaces. In this paper, we would like to study quantum mechanics in the light of the canonical quantization on a doubly connected space. We give proofs of the statements appearing in [13].

In physics, when we consider the canonical quantization on Euclidean space $\mathbb{R}^N$, we usually use the orthogonal coordinates $Ox_1x_2\cdots x_N$. It is well-known that we are faced with a difficulty when we construct quantum mechanics by using the polar coordinates. However, it is hardly possible that the construction of quantum mechanics depends on specific coordinates. How does the canonical quantization in quantum mechanics select representation of coordinates?

In the standard quantum mechanics in $\mathbb{R}^N$, the canonical momentum $p_j \equiv -i\partial/\partial x_j$ and position operators $q_j \equiv x_j\cdot$ $(j=1,2,\cdots)$ are defined to be self-adjoint operators, p_j and q_j, acting in a Hilbert space $\mathcal{H}$ and satisfying Heisenberg's canonical commutation relations (CCR):

$$\begin{cases} [p_j, q_{j'}] = -i\hbar\delta_{jj'} \\ [p_j, p_{j'}] = 0 = [q_j, q_{j'}]\,, \quad j, j' = 1, 2, \cdots, N;\ N \in \mathbb{N}. \end{cases} \tag{1.1}$$

on a dense subspace in $\mathcal{H}$, where $\hbar$ is the Planck constant divided by 2π, and δ_{ij} the Kronecker delta. Although Schmüdtgen studied classes, $\mathcal{K}$ and $\mathcal{C}$, of representation of the Heisenberg's CCR in [22, 23], we here employed the definition of Heisenberg's CCR in [1, 436.C].

It is well-known that the Weyl's CCR for strongly continuous one-parameter groups, $\{e^{isp_j}\}_{-\infty<s<\infty}$, $\{e^{itq_j}\}_{-\infty<t<\infty}$,

$$\begin{cases} e^{itq_j}e^{isp_k} = \exp\left[-ist\hbar\delta_{jk}\right]e^{isp_k}e^{itq_j}, \\ e^{itq_j}e^{isq_k} = e^{isq_k}e^{itq_j}, \quad e^{itp_j}e^{isp_k} = e^{isp_k}e^{itp_j} \end{cases} \tag{1.2}$$

($s, t \in \mathbb{R}$; $j, k = 1, \cdots, N$) determines $\{p_j, q_j\}$ uniquely up to unitary equivalence. The irreducible representation of p_j, q_j are unitary equivalent to the Schrödinger representation by von Neumann's uniqueness theorem (see [18, Theorem VIII.14]). Thus, these p_j, q_j satisfy Heisenberg's CCR. Conversely, those self-adjoint operators p_j, q_j satisfying Heisenberg's CCR lead to Weyl's.

Whether the two CCR's, Heisenberg's and Weyl's, are equivalent or not presents a problem when the underlying space $\mathbb{R}^N$ is replaced by a multiply connected one. While Weyl's CCR implies Heisenberg's (see [18, Corollary of Theorem VIII.14]), the converse is not necessarily true. Nelson gave a mathematical example in non-Euclidean space to show it is not true [18, Corollary and Nelson's example on p.275], Schmüdtgen gave such examples that they prove the two CCR's are not equivalent [23, §3]. More realistic examples were given by Reeh[20] and Arai[4] for the case of the Aharonov-Bohm effect with a string of magnetic field of zero radius. They set the orthogonal coordinates Ox_1x_2 in the two dimensional Euclidean space, and considered a particle moving on a plane with holes of the 0-radius. Though it must be remarked that what they called momenta were actually the mass-times-velocity operators, $P_j \equiv p_j - \mathsf{q}A_j$, which is called the mv-momentum (kinetic momentum) by Feynman (see [7, (21.14)]) where $p_j \equiv -i\hbar\partial/\partial x_j$ $(j = 1, 2;\ (x_1\,,\, x_2) \in \mathbb{R}^2)$; and q is a charge, these operators do satisfy Heisenberg's CCR but do not Weyl's if the magnetic flux going through the interior of every rectangular closed curves in $\mathbb{R}^2$ does not have a value of an integer multiple of $2\pi\hbar c/\mathsf{q}$.

Indeed P_j, q_j do not satisfy Weyl's CCR under the condition above, physicists are not surprised at this fact because they do not regard P_j as the canonical momenta. So in this paper, we show, whenever we employ 2-dimensional orthogonal coordinates Ox_1x_2 on $\Omega_R \equiv \mathbb{R}^2 \setminus D_R$, $D_R \equiv \{(x_1\,,\, x_2)\,|\, x_1^2 + x_2^2 \le R^2\}$ $(R > 0)$, the operators p_j, q_j do not satisfy Weyl's CCR even if we consider any self-adjoint extensions of p_j. Namely, we cannot construct the Schrödinger representation by employing Ox_1x_2 on Ω_R. Thus, we construct the Schrödinger representation, that is the canonical momentum and position operators satisfying Weyl's CCR on Ω_R, by employing another coordinates. And we shall extend Reeh's and Arai's results with $R = 0$ to the case of $R > 0$. Our method is to reduce the disc by a conformal mapping to a line segment, and invoke Arai's argument using the fact that the segment has Lebesgue measure zero as dose Arai's point hole.

In §2, we shall show that there are uncountably many different self-adjoint extensions of $-i\partial/\partial x_j$ $(j = 1, 2)$ with suitable boundary conditions on ∂D_R. However, in §3 it turns out that none of the self-adjoint extensions of $-i\partial/\partial x_j$

$(j = 1, 2)$ satisfy Weyl's CCR though any extensions satisfy Heisenberg's. In §4, therefore, we define momentum operators as generators of shifts along the stream lines of an incompressible vortex-free flow passing by the disc D_R. The position operators are defined in the standard way as multiplication by stream-line coordinates. The author would like to mention that Prof. Ezawa of Gakushuin University remarked to him that Tomonaga once tried such a construction using stream lines in physics to extract a collective mode of motion of a many-particle system [24].

To establish that the canonical pairs so defined have unique self-adjoint extensions satisfying Weyl's CCR, we use a conformal (Joukowski) transformation to reduce the disc D_R to line segment $[-2R\,,\,2R]$. Of course, the canonical pairs satisfy Heisenberg's CCR also.

In §5, in the same way as Arai's argument using the fact that the holes have Lebesgue measure zero, we shall introduce magnetic flux to show that Weyl's CCR for the mv-momentum with respect to the new coordinates is destroyed by the Aharonov-Bohm phase, while Heisenberg's remains valid. The canonical pairs are the same as those in §4, and both CCR's, Weyl's and Heisenberg's, are valid. We shall watch how the Joukowski transformation maps poles of the gauge potential. Then, the point we wish to make here is that the Aharonov-Bohm effect shows itself in the algebra of operators besides the well-known change in the interference pattern. As a conclusion of our assertion in this paper, we shall show in §5 that the Aharonov-Bohm phase appears in Weyl's CCR for just coordinates by the Joukowski transformation, not the orthogonal coordinates Ox_1x_2, which is caused by inequivalence between Weyl's CCR and Heisenberg's.

2 Self-Adjoint Extensions of p_j $(j = 1, 2)$ on Ω_R

We deal with the case of $\Omega_R \stackrel{\text{def}}{=} \mathbb{R}^2 \setminus D_R$, where $D_R \stackrel{\text{def}}{=} \{(x_1\,,\,x_2)\,|\,x_1^2 + x_2^2 \leq R^2\}$, for the fixed radius $R > 0$. We denote $\mathbb{R}^2 \setminus \{(0\,,\,0)\}$ by Ω_0. We set $\mathsf{m} = \hbar = c = 1$.

In this section, we shall find that there are uncountably many self-adjoint extensions of $-i\partial/\partial x_j$ $(j = 1, 2)$ on $L^2(\Omega_R)$ although $-i\partial/\partial x_j$ $(j = 1, 2)$ are essentially self-adjoint on $L^2(\Omega_0)$ (see Refs.[4, 20]).

For $t \in [-R\,,\,R]$, we define a function $w_\pm : [-R\,,\,R] \to \mathbb{R}$ by $w_\pm(t) \stackrel{\text{def}}{=} \pm\sqrt{R^2 - t^2}$. Then, we define x_j-section $S(x_j)$ $(j = 1, 2)$ of $\mathbb{R}$ for every $x_j \in \mathbb{R}$ by $S(x_j) \stackrel{\text{def}}{=} (-\infty\,,\,\infty)$ if $R < |x_j|$; $(-\infty\,,\,w_-(x_j)) \cup (w_+(x_j)\,,\,\infty)$ if $|x_j| \leq R$.

We define two sets $AC^1_{loc}(\Omega_R)$ and $AC^2_{loc}(\Omega_R)$ of functions on Ω_R by

$$AC^1_{loc}(\Omega_R) \stackrel{\text{def}}{=} \left\{ f \in L^2(\Omega_R) \,\middle|\, \begin{array}{l} \text{for almost all } x_2 \in \mathbb{R},\ f(\cdot\,,\,x_2) \text{ is absolutely continuous} \\ \text{on arbitrary closed interval } [c\,,\,c'] \text{ contained in } S(x_2) \text{ such that} \\ \dfrac{\partial f}{\partial x_1} \in L^2(\Omega_R) \end{array} \right\},$$

and $AC^2_{loc}(\Omega_R)$ is defined similarly by replacing $f(\cdot\,,\,x_2)$ by $f(x_1\,,\,\cdot)$.

Remark 2.1. Let f_1 be in $AC^1_{loc}(\Omega_R)$, and f_2 be in $AC^2_{loc}(\Omega_R)$. Then, we obtain $D_1 f_1 = \partial f_1/\partial x_1$, and $D_2 f_2 = \partial f_2/\partial x_2$, where D_j $(j = 1, 2)$ denotes derivative in the sense of distribution with test functions in $C_0^\infty(\Omega_R)$ which denotes the set of all $C^\infty(\Omega_R)$-functions with compact support in Ω_R.

Let $\widetilde{j} \equiv 2$ if $j = 1$; 1 if $j = 2$. We define two operators p_j $(j = 1, 2)$ by

$$
\begin{aligned}
p_j &\overset{\text{def}}{=} \frac{1}{i} D_j \quad \left(= \frac{1}{i}\frac{\partial}{\partial x_j} \right), \qquad (2.1)\\
D(p_j) &\overset{\text{def}}{=} \Big\{ f \in AC^j_{loc}(\Omega_R) \,\Big|\, \text{for almost all } |x_{\widetilde{j}}| \le R \\
&\qquad \lim_{x_j \to w_-(x_{\widetilde{j}})} f(x_1\,,\, x_2) = 0 = \lim_{x_j \to w_+(x_{\widetilde{j}})} f(x_1\,,\, x_2) \Big\}.
\end{aligned}
$$

Here, in general, we denote by $D(T)$ the domain of operators T.

Remark 2.2. $\quad C_0^\infty(\Omega_R) \subset D(p_j), \qquad j = 1, 2.$

The position operators q_j $(j = 1, 2)$ are realized as self-adjoint operators:

$$
\begin{aligned}
q_j &\overset{\text{def}}{=} x_j \quad (\text{the multiplication by } x_j),\\
D(q_j) &\overset{\text{def}}{=} \left\{ f \in L^2(\Omega_R) \,\middle|\, \int_{\Omega_R} dx_1 dx_2 |x_j f(x_1\,,\, x_2)|^2 < \infty \right\},
\end{aligned}
$$

(see Example 5.11 and its Remark 1 in [8] or Proposition 1 and the proof of Proposition 3 in §VIII.3 of [18]).

It is easy to prove the following proposition:

PROPOSITION 2.1. $\{p_j, q_j\}_{j=1,2}$ *satisfies Heisenberg's CCR on* $C_0^\infty(\Omega_R)$.

Although it is easy to prove the following lemma, we note fundamental facts concerning functions in $L^2(S(x_j))$ and and $AC^j_{loc}(\Omega_R)$ $(j = 1, 2)$:

LEMMA 2.2.

(a) *If* $f \in L^2(\Omega_R)$, *then* $f \in L^2(S(x_j))$ *for almost all* $x_j \in \mathbb{R}$.

(b) *If* $f \in AC^j_{loc}(\Omega_R)$ $(j = 1, 2)$, *then* $\lim_{x_j \to \pm\infty} f(x_1\,,\, x_2) = 0$ *for almost all* $x_{\widetilde{j}} \in \mathbb{R}$, *and* $\lim_{x_j \to w_\pm(x_{\widetilde{j}})} f(x_1\,,\, x_2)$ *exists for almost all* $|x_{\widetilde{j}}| \le R$.

Proof. It is easy to prove part (a) in a similar way to Fubini's theorem. The first statement of part (b) is proven by using Schwarz' inequality and the Riemann-Lebesgue theorem. We note here for every x, x' with $x' < x < w_-(x_2)$ or $w_+(x_2) < x' < x$

$$
f(x\,,\, x_2) - f(x'\,,\, x_2) = \int_{S(x_2)} dt \frac{\partial f}{\partial t}(t\,,\, x_2)\chi_{[x',x]}(t),
$$

where $\chi_{[x',x]}$ is the characteristic function of $[x'\,,\, x]$. Then, the last statement of part (b) follows from Schwarz' inequality, and Cauchy's theorem. In the same way as $j = 1$, we can prove part (b) for $j = 2$. Q.E.D.

By Lemma 2.2(b), from now on, we set

$$f(w_\pm(x_2)\,,\,x_2) \stackrel{\text{def}}{=} \lim_{x_1 \to w_\pm(x_2)} f(x_1\,,\,x_2)$$

for $f \in AC^1_{loc}(\Omega_R)$, and

$$f(x_1\,,\,w_\pm(x_1)) \stackrel{\text{def}}{=} \lim_{x_2 \to w_\pm(x_1)} f(x_1\,,\,x_2)$$

for $f \in AC^2_{loc}(\Omega_R)$.

There are many ways to extend p_j $(j = 1, 2)$ to self-adjoint operators. We shall show it in the following theorem for comprehending all self-adjoint extensions of p_j $(j = 1, 2)$.

THEOREM 2.3. *The operator p_j $(j = 1, 2)$ is closed symmetric, but not essentially self-adjoint. Moreover, p_j has uncountably many different self-adjoint extensions.*

Proof. Let $f_n \in D(p_1)$ and $f, g \in L^2(\Omega_R)$ satisfying $f_n \to f$ and $p_1 f_n \to g$ in $L^2(\Omega_R)$ as $n \to \infty$. Then, applying integration by parts to every test function in $C_0^\infty(\Omega_R)$ and $f_n \in D(p_1)$, furthermore taking $n \to \infty$, we have $D_1 f = ig \in L^2(\Omega_R)$, which implies that f is absolutely continuous on arbitrary closed interval contained in $S(x_2)$, and $\partial f/\partial x_1 = D_1 f = ig \in L^2(\Omega_R)$.

Let B^+ be $\{(x_1\,,\,x_2)\,|\,-R \le x_2 \le R,\ w_+(x_2) \le x_1\}$. Since $\|f_n - f\|_{L^2(\Omega_R)} \to 0$ as $n \to \infty$, we have

$$f_n \to f \qquad \text{in} \quad L^2(B^+) \tag{2.2}$$

as $n \to \infty$. In the same way, since $\|\partial f_n/\partial x_1 - ig\|_{L^2(\Omega_R)} \to 0$ as $n \to \infty$ and $\partial f/\partial x_1 = ig$, we have

$$\frac{\partial f_n}{\partial x_1} \to \frac{\partial f}{\partial x_1} \qquad \text{in} \quad L^2(B^+) \tag{2.3}$$

as $n \to \infty$. By applying integration by parts, (2.2), (2.3), and Lemma 2.2(b) to $f_n \in D(p_1)$ and f, we have $\displaystyle\int_{-R}^{R} dx_2 \lim_{x_1 \to w_+(x_2)} |f(x_1, x_2)|^2 = 0$. Namely, for almost all $x_2 \in [-R, R]$, $\displaystyle\lim_{x_1 \to w_+(x_2)} f(x_1, x_2) = 0$. Similarly, we have $\displaystyle\lim_{x_1 \to w_-(x_2)} f(x_1\,,\,x_2) = 0$. Hence it follows that $f \in D(p_1)$ and $p_1 f = g$, which implies that p_1 is closed. In the same way as the above, p_2 is closed.

Let b_1 be $-R$. Fix arbitrarily b_2 with $-R < b_2 < R$. We define a sequence b_n $(n \in \mathbb{N})$ by $b_n \stackrel{\text{def}}{=} (R + b_{n-1})/2$. So we have $b_1 = -R$, $b_2 = 0$, $\cdots$, $b_n = 2^{2-n}(2^{n-2} - 1)R$ $(n = 3, 4, \cdots)$. We set $B_n^+ \stackrel{\text{def}}{=} \left\{(x_1\,,\,x_2)\,|\,b_n \le x_2 < b_{n+1},\quad \sqrt{R^2 - x_2^2} \le x_1\right\}$ and $B_n^- \stackrel{\text{def}}{=} \left\{(x_1\,,\,x_2)\,|\,b_n \le x_2 < b_{n+1},\quad x_1 \le -\sqrt{R^2 - x_2^2}\right\}$. We define functions $f_n^\pm \in L^2(\Omega_R)$ $(n \in \mathbb{N})$ by $f_n^\pm(x_1\,,\,x_2) \stackrel{\text{def}}{=} \exp\left[\mp x_1\right] \chi_{B_n^\pm}(x_1\,,\,x_2)$, where $\chi_{B_n^\pm}(x_1\,,\,x_2)$ denotes the characteristic function of $B_n^\pm$. For every $f \in D(p_1)$, since $f(w_-(x_2)\,,\,x_2) = 0 = f(w_+(x_2)\,,\,x_2)$ and Lemma 2.2(b) holds, by using the integration by parts, we have $p_1^* f_n^\pm = \pm i f_n^\pm$ for each $n \in \mathbb{N}$. Therefore, p_1 is not essentially self-adjoint by [18, Corollary after Theorem VIII.3]. Besides, if $m \neq n$ then $f_m^\pm$ and $f_n^\pm$ intersect orthogonally. So, it follows that $\dim \mathrm{Ker}(p_1^* - i) = \aleph_0 = \dim \mathrm{Ker}(p_1^* + i)$, which implies our theorem by [5, Corollary on p.1230] or [19, Corollary on p.141]. Similarly, we can prove the result for p_2. Q.E.D.

By Theorem 2.3, we have known that there are uncountably many self-adjoint extensions of p_j $(j = 1, 2)$. In order for momentum operators to generate shifts, we wish there were a pair of self-adjoint extensions of p_j $(j = 1, 2)$ satisfying Weyl's CCR. As a matter of fact, we shall realize that there is no such pair in the self-adjoint extensions of p_j $(j = 1, 2)$. After here and in the next section, we shall see it.

The following corollary follows immediately from [19, Corollary on p.141], which is the first step for comprehending the domain of self-adjoint extensions of p_j $(j = 1, 2)$:

Corollary. *For each $j = 1, 2$, there is a one-one correspondence between self-adjoint extension of p_j and unitary operators from* $\mathrm{Ker}\left(p_j^* - i\right)$ *onto* $\mathrm{Ker}\left(p_j^* + i\right)$. *Let $U_j : \mathrm{Ker}\left(p_j^* - i\right) \to \mathrm{Ker}\left(p_j^* + i\right)$ $(j = 1, 2)$ be an arbitrary unitary operator, and p_{U_j} be the self-adjoint extension of p_j corresponding to U_j $(j = 1, 2)$, mentioned above. Then,*

$$D(p_{U_j}) = \left\{\varphi_0 + \varphi_+ + U_j\varphi_+ \,|\, \varphi_0 \in D(p_j), \varphi_+ \in \mathrm{Ker}(p_j^* - i)\right\},$$
$$p_{U_j}(\varphi_0 + \varphi_+ + U_j\varphi_+) = p_j\varphi_0 + i\varphi_+ - iU_j\varphi_+.$$

To obtain description of domains of self-adjoint extensions p_{U_j} $(j = 1, 2)$ which is appropriate for their boundary conditions on ∂D_R, we have exactly the form of adjoint operators of p_j as the following proposition. Its proof follows from [18, Example in VIII.2]:

Proposition 2.4. *The adjoint operators p_j^* of p_j $(j = 1, 2)$ are given as follows:*

$$p_j^* = -i\partial/\partial x_j \qquad \textit{with} \quad D(p_j^*) = AC_{loc}^j(\Omega_R).$$

Proof. We prove our assertion by extending the proof for 1 - dimensional case in [18, Example in VIII.2] to that for 2 - dimensional case in our proposition. Fix $\Psi \in D(p_1^*) \subset L^2(\Omega_R)$. Let j^2 be an arbitrary function in $C_0^\infty(\mathbb{R})$. And let j^1 be an arbitrary function in $C_0^\infty(\mathbb{R})$ such that $j^1 \geq 0$; $\mathrm{supp}\, j^1 \times \mathrm{supp}\, j^2 \subset \Omega_R$; and $\int_{-\infty}^{\infty} dx_1\, j^1(x_1) = 1$. Defining $j(x_1\,,\,x_2) \stackrel{\text{def}}{=} j^1(x_1)j^2(x_2)$, we have $j \in C_0^\infty(\Omega_R)$. For $\varepsilon > 0$, we define a function j_ε by $j_\varepsilon(x_1\,,\,x_2) \stackrel{\text{def}}{=} \varepsilon^{-1}j(x_1/\varepsilon\,,\,x_2)$. Let α and β be continuous functions of x_2 such that $[\alpha(x_2)\,,\,\beta(x_2)] \subset S(x_2)$ for all $x_2 \in \mathbb{R}$;

$\sup_{x_2\in\mathbb{R}}(\beta(x_2)-\alpha(x_2))<\gamma$ for some $\gamma>0$; and

$$\delta_{\alpha,\beta}\stackrel{\rm def}{=}\min\left\{\inf_{|x_2|\le R}|w_\pm(x_2)-\alpha(x_2)|,\ \inf_{|x_2|\le R}|w_\pm(x_2)-\beta(x_2)|\right\}>0.$$

We define two functions, $f_\varepsilon^{\alpha,\beta}$ and $g_\varepsilon^{\alpha,\beta}$ by

$$\begin{aligned} f_\varepsilon^{\alpha,\beta}(x_1\,,\,x_2) &\stackrel{\rm def}{=} j_\varepsilon(x_1-\beta(x_2)\,,\,x_2)-j_\varepsilon(x_1-\alpha(x_2)\,,\,x_2),\\ g_\varepsilon^{\alpha,\beta}(x_1\,,\,x_2) &\stackrel{\rm def}{=} \int_{-\infty}^{x_1}dt\,f_\varepsilon^{\alpha,\beta}(t\,,\,x_2), \end{aligned}$$

respectively. Let $v_{k,\pm}$ $(k=1,2)$ be non-zero numbers satisfying

$$v_{k,-}\le\inf\operatorname{supp}j^k<\sup\operatorname{supp}j^k\le v_{k,+}$$

for $k=1,2$. Then, $I_k\equiv\operatorname{supp}j^k\subset[v_{k,-}\,,\,v_{k,+}]$ $(k=1,2)$.

Fix an arbitrary ε' with $0<\varepsilon'<\delta_{\alpha,\beta}$. For all x_1 with $x_1<\alpha(x_2)-\varepsilon'$, and ε with $0<\varepsilon<\varepsilon'/|v_{1,-}|$, the inequality $(x_1-\beta(x_2))/\varepsilon\le(x_1-\alpha(x_2))/\varepsilon<v_{1,-}$ holds. For all x_1 with $x_1>\beta(x_2)+\varepsilon'$, and ε with $0<\varepsilon<\varepsilon'/|v_{1,+}|$, the inequality $(x_1-\alpha(x_2))/\varepsilon\ge(x_1-\beta(x_2))/\varepsilon>v_{1,+}$ holds. Hence it follows that $\operatorname{supp}g_\varepsilon^{\alpha,\beta}(\cdot\,,\,x_2)\subset[\alpha(x_2)-\varepsilon'\,,\,\beta(x_2)+\varepsilon']\subset S(x_2)$ for every $x_2\in\mathbb{R}$, ε' with $0<\varepsilon'<\delta_{\alpha,\beta}$, and ε with $0<\varepsilon\le\delta_1(\varepsilon')\equiv\min\{\varepsilon'/|v_{1,-}|\,,\,\varepsilon'/|v_{1,+}|\}$. It is clear that $\operatorname{supp}g_\varepsilon^{\alpha,\beta}(x_1\,,\,\cdot)\subset I_2$ for every $x_1\in\mathbb{R}$, so $\operatorname{supp}g_\varepsilon^{\alpha,\beta}\subset\{(x_1\,,\,x_2)\,|\,x_1\in[\alpha(x_2)-\varepsilon'\,,\,\beta(x_2)+\varepsilon']\,,x_2\in I_2\}\subset\Omega_R$ for $0<\varepsilon\le\delta_1(\varepsilon')$. Thus, we have

$$g_\varepsilon^{\alpha,\beta}\in C_0^\infty(\Omega_R)\qquad\text{for}\quad 0<\varepsilon\le\delta_1(\varepsilon'). \tag{2.4}$$

For instance, let ε be so small that $0<\varepsilon<\delta_{\alpha,\beta}/\max\{|v_{1,-}|\,,\,|v_{1,+}|\}$, and we set $\varepsilon'\equiv\max\{|v_{1,-}|\,,\,|v_{1,+}|\}\,\varepsilon$. Then, $0<\varepsilon'<\delta_{\alpha,\beta}$ and $0<\varepsilon<\delta_1(\varepsilon')$ hold. Thus, we take such ε and ε' from now on.

We note here that $j^2(x_2)=\displaystyle\int_{-\infty}^{v_{1,+}}ds\,j(s\,,\,x_2)$ for every $x_2\in\mathbb{R}$. Let $\chi_{\alpha\,,\,\beta}$ be a characteristic function of x_1 on $[\alpha(x_2)\,,\,\beta(x_2)]$, then we have

$$\begin{aligned} &\|-g_\varepsilon^{\alpha,\beta}-\chi_{\alpha\,,\,\beta}j^2\|_{L^2(\Omega_R)}^2\\ \le\ & 2\int_{I_2}dx_2\int_{\alpha(x_2)}^{\beta(x_2)}dx_1\left(\left|\int_{v_{1,+}}^{(x_1-\alpha(x_2))/\varepsilon}ds\,j^1(s)j^2(x_2)\right|^2+\left|\int_{-\infty}^{(x_1-\beta(x_2))/\varepsilon}ds\,j^1(s)j^2(x_2)\right|^2\right)\\ &+\int_{I_2}dx_2\left(\int_{\alpha(x_2)-\varepsilon'}^{\alpha(x_2)}dx_1+\int_{\beta(x_2)}^{\beta(x_2)+\varepsilon'}dx_1\right)|g_\varepsilon^{\alpha,\beta}(x_1\,,\,x_2)|^2, \end{aligned}$$

and besides, for $x_1\in[\alpha(x_2)\,,\,\beta(x_2)]$, we have

$$\begin{aligned} 0\ &<\ \left|\int_{v_{1,+}}^{(x_1-\alpha(x_2))/\varepsilon}ds\,j^1(s)j^2(x_2)\right|\quad\text{and}\quad\left|\int_{-\infty}^{(x_1-\beta(x_2))/\varepsilon}ds\,j^1(s)j^2(x_2)\right|\\ &\le\ \int_{-\infty}^{\infty}|j(s\,,\,x_2)|ds\le K_1\equiv\sup_{x_2\in I_2}|j^2(x_2)|<\infty \end{aligned}$$

by the assumptions about j^1. Moreover, we have $\int_{I_2} dx_2 \int_{\alpha(x_2)}^{\beta(x_2)} dx_1\, K_1$ $\leq \gamma(v_{2,+} - v_{2,-})K_1$ by the assumption $\sup_{x_2\in\mathbb{R}}(\beta(x_2) - \alpha(x_2)) < \gamma$, and

$$\lim_{\varepsilon\downarrow 0} \int_{v_{1,+}}^{(x_1-\alpha(x_2))/\varepsilon} ds j^1(s) = 0 = \lim_{\varepsilon\downarrow 0} \int_{-\infty}^{(x_1-\beta(x_2))/\varepsilon} ds j^1(s)$$

for $\alpha(x_2) < x_1 < \beta(x_2)$. By Lebesgue's dominated convergence theorem, we have $-g_\varepsilon^{\alpha,\beta} \longrightarrow \chi_{\alpha\,,\beta} j^2$ as $\varepsilon \downarrow 0$ in $L^2(\Omega_R)$. Since $\Psi \in D(p_1^*)$ by the first assumption, and $p_1 g_\varepsilon^{\alpha,\beta}$ is well-defined by (2.4), the equality $< p_1 g_\varepsilon^{\alpha,\beta}\,,\, \Psi >_{L^2(\Omega_R)} =$ $< g_\varepsilon^{\alpha,\beta}\,,\, p_1^*\Psi >_{L^2(\Omega_R)}$ holds for $0 < \varepsilon \leq \delta_1(\varepsilon')$. Thus, we have

$$\lim_{\varepsilon\downarrow 0} < p_1 g_\varepsilon^{\alpha,\beta}\,,\, \Psi >_{L^2(\Omega_R)} = -\int_{I_2} dx_2 j^2(x_2) \int_{\alpha(x_2)}^{\beta(x_2)} dx_1 \left(p_1^*\Psi\right)(x_1\,,\, x_2). \tag{2.5}$$

For every $\varphi \in C_0^\infty(\Omega_R)$, we define an operator J_ε by

$$\left(J_\varepsilon\varphi\right)(x_1\,,\, x_2) \stackrel{\text{def}}{=} \frac{1}{\varepsilon}\int_{S(x_2)} dt\, j^1\left(\frac{t-x_1}{\varepsilon}\right)\varphi(t\,,\, x_2),$$

and set $\varphi_\varepsilon^J(x_1\,,\, x_2) \stackrel{\text{def}}{=} \left(J_\varepsilon\varphi\right)(x_1\,,\, x_2)$. Then, we can calculate φ_ε^J as

$$\varphi_\varepsilon^J(x_1\,,\, x_2) = \begin{cases} \displaystyle\int_{I_1} ds\, j^1(s)\varphi(x_1+\varepsilon s\,,\, x_2) & \text{if } |x_2| > R, \\ \\ \displaystyle\int_{-\infty}^{(w_-(x_2)-x_1)/\varepsilon} ds\, j^1(s)\varphi(x_1+\varepsilon s\,,\, x_2) & \\ \displaystyle\quad + \int_{(w_+(x_2)-x_1)/\varepsilon}^{\infty} ds\, j^1(s)\varphi(x_1+\varepsilon s\,,\, x_2) & \text{if } |x_2| \leq R. \end{cases} \tag{2.6}$$

So, we obtain

$$|\varphi_\varepsilon^J(x_1\,,\, x_2)| \leq \sup_{(x\,,\,y)\in\operatorname{supp}\varphi} |\varphi(x\,,\, y)| \tag{2.7}$$

by (2.6).

Let $\varepsilon'' > 0$. For ε with $0 < \varepsilon < \delta_1(\varepsilon'') \equiv \min\{\varepsilon''/|v_{1,-}|\,,\, \varepsilon''/|v_{1,+}|\}$, and every x_2 with $|x_2| \leq R$, if $x_1 < w_-(x_2) - \varepsilon''$, then $\dfrac{w_+(x_2)-x_1}{\varepsilon} \geq \dfrac{w_-(x_2)-x_1}{\varepsilon} > v_{1,+}$, and if $x_1 > w_+(x_2) + \varepsilon''$, then $\dfrac{w_-(x_2)-x_1}{\varepsilon} \leq \dfrac{w_+(x_2)-x_1}{\varepsilon} < v_{1,-}$. Since we assumed $(\operatorname{supp} j^1 \times \operatorname{supp} j^2) \subset \Omega_R$ and Ω_R is open, by making ε'' less than the distance $dist(D_R\,,\, \operatorname{supp}\varphi)$ between D_R and $\operatorname{supp}\varphi$, we have

$$|\varphi_\varepsilon^J(x_1\,,\, x_2) - \varphi(x_1\,,\, x_2)| \leq \sup_{(h,x_1,x_2)\in\mathcal{S}} |\varphi(x_1+h\,,\, x_2) - \varphi(x_1\,,\, x_2)| \longrightarrow 0$$
$$\text{for all } x_1 \text{ with } \; x_1 < w_-(x_2) - \varepsilon'' \; \text{ or } \; x_1 > w_+(x_2) + \varepsilon'' \tag{2.8}$$

as $\varepsilon \downarrow 0$, where $v_1 \equiv \max\{|v_{1,-}|\,,\, |v_{1,+}|\}$ and $\mathcal{S} \equiv \{(h, x_1, x_2) \,|\, |h| \leq \varepsilon v_1;\, (x_1\,,\, x_2),$ $(x_1+h\,,\, x_2) \in \operatorname{supp}\varphi\}$ because φ is uniformly continuous, $\operatorname{supp}\varphi$ being compact and φ continuous.

For $f, \varphi \in C_0^\infty(\Omega_R)$, $j^1 \geq 0$ and (2.6), we have

$$\begin{aligned}
& |\langle f\,,\, J_\varepsilon\varphi\rangle_{L^2(\Omega_R)}| \\
\leq\ & \int_{I_1} ds j^1(s) \int\int_{\Omega_R} dx_1 dx_2 |\overline{f(x_1\,,\,x_2)}\varphi(x_1+\varepsilon s\,,\,x_2)| \\
\leq\ & \int_{I_1} ds j^1(s) \left(\int\int_{\Omega_R} dx_1 dx_2 |f(x_1\,,\,x_2)|^2\right)^{1/2} \\
& \qquad \left(\int\int_{\Omega_R} dx_1 dx_2 |\varphi(x_1+\varepsilon s\,,\,x_2)|^2\right)^{1/2}. \qquad (2.9)
\end{aligned}$$

It is clear that

$$\int_{-R}^{R} dx_2 \int_{w_+(x_2)+\varepsilon s}^{\infty} dx_1 |\varphi(x_1\,,\,x_2)|^2 \leq \int_{-R}^{R} dx_2 \int_{w_+(x_2)}^{\infty} dx_1 |\varphi(x_1\,,\,x_2)|^2,$$

and, since $\operatorname{supp}\varphi \subset \Omega_R$,

$$\int_{-R}^{R} dx_2 \int_{-\infty}^{w_-(x_2)+\varepsilon s} dx_1 |\varphi(x_1\,,\,x_2)|^2 = \int_{-R}^{R} dx_2 \int_{-\infty}^{w_-(x_2)} dx_1 |\varphi(x_1\,,\,x_2)|^2.$$

So, by the change of variable, $x_1 + \varepsilon s \to x_1'$, we have

$$\begin{aligned}
& \int\int_{\Omega_R} dx_1 dx_2 |\varphi(x_1+\varepsilon s, x_2)|^2 \\
=\ & \left(\int_{|x_2|>R} dx_2 \int_{-\infty}^{\infty} dx_1' + \int_{-R}^{R} dx_2 \int_{-\infty}^{w_-(x_2)+\varepsilon s} dx_1' + \int_{-R}^{R} dx_2 \int_{w_+(x_2)+\varepsilon s}^{\infty} dx_1'\right) |\varphi(x_1'\,,\,x_2)|^2 \\
\leq\ & \left(\int_{|x_2|>R} dx_2 \int_{-\infty}^{\infty} dx_1' + \int_{-R}^{R} dx_2 \int_{-\infty}^{w_-(x_2)} dx_1' + \int_{-R}^{R} dx_2 \int_{w_+(x_2)}^{\infty} dx_1'\right) |\varphi(x_1'\,,\,x_2)|^2 \\
=\ & \int\int_{\Omega_R} dx_1 dx_2 |\varphi(x_1\,,\,x_2)|^2. \qquad (2.10)
\end{aligned}$$

By (2.9) and (2.10), we have $| < f, J_\varepsilon\varphi >_{L^2(\Omega_R)} | \leq \|f\|_{L^2(\Omega_R)} \|\varphi\|_{L^2(\Omega_R)}$. Thus, since $C_0^\infty(\Omega_R)$ is dense in $L^2(\Omega_R)$, J_ε is a bounded operator on $L^2(\Omega_R)$ with $\|J_\varepsilon\| \leq 1$.

By (2.6), $|\varphi_\varepsilon^J(x_1\,,\,x_2)| \leq \sup_{(x,y)\in \operatorname{supp}\varphi} |\varphi(x\,,\,y)|$, and (2.8), we can estimate

$\|J_\varepsilon\varphi-\varphi\|_{L^2(\Omega_R)}$ as

$$\begin{aligned}
&\|J_\varepsilon\varphi-\varphi\|^2_{L^2(\Omega_R)}\\
=\;& \int_{|x_2|>R} dx_2 \int_{-\infty}^{\infty} dx_1 |\varphi^J_\varepsilon(x_1\,,\,x_2)-\varphi(x_1\,,\,x_2)|^2\\
&+\int_{-R}^{R} dx_2 \Bigg(\int_{-\infty}^{w_-(x_2)-\varepsilon''} dx_1 + \int_{w_-(x_2)-\varepsilon''}^{w_-(x_2)} dx_1\\
&\qquad\qquad + \int_{w_+(x_2)}^{w_+(x_2)+\varepsilon''} dx_1 + \int_{w_+(x_2)+\varepsilon''}^{\infty} dx_1\Bigg) |\varphi^J_\varepsilon(x_1\,,\,x_2)-\varphi(x_1\,,\,x_2)|^2\\
\le\;& \int\int_{\mathrm{supp}\,\varphi^J_\varepsilon \cup \mathrm{supp}\varepsilon} \left(\int_{I_1} ds\, j^1(s)|\varphi(x_1+\varepsilon s\,,\,x_2)-\varphi(x_1\,,\,x_2)\right)^2\\
&+\int_{-R}^{R} dx_2 \left(\int_{-\infty}^{w_-(x_2)-\varepsilon''} dx_1 + \int_{w_+(x_2)+\varepsilon''}^{+\infty} dx_1\right) |\varphi^J_\varepsilon(x_1\,,\,x_2)-\varepsilon(x_1\,,\,x_2)|^2\\
&+\int_{-R}^{R} dx_2 \left(\int_{w_-(x_2)-\varepsilon''}^{w_-(x_2)} dx_1 + \int_{w_+(x_2)}^{w_+(x_2)+\varepsilon''} dx_1\right)\left(2\sup_{(x,y)\in\mathrm{supp}\,\varphi}|\varphi(x\,,\,y)|\right)^2\\
\le\;& \int\int_{\mathrm{supp}\,\varphi^J_\varepsilon \cup \mathrm{supp}\,\varepsilon} dx_1dx_2 \left(\int_{I_1} ds\, j^1(s)|\varphi(x_1+\varepsilon s\,,\,x_2)-\varphi(x_1\,,\,x_2)|\right)^2\\
&+2m_L^{(2)}\left(\mathrm{supp}\,\varphi^J_\varepsilon\cup\mathrm{supp}\,\varphi\right)\left(\sup_{|h|\le\varepsilon v_1;(x,y),(x+h,y)\in\mathrm{supp}\varphi}|\varphi(x+\varepsilon s\,,\,y)-\varphi(x\,,\,y)|\right)^2\\
&+16R\varepsilon''\left(\sup_{(x,y)\in\mathrm{supp}\,\varphi}|\varphi(x\,,\,y)|\right)^2
\end{aligned}$$

for every ε'' and ε with $0<\varepsilon''<dist(D_R\,,\,\mathrm{supp}\,\varphi)$ and $0<\varepsilon<\delta_1(\varepsilon'')$, where $m_L^{(2)}$ denote the Lebesgue measure on $\mathbb{R}^2$. So, by Lebesgue dominated convergence theorem, taking $\varepsilon\downarrow 0$ first, and next $\varepsilon''\downarrow 0$, we have $\lim_{\varepsilon\downarrow 0} J_\varepsilon\varphi=\varphi$ in $L^2(\Omega_R)$, i.e., $\mathrm{l.i.m.}_{\varepsilon\downarrow 0}J_\varepsilon\varphi=\varphi$, which can be extended to $\varphi\in L^2(\Omega_R)$ because $C_0^\infty(\Omega_R)$ is dense in $L^2(\Omega_R)$. Thus, for $\varphi\in L^2(\Omega_R)$ there exists a sequence $\{\varepsilon_\nu\}_{\nu\in\mathbb{N}}$ such that

$$\lim_{\varepsilon\downarrow 0}\left(J_{\varepsilon_\nu}\varphi\right)(x_1\,,\,x_2)=\varphi(x_1\,,\,x_2)\qquad\text{for almost all } (x_1\,,\,x_2)\in\Omega_R. \tag{2.11}$$

By (2.4), for sufficiently large $\nu\in\mathbb{N}$,

$$\begin{aligned}
< p_1 g^{\alpha,\beta}_{\varepsilon_\nu}\,,\,\Psi >_{L^2(\Omega_R)} \;=\;& -\frac{1}{i}\int_{I_2} dx_2\overline{j^2(x_2)}\left(J_{\varepsilon_\nu}\Psi\right)(\beta(x_2)\,,\,x_2)\\
&+\frac{1}{i}\int_{I_2} dx_2\overline{j^2(x_2)}\left(J_{\varepsilon_\nu}\Psi\right)(\alpha(x_2)\,,\,x_2).
\end{aligned}\tag{2.12}$$

By applying Lebesgue's dominated convergence theorem to (2.7), (2.11) and (2.12), we have

$$\begin{aligned}
&\lim_{\nu\to\infty}\langle p_1 g^{\alpha,\beta}_{\varepsilon_\nu}\,,\,\Psi\rangle_{L^2(\Omega_R)}\\
=\;&\frac{1}{i}\int_{-\infty}^{\infty} dx_2 j^2(x_2)\left\{\Psi(\alpha(x_2)\,,\,x_2)-\Psi(\beta(x_2)\,,\,x_2)\right\}.
\end{aligned}\tag{2.13}$$

By (2.5) and (2.13), we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty} dx_2 j^2(x_2)\left\{\frac{1}{i}\left(\Psi\left(\alpha\left(x_2\right)\,,\,x_2\right)-\Psi\left(\beta\left(x_2\right)\,,\,x_2\right)\right)+\int_{\alpha(x_2)}^{\beta(x_2)} dx_1\left(p_1^*\Psi\right)\left(x_1\,,\,x_2\right)\right\}\\ &= 0. \end{aligned} \tag{2.14}$$

Here, set

$$\Xi(y) \stackrel{\text{def}}{=} \overline{\frac{1}{i}\left(\Psi\left(\alpha\left(y\right)\,,\,y\right)-\Psi\left(\beta\left(y\right)\,,\,y\right)\right)+\int_{\alpha(y)}^{\beta(y)} dx\left(p_1^*\Psi\right)\left(x\,,\,y\right)}, \qquad y\in\mathbb{R}.$$

Remember here j^2 is an arbitrary function in $C_0^\infty(\mathbb{R})$. So, we can define a bounded linear operator T from $C_0^\infty(\mathbb{R})$ to $\mathbb{C}$ by $Tj^2 \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} dy\,\overline{\Xi(y)}j^2(y)$, i.e., $T\in\mathbf{B}(C_0^\infty(\mathbb{R}),\mathbb{C})$, where $\mathbf{B}(H,\mathbb{C})$ denotes a set of all bounded linear operators from a dense subspace H of a Hilbert space to $\mathbb{C}$, and $\|\cdot\|_{\mathbf{B}(H,\mathbb{C})}$ denotes an operator norm for $\mathbf{B}(H,\mathbb{C})$. Thus, we have a unique bounded operator $\widetilde{T}$ from $L^2(\mathbb{R})$ to $\mathbb{C}$ with $\|T\|_{\mathbf{B}(C_0^\infty(\mathbb{R}),\mathbb{C})}=0=\|\widetilde{T}\|_{\mathbf{B}(L^2(\mathbb{R}),\mathbb{C})}$ by (2.14), which implies that

$$\begin{aligned}&\int_{-\infty}^{\infty} dx_2 f(x_2)\left\{\frac{1}{i}\left(\Psi\left(\alpha\left(x_2\right)\,,\,x_2\right)-\Psi\left(\beta\left(x_2\right)\,,\,x_2\right)\right)+\int_{\alpha(x_2)}^{\beta(x_2)} dx_1\left(p_1^*\Psi\right)\left(x_1\,,\,x_2\right)\right\}\\ &= \widetilde{T}f=0 \end{aligned} \tag{2.15}$$

for every $f\in L^2(\mathbb{R})$.

Since $\Psi, p_1^*\Psi\in L^2(\Omega_R)$, α and β are continuous, Ξ is a measurable function on $\mathbb{R}$. Remember here we selected j^2 in $C_0^\infty(\mathbb{R})$ arbitrary. So, if necessary, we can modify Ξ such that $|\Xi(x_2)|<\infty$ for almost all $x_2\in\mathbb{R}$ by (2.14).

For each $n\in\mathbb{N}$, $\mathbb{R}_n$ denotes $\mathbb{R}_n \stackrel{\text{def}}{=} \{y\,|\,|y|<n, |\Xi(y)|<n\}$, which is a measurable subset of $\mathbb{R}$. We define a function Ξ_n by $\Xi_n(y) \stackrel{\text{def}}{=} \chi_{\mathbb{R}_n}(y)\Xi(y)$ $(y\in\mathbb{R})$, where $\chi_{\mathbb{R}_n}$ is the characteristic function of $\mathbb{R}_n$. It is clear that $\Xi_n\in L^2(\mathbb{R})$. We define a linear functional T_n on $L^2(\mathbb{R})$ by $T_n f \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} dy\,\overline{\Xi_n(y)}f(y)$ for every $f\in L^2(\mathbb{R})$. By Riesz' theorem, we have $\|\Xi_n\|_{L^2(\mathbb{R})}=\|T_n\|_{L^2(\mathbb{R})^*}$, where $L^2(\mathbb{R})^* \stackrel{\text{def}}{=} \mathbf{B}(L^2(\mathbb{R}),\mathbb{C})$. Fix $f\in L^2(\mathbb{R})$ arbitrarily. By (2.15), $|\overline{\Xi}f|$ is integrable. So, we have

$$\sup_{n\in\mathbb{N}}|T_n(f)|\le M_f \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} dy\,|\overline{\Xi(y)}f(y)|<\infty.$$

Thus, by principle of uniform boundedness (see [18, Theorem III.9]), there exists $M\ge 0$ such that $\|T_n\|_{L^2(\mathbb{R})^*}=\|\Xi_n\|_{L^2(\mathbb{R})}\le M$ for all $n\in\mathbb{N}$. It is clear that $|\Xi_n(x_2)|^2$ $(n\in\mathbb{N})$ is a non-negative and non-decreasing sequence of functions, and $\Xi_n(x_2)\longrightarrow\Xi(x_2)$. Thus, by Levi's convergence theorem and $\|T_n\|_{L^2(\mathbb{R})^*}\le M$, we have $\int_{-\infty}^{\infty} dx_2|\Xi(x_2)|^2=\lim_{n\to\infty}\int_{-\infty}^{\infty} dx_2|\Xi_n(x_2)|^2\le M^2$, which means that $\Xi\in$

$L^2(\mathbb{R})$. By (2.14) and $\Xi \in L^2(\mathbb{R})$, we have

$$\frac{1}{i}\left\{\Psi\left(\beta\left(y\right)\,,\,y\right)-\Psi\left(\alpha\left(y\right)\,,\,y\right)\right\}=\int_{\alpha(y)}^{\beta(y)} dx_1\,\left(p_1^*\Psi\right)\left(x_1\,,\,y\right) \tag{2.16}$$

for almost all $y \in \mathbb{R}$.

Fix $y_0, c, x_1 = c' \in \mathbb{R}$ such that $[c\,,\,c'] \subset S(y_0)$. We can select functions α and β of y such that $[\alpha(y)\,,\,\beta(y)] \subset S(y)$ for all $y \in \mathbb{R}$; $\sup_{y\in\mathbb{R}}\left(\beta(y)-\alpha(y)\right) < \gamma$ for some $\gamma > 0$; $\delta_{\alpha,\beta} > 0$; and $\alpha(y_0) = c$, $\beta(y_0) = c' = x_1$. Then, by (2.16) $\Psi(\cdot\,,\,y_0)$ is absolutely continuous on $[c\,,\,c']$ and $L^2(\Omega_R) \ni p_1^*\Psi = -i\partial\Psi/\partial x_1$. Hence it follows that $D(p_1^*) \subset AC^1_{loc}(\Omega_R)$. It is clear that $D(p_1^*) \supset AC^1_{loc}(\Omega_R)$. Therefore, we obtain part (a). Similarly, we have part (b). Q.E.D.

We prepare some lemmas for a while in order to investigate boundary conditions on ∂D_R for functions in $D(p_{U_j})$ $(j = 1, 2)$.

We define $WS_j^{\pm}(\Omega_R)$ for $j = 1, 2$, the vector space of weak solutions for $D_j f = \pm f$, by

$$WS_j^{\pm}(\Omega_R) \stackrel{\text{def}}{=} \{f \in L^2(\Omega_R) \,|\, D_j f = \pm f\}.$$

It is evident that the following lemma holds.

LEMMA 2.5. *If $\varphi \in \mathrm{Ker}(p_j^* \pm i)$ for $j = 1, 2$, then $\overline{\varphi} \in WS_j^{\pm}(\Omega_R)$.*

Since Ω_R is open, for every $(x_1\,,\,x_2) \in \Omega_R$, there exists $\delta_{1,2} > 0$ such that $Ball\left((x_1\,,\,x_2)\,,\,\delta_{1,2}\right) \subset \Omega_R$, where $Ball\left((x_1\,,\,x_2)\,,\,\delta_{1,2}\right)$ denotes the open ball with center $(x_1\,,\,x_2)$ and radius $\delta_{1,2}$. And $\overline{Ball\left((x_1\,,\,x_2)\,,\,\delta_{1,2}/2\right)} \subset Ball\left((x_1\,,\,x_2)\,,\,\delta_{1,2}\right)$. Here, for a set S, we denote the closure of S by $\overline{S}$. There exists an open perfect square $J(x_1\,,\,x_2)$ with center $(x_1\,,\,x_2)$ such that $\overline{J(x_1\,,\,x_2)} \subset \overline{Ball\left((x_1\,,\,x_2)\,,\,\delta_{1,2}/2\right)}$. So, we have

$$\overline{J(x_1\,,\,x_2)} \subsetneq \Omega_R \tag{2.17}$$

$$\Omega_R = \bigcup_{(x_1\,,x_2)\in\Omega_R} J(x_1\,,\,x_2). \tag{2.18}$$

We denote by $\mathbf{J}$ the set $\{J(x_1\,,\,x_2) \,|\, (x_1\,,\,x_2) \in \Omega_R\}$.

Let $\rho_\varepsilon *$ $(\varepsilon > 0)$ be the Friedrichs mollifier. And we set $\varphi_\varepsilon \equiv \rho_\varepsilon * \varphi$ for $\varphi \in WS_j^{\pm}(\Omega_R)$ $(j = 1, 2)$. To be exact, let φ^{ext} be a function which is defined by φ on Ω_R, and 0 on $\mathbb{R}^2 \setminus \Omega_R$. We define φ_ε by

$$\varphi_\varepsilon(x_1\,,\,x_2) \stackrel{\text{def}}{=} \int\int_{\mathbb{R}^2} dx_1'dx_2'\,\rho_\varepsilon(x_1 - x_1'\,,\,x_2 - x_2')\varphi^{\mathrm{ext}}(x_1'\,,\,x_2').$$

The following fact can be proved easily.

LEMMA 2.6.

(a) *$\varphi_\varepsilon \to \varphi$ as $\varepsilon \downarrow 0$ in $L^2(\Omega_R)$.*

(b) $\varphi_\varepsilon \in C^\infty(\Omega_R)$.

(c) *For every* $J = J_1 \times J_2 \in \mathbf{J}$ *and* $\varphi^\pm \in WS_j^\pm(\Omega_R)$ $(j = 1, 2)$, *there exists* $g^j_{J,\varepsilon} \in C^\infty(J_j)$ *such that* $\varphi_\varepsilon^\pm(x_1\,,\,x_2) = \exp[\pm x_j]\, g^j_{J,\varepsilon}(x_{\widetilde{j}})$ *for* $(x_1\,,\,x_2) \in J$. *Here* $g^j_{J,\varepsilon}$ *may be the zero-valued function or not.*

Proof. The parts (a) and (b) are well-known facts. We here prove the part (c) only. Let $\varphi^\pm \in WS_j^\pm(\Omega_R)$. Then, it follows immediately that $\varphi^\pm \in L^2(\Omega_R)$ from the definition of $WS_j^\pm(\Omega_R)$. Since $\varphi^\pm \in L^2(\Omega_R)$, and part (b) holds, we have $\varphi_\varepsilon^\pm \in C^\infty(\Omega_R) \cap L^2(\Omega_R)$. Since $\dfrac{\partial \rho_\varepsilon}{\partial x_j}(x_1 - x_1'\,,\,x_2 - x_2') = -\dfrac{\partial \rho_\varepsilon}{\partial x_j'}(x_1 - x_1'\,,\,x_2 - x_2')$, we have

$$\begin{aligned} &\frac{\partial \varphi_\varepsilon^\pm}{\partial x_j}(x_1\,,\,x_2) \\ &= -\int\!\!\int_{\mathbb{R}^2} dx_1 dx_2 \left(\frac{\partial \rho_\varepsilon}{\partial x_j'}(x_1 - x_1'\,,\,x_2 - x_2')\right) \varphi^{\pm\,\mathrm{ext}}(x_1'\,,\,x_2') \end{aligned} \tag{2.19}$$

for every $(x_1\,,\,x_2) \in J$. Define $\rho_{\varepsilon,(x_1\,,x_2)}(x_1'\,,\,x_2') \stackrel{\mathrm{def}}{=} \rho_\varepsilon(x_1 - x_1'\,,\,x_2 - x_2')$ for $(x_1\,,\,x_2) \in J$ and $(x_1'\,,\,x_2') \in \mathbb{R}^2$. We note that $|x_1 - x_1'| \le \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2} \le \varepsilon$ implies $x_1 - \varepsilon \le x_1' \le x_1 + \varepsilon$. Similarly, $x_2 - \varepsilon \le x_2' \le x_2 + \varepsilon$. Thus, the support of $\rho_{\varepsilon,(x_1\,,x_2)}$ is contained inside the ε-neighborhood of J. So $\mathrm{supp}\,\rho_{\varepsilon,(x_1\,,x_2)} \subset \Omega_R$ for sufficiently small $\varepsilon > 0$ by (2.17). Hence it follows that $\rho_{\varepsilon,(x_1\,,x_2)} \in C_0^\infty(\Omega_R)$. And, immediately, it follows that

$$\frac{\partial \varphi_\varepsilon^\pm}{\partial x_j}(x_1\,,\,x_2) = \pm \int\!\!\int_{\mathbb{R}^2} dx_1 dx_2\, \rho_{\varepsilon,(x_1\,,x_2)}(x_1'\,,\,x_2') \left(\varphi^\pm\right)^{\mathrm{ext}}(x_1'\,,\,x_2')$$

from (2.19), $\varphi^\pm \in WS_j(\Omega_R)$ and $\rho_{\varepsilon,(x_1\,,x_2)} \in C_0^\infty(\Omega_R)$. Namely, we have $\dfrac{\partial \varphi_\varepsilon^\pm}{\partial x_j}(x_1\,,\,x_2) = \pm\varphi_\varepsilon^\pm(x_1\,,\,x_2)$ on J. So, $\varphi_\varepsilon^\pm \in C^\infty(\Omega_R) \cap L^2(\Omega_R)$ and $\dfrac{\partial \varphi_\varepsilon^\pm}{\partial x_j} = \pm\varphi_\varepsilon^\pm$ on Ω_R by (2.18) lead us to the part (c). Q.E.D.

Let $\{e_n\}_{n\in\mathbb{N}}$ be a complete orthonormal basis of $L^2((-R, R))$. We define functions $f_n^{j,\pm}$ $(j = 1, 2;\ n \in \mathbb{N})$ on $\overline{\Omega_R}$ by

$$f_n^{j,\pm}(x_1\,,\,x_2) \stackrel{\mathrm{def}}{=} \sqrt{2} e^{\mp x_j} \chi_{S^\pm(x_{\widetilde{j}})}(x_j) e^{\sqrt{R^2 - x_{\widetilde{j}}^2}} e_n(x_{\widetilde{j}}), \tag{2.20}$$

where $\widetilde{j} = 2$ if $j = 1$; $\widetilde{j} = 1$ if $j = 2$; and $\chi_{S^+(x_{\widetilde{j}})}(x_j) \stackrel{\mathrm{def}}{=} 1$ if $|x_{\widetilde{j}}| \le R$ and $w_+(x_{\widetilde{j}}) \le x_j$; 0 otherwise, and $\chi_{S^-(x_{\widetilde{j}})}(x_j) \stackrel{\mathrm{def}}{=} 1$ if $|x_{\widetilde{j}}| \le R$ and $x_j \le w_-(x_{\widetilde{j}})$; 0 otherwise.

Then we get the following lemma:

LEMMA 2.7.

(a) $\{f_n^{j,+}\}_{n\in\mathbb{N}}$ *(resp.* $\{f_n^{j,-}\}_{n\in\mathbb{N}}$*) is a complete orthonormal basis of* $\mathrm{Ker}(p_j^* - i)$ *(resp.* $\mathrm{Ker}(p_j^* + i)$*) for* $j = 1, 2$.

(b) *For every* $f_+, g_+ \in \mathrm{Ker}(p_j^* - i)$,

$$\begin{aligned}
&\sum_{m=1}^{\infty} \overline{\langle f_m^{j,+}\,,\, g_+\rangle_{L^2(\Omega_R)}}\langle f_m^{j,+}\,,\, f_+\rangle_{L^2(\Omega_R)} \\
= &\sum_{m=1}^{\infty} \overline{\langle f_m^{j,-}\,,\, U_j g_+\rangle_{L^2(\Omega_R)}}\langle f_m^{j,-}\,,\, U_j f_+\rangle_{L^2(\Omega_R)}
\end{aligned}$$

for $j = 1, 2$.

(c) *For every* $\varphi \in D(p_{U_1})$ *(resp.* $D(p_{U_2})$*),*

$$\begin{aligned}
&\varphi(w_+(x_2)\,,\, x_2) = \varphi_+(w_+(x_2)\,,\, x_2) \\
= \;&\gamma_{U_1}(\varphi_+; x_2)\,(U_1\varphi_+)\,(w_-(x_2)\,,\, x_2), = \gamma_{U_1}(\varphi_+; x_2)\varphi(w_-(x_2)\,,\, x_2),
\end{aligned}$$

$$\begin{aligned}
(\textit{resp.} \qquad &\varphi(x_1\,,\, w_+(x_1)) = \varphi_+(x_1\,,\, w_+(x_1)) \\
= \;&\gamma_{U_2}(\varphi_+; x_1)\,(U_2\varphi)\,(x_1\,,\, w_-(x_1)) = \gamma_{U_2}(\varphi_+; x_1)\varphi(x_1\,,\, w_-(x_1)), \qquad)
\end{aligned}$$

where

$$\gamma_{U_j}(\varphi_+; x_{\tilde{j}}) = \frac{\displaystyle\sum_{m=1}^{\infty}\langle f_m^{j,+}\,,\, \varphi_+\rangle_{L^2(\Omega_R)} e_m(x_{\tilde{j}})}{\displaystyle\sum_{n=1}^{\infty}\langle f_n^{j,-}\,,\, U_j\varphi_+\rangle_{L^2(\Omega_R)} e_n(x_{\tilde{j}})}.$$

Furthermore,

$$\begin{aligned}
&\int_{-R}^{R} dx_2\, |\varphi(w_+(x_2)\,,\, x_2)|^2 = \int_{-R}^{R} dx_2\, |\varphi(w_-(x_2)\,,\, x_2)|^2 < \infty. \\
(\textit{resp.} \quad &\int_{-R}^{R} dx_1\, |\varphi(x_1\,,\, w_+(x_1))|^2 = \int_{-R}^{R} dx_1\, |\varphi(x_1\,,\, w_-(x_1))|^2 < \infty.)
\end{aligned}$$

Proof. Since

$$\begin{aligned}
\int\int_{\Omega_R} dx_1 dx_2\, \overline{f_n^{j,+}(x_1\,,\, x_2)} f_m^{j,+}(x_1\,,\, x_2) &= 2\int_{-R}^{R} dx_{\tilde{j}}\, e^{2\sqrt{R^2 - x_{\tilde{j}}^2}}\overline{e_n(x_{\tilde{j}})} e_m(x_{\tilde{j}}) \int_{w_+(x_{\tilde{j}})}^{\infty} dx_j\, e^{-2x_j} \\
&= \int_{-R}^{R} dx_{\tilde{j}} \overline{e_n(x_{\tilde{j}})} e_m(x_{\tilde{j}}) = \delta_{mn},
\end{aligned}$$

$\{f_n^{j,+}\}_{n\in\mathbb{N}}$ is an orthonormal system of $L^2(\Omega_R)$ for each $j = 1, 2$. It is clear that each $f_n^{j,+}$ is in $\mathrm{Ker}(p_j^* - i)$ by Proposition 2.4 and the facts that $f_n^{j,+} \in AC_{loc}^j(\Omega_R)$.

Let $\varphi_+ \in \mathrm{Ker}(p_j^* - i)$. By Lemmas 2.5 and 2.6(c), we can know that φ_+ is constructed by connection of 0 and e^{-x_j} along lines parallel to the x_j-axis. However, we cannot always do it everywhere on the line parallel to the x_j-axis. For instance, if we construct φ_+ by connecting 0 and e^{-x_j} at $x_j = a$, we have $D_j\varphi_+ = -\varphi_+ + \delta_a$, where δ_a denotes the delta function with mass at $x_j = a$, so $\varphi_+ \notin \mathrm{Ker}(p_j^* - i)$. Thus, there exists function g_{φ_+} of $x_{\tilde{j}} \in (-R\,,\, R)$ such that $\varphi_+(x_1\,,\,x_2) = e^{-x_j}\chi_{S^+(x_{\tilde{j}})}(x_j)g_{\varphi_+}(x_{\tilde{j}})$ and we can select g_{φ_+} with $\mathrm{supp}\, g_{\varphi_+} \subset [-R\,,\,R]$. We can apply the same consideration as we had above for φ_+ to the case for $U_1\varphi_+$. So, let $\varphi \in D(p_{U_j})$, then we obtain the boundary condition for $j=1$ as

$$\varphi(w_+(x_2)\,,\,x_2) = \varphi_+(w_+(x_2)\,,\,x_2), \tag{2.21}$$

and for $j=2$

$$\varphi(x_1\,,\,w_+(x_1)) = \varphi_+(x_1\,,\,w_+(x_1)), \tag{2.22}$$

where φ_+ means the function in $\mathrm{Ker}(p_j^* - i)$ in our corollary.

Since $\varphi_+ \in L^2(\Omega_R)$, we have

$$\infty > \int\int_{\Omega_R} dx_1 dx_2\,|\varphi_+(x_1\,,\,x_2)|^2 = \frac{1}{2}\int_{-R}^{R} dx_{\tilde{j}}\,|g_{\varphi_+}(x_{\tilde{j}})|^2 e^{-2\sqrt{R^2-x_{\tilde{j}}^2}}.$$

Namely, we have $g_{\varphi_+}(\cdot)e^{-\sqrt{R^2-(\cdot)^2}} \in L^2((-R,R))$. So,

$$\begin{aligned} g_{\varphi_+}(x_{\tilde{j}})e^{-\sqrt{R^2-x_{\tilde{j}}^2}} &= \sum_{n=1}^{\infty} G_n e_n(x_{\tilde{j}}), \\ G_n &= \int_{-R}^{R} dx_{\tilde{j}}\,\overline{e_n(x_{\tilde{j}})}g_{\varphi_+}(x_{\tilde{j}})e^{-\sqrt{R^2-x_{\tilde{j}}^2}}. \end{aligned}$$

Here we can calculate

$$\begin{aligned} &\sum_{n=1}^{\infty}\left(\int\int_{\Omega_R} dx_1' dx_2'\,\overline{f_n^{j,+}(x_1'\,,\,x_2')}\varphi_+(x_1'\,,\,x_2')\right) f_n^{j,+}(x_1\,,\,x_2) \\ = &\left(\sum_{n=1}^{\infty} G_n e_n(x_{\tilde{j}})\right) e^{-x_j}\chi_{S^+(x_{\tilde{j}})}(x_j)e^{\sqrt{R^2-x_{\tilde{j}}^2}} \\ = &\ \varphi_+(x_1\,,\,x_2). \end{aligned}$$

Thus, $\{f_n^{j,+}\}_{n\in\mathbb{N}}$ is a complete orthonormal system of $\mathrm{Ker}(p_j^* - i)$. Similarly, $\{f_n^{j,-}\}$ is a complete orthonormal system of $\mathrm{Ker}(p_j^* + i)$.

We define $\alpha_n^j(\varphi_+)$ for each $j=1,2$ by $\alpha_n^j(\varphi_+) \stackrel{\mathrm{def}}{=} \langle f_n^{j,+}\,,\,\varphi_+\rangle_{L^2(\Omega_R)}$ for $\varphi_+ \in \mathrm{Ker}(p_j^* - i)$. Then, we have

$$\begin{aligned} \varphi_+(x_1\,,\,x_2) &= \sum_{m=1}^{\infty}\alpha_m^j(\varphi_+)f_m^{j,+}(x_1\,,\,x_2) && (2.23) \\ &= \sqrt{2}e^{-x_j}\chi_{S^+(x_{\tilde{j}})}(x_j)e^{\sqrt{R^2-x_{\tilde{j}}^2}}\sum_{m=1}^{\infty}\alpha_m^j(\varphi_+)e_m(x_{\tilde{j}}). && (2.24) \end{aligned}$$

Defining $u^j_{m,n} \stackrel{\rm def}{=} \langle f^-_n\,,\, U_j f^{j,+}_m\rangle_{L^2(\Omega_R)}$, we have

$$U_j f^{j,+}_m = \sum_{n=1}^{\infty} u^j_{m,n} f^{j,-}_n, \tag{2.25}$$

$$\left(U_j\varphi_+\right)(x_1\,,\,x_2) = \sum_{\mu=1}^{\infty}\left(\sum_{m=1}^{\infty}\alpha^j_\mu\left(\varphi_+\right)u^j_{\mu,m}f^{j,-}_m\left(x_1\,,\,x_2\right)\right) \tag{2.26}$$

$$= \sqrt{2}e^{x_j}\chi_{S^-(x_{\tilde{j}})}(x_j)e^{\sqrt{R^2-x^2_{\tilde{j}}}}\sum_{\mu=1}^{\infty}\left(\sum_{m=1}^{\infty}\alpha^j_\mu\left(\varphi_+\right)u^j_{\mu,m}e_m\left(x_{\tilde{j}}\right)\right). \tag{2.27}$$

We obtain that

$$\begin{aligned}
&\sum_{m=1}^{\infty}\left(\overline{\langle f^{j,-}_m\,,\,U_j g_+\rangle_{L^2(\Omega_R)}}\langle f^{j,-}_m\,,\,U_j f_+\rangle_{L^2(\Omega_R)} - \overline{\langle f^{j,+}_m\,,\,g_+\rangle_{L^2(\Omega_R)}}\langle f^{j,+}_m\,,\,f_+\rangle_{L^2(\Omega_R)}\right)\\
=&\sum_{m=1}^{\infty}\left(\overline{\langle\langle f^{j,-}_m\,,\,U_j f_+\rangle_{L^2(\Omega_R)}f^{j,-}_m\,,\,U_j g_+\rangle_{L^2(\Omega_R)}} - \overline{\langle\langle f^{j,+}_m\,,\,f_+\rangle_{L^2(\Omega_R)}f^{+}_m\,,\,g_+\rangle_{L^2(\Omega_R)}}\right)\\
=&\overline{\langle U_j f_+\,,\,U_j g_+\rangle_{L^2(\Omega_R)}} - \overline{\langle f_+\,,\,g_+\rangle_{L^2(\Omega_R)}} = 0,
\end{aligned}$$

which implies part (b).

Let $\varphi\in D(p_{U_1})$. By (2.24) and (2.27), we have

$$\varphi(w_+(x_2)\,,\,x_2) = \sqrt{2}\sum_{m=1}^{\infty}\alpha^1_m(\varphi_+)e_m(x_2), \tag{2.28}$$

$$\varphi(w_-(x_2)\,,\,x_2) = \sqrt{2}\sum_{\mu=1}^{\infty}\left(\sum_{m=1}^{\infty}\alpha^1_\mu\left(\varphi_+\right)u^1_{\mu,m}e_m\left(x_2\right)\right). \tag{2.29}$$

he first statement of part (c) follows from (2.21), (2.22), (2.28) and (2.29). .

Since $\sum_{m=1}^{\infty}|\alpha^1_m(\varphi_+)|^2 = \|\varphi_+\|^2_{L^2(\Omega_R)}$ (i.e., $\{\alpha^1_m(\varphi_+)\}_{m\in\mathbb{N}}$ is a square summable sequence), we have $\psi_+ \equiv \sum_{m=1}^{\infty}\alpha^1_m(\varphi_+)e_m \in L^2((-R\,,\,R))$. So, by (2.28) we obtain

$$\varphi(w_+(\cdot)\,,\,\cdot) = \sqrt{2}\psi_+ \in L^2((-R\,,\,R))$$

and

$$\begin{aligned}
\int_{-R}^{R}dx_2\,|\varphi(w_+(x_2)\,,\,x_2)|^2 &= 2\int_{-R}^{R}dx_2\,|\psi(x_2)|^2\\
&= 2\|\varphi_+\|^2_{L^2(\Omega_R)}.
\end{aligned} \tag{2.30}$$

Since $\left\{f^{1,+}_\mu\right\}_{\mu\in\mathbb{N}}$ and $\left\{U_1^{-1}f^{1,-}_m\right\}_{m\in\mathbb{N}}$ are complete orthonormal systems of

$L^2(\Omega_R)$, we have

$$\begin{aligned}\sum_{m=1}^{\infty}\Big|\sum_{\mu=1}^{\infty}\alpha_{\mu}^{1}(\varphi_+)u_{\mu,m}^{1}\Big|^2 &= \sum_{m=1}^{\infty}\Big|\langle\sum_{\mu=1}^{\infty}\langle f_{\mu}^{1,+}\,,\,U_1^{-1}f_m^{1,-}\rangle_{L^2(\Omega_R)}f_{\mu}^{1,+}\,,\,\varphi_+\rangle_{L^2(\Omega_R)}\Big|^2 \\ &= \sum_{m=1}^{\infty}\Big|\langle U_1^{-1}f_m^{1,-}\,,\,\varphi_+\rangle_{L^2(\Omega_R)}\Big|^2 \\ &= \|\varphi_+\|_{L^2(\Omega_R)}^2 < \infty. \qquad (2.31)\end{aligned}$$

Namely, $\left\{\sum_{\mu=1}^{\infty}\alpha_{\mu}^{1}(\varphi_+)u_{\mu,m}^{1}\right\}_{m\in\mathbb{N}} = \left\{\langle f_m^{1,-}\,,\,U_1\varphi_+\rangle_{L^2(\Omega_R)}\right\}_{m\in\mathbb{N}}$ is a square summable sequence. So, we have

$$\begin{aligned}\psi_- &\equiv \sum_{m=1}^{\infty}\left(\sum_{\mu=1}^{\infty}\alpha_{\mu}^{1}(\varphi_+)u_{\mu,m}^{1}\right)e_m \\ &= \sum_{m=1}^{\infty}\langle f_m^{1,-}\,,\,U_1\varphi_+\rangle_{L^2(\Omega_R)}e_m \qquad \in L^2((-R\,,\,R)) \qquad (2.32)\end{aligned}$$

and

$$\begin{aligned}\psi_N &\equiv \sum_{\mu=1}^{N}\left(\sum_{m=1}^{N}\alpha_{\mu}^{1}(\varphi_+)u_{\mu,m}^{1}\right)e_m \\ &= \sum_{m=1}^{N}\left(\sum_{\mu=1}^{N}\alpha_{\mu}^{1}(\varphi_+)u_{\mu,m}^{1}\right)e_m \\ &\longrightarrow \psi_- \quad \text{in } L^2((-R\,,\,R))\end{aligned}$$

as $N\to\infty$. Thus, there exists a subsequence $\{\psi_{N_k}\}_{k\in\mathbb{N}}$ of $\{\psi_N\}_{N\in\mathbb{N}}$ such that $\psi_{N_k}\longrightarrow\psi_-$ (a.e.). By (2.29) and (2.31), we have

$$\varphi(w_-(\cdot)\,,\,\cdot) = \sqrt{2}\psi_- \in L^2((-R\,,\,R)) \qquad (2.33)$$

and

$$\begin{aligned}\int_{-R}^{R}dx_2\,|\varphi(w_-(x_2)\,,\,x_2)|^2 &= 2\int_{-R}^{R}dx_2\,|\psi_-(x_2)|^2 \\ &= 2\|\varphi_+\|_{L^2(\Omega_R)}^2. \qquad (2.34)\end{aligned}$$

Thus, we have the last statement of part (c) by (2.30) and (2.34).

In the same way, we have part (c) in the case when $\varphi\in D(p_{U_2})$. Q.E.D.

Remark 2.4. We here note that γ_{U_j} $(j=1,2)$ is depend on φ_+ and $x_{\tilde{j}}$, which is different from the 1-dimensional case (see Example 1 in X.1 in [19]). Then, why does p_{U_j} keep symmetry? The reason is as follows: for instance, let $f=$

$f_0 + f_+ + U_1 f_+,\ g = g_0 + g_+ + U_1 g_+ \in D(p_{U_1})$. Then we have

$$
\begin{aligned}
& \langle g\,,\,p_{U_x} f\rangle_{L^2(\Omega_R)} - \langle p_{U_1} g\,,\,f\rangle_{L^2(\Omega_R)} \\
= \ & \frac{1}{i}\int_{-R}^{R} dx_2 \left\{\overline{g(w_-(x_2)\,,\,x_2)} f(w_-(x_2)\,,\,x_2) - \overline{g(w_+(x_2)\,,\,x_2)} f(w_+(x_2)\,,\,x_2)\right\} \\
& \text{by Lemma 2.2(b), Proposition 2.4 with } p_{U_1} \subset p_1^* \\
= \ & \frac{2}{i}\int_{-R}^{R} dx_2 \left\{\sum_{m=1}^{\infty}\left(\sum_{\mu=1}^{\infty}\overline{\alpha_\mu^1(g_+)u_{\mu,m}^1}\right)\overline{e_m(x_2)}\sum_{n=1}^{\infty}\left(\sum_{\nu=1}^{\infty}\alpha_\nu^1(f_+)u_{\nu,n}^1\right)e_n(x_2)\right. \\
& \qquad\qquad \left. - \sum_{m=1}^{\infty}\overline{\alpha_m^1(g_+)e_m(x_2)}\sum_{n=1}^{\infty}\alpha_n^1(f_+)e_n(x_2)\right\} \\
& \text{by (2.28), (2.29) and (2.33)} \\
= \ & \frac{2}{i}\int_{-R}^{R} dx_2 \left\{\sum_{m=1}^{\infty}\overline{\langle U_1^{-1} f_m^{1,-}\,,\,g_+\rangle_{L^2(\Omega_R)} e_m(x_2)}\sum_{n=1}^{\infty}\langle U_1^{-1} f_n^{1,-}\,,\,f_+\rangle_{L^2(\Omega_R)} e_n(x_2)\right. \\
& \qquad\qquad \left. - \sum_{m=1}^{\infty}\overline{\alpha_m^1(g_+)e_m(x_2)}\sum_{n=1}^{\infty}\alpha_n^1(f_+)e_n(x_2)\right\} \\
= \ & \frac{2}{i}\left\{\sum_{m=1}^{\infty}\overline{\langle f_m^{1,-}\,,\,U_1 g_+\rangle_{L^2(\Omega_R)}}\sum_{n=1}^{\infty}\langle f_n^{1,-}\,,\,U_1 f_+\rangle_{L^2(\Omega_R)}\delta_{mn}\right. \\
& \qquad \left. - \sum_{m=1}^{\infty}\overline{\langle f_m^{1,+}\,,\,g_+\rangle_{L^2(\Omega_R)}}\sum_{n=1}^{\infty}\langle f_n^{1,+}\,,\,f_+\rangle_{L^2(\Omega_R)}\delta_{mn}\right\} \\
& \text{by (2.32) and the continuity of the inner product} \\
= \ & \frac{2}{i}\sum_{m=1}^{\infty}\left\{\overline{\langle f_m^{1,-}\,,\,U_1 g_+\rangle_{L^2(\Omega_R)}}\langle f_m^{1,-}\,,\,U_1 f_+\rangle_{L^2(\Omega_R)} - \overline{\langle f_m^{1,+}\,,\,g_+\rangle_{L^2(\Omega_R)}}\langle f_m^{1,+}\,,\,f_+\rangle_{L^2(\Omega_R)}\right\} \\
= \ & 0 \qquad \text{by Lemma 2.7(b).}
\end{aligned}
$$

Now that we have Proposition 2.4 and Lemmas 2.5 - 2.7, we can characterize the domains of p_{U_j} $(j = 1, 2)$ with the suitable boundary conditions on ∂D_R, respectively:

THEOREM 2.8.
(a) $\quad p_{U_1} = \dfrac{1}{i}\dfrac{\partial}{\partial x_1}$ *with*

$$\begin{aligned} D(p_{U_1}) &= \Bigg\{ f \in AC^1_{loc}(\Omega_R) \,\Bigg|\, \int_{-R}^{R} dx_2 |f(w_\pm(x_2)\,,\, x_2)|^2 < \infty, \\ &\qquad \textit{and there exists } f_{\rm pls} \in \mathrm{Ker}(p_1^* - i) \textit{ such that} \\ &\qquad f(w_+(x_2)\,,\, x_2) = f_{\rm pls}(w_+(x_2)\,,\, x_2), \\ &\qquad f(w_+(x_2)\,,\, x_2) = \gamma_{U_1}(f_{\rm pls}\,;\, x_2) f(w_-(x_2)\,,\, x_2) \\ &\qquad \textit{for almost all } -R < x_2 < R \Bigg\}. \end{aligned}$$

(b) $\quad p_{U_2} = \dfrac{1}{i}\dfrac{\partial}{\partial x_2}$ *with*

$$\begin{aligned} D(p_{U_2}) &= \Bigg\{ f \in AC^2_{loc}(\Omega_R) \,\Bigg|\, \int_{-R}^{R} dx_1 |f(x_1\,,\, w_\pm(x_1))|^2 < \infty, \\ &\qquad \textit{and there exists } f_{\rm pls} \in \mathrm{Ker}(p_2^* - i) \textit{ such that} \\ &\qquad f(x_1\,,\, w_+(x_1)) = f_{\rm pls}(x_1\,,\, w_+(x_1)), \\ &\qquad f(x_1\,,\, w_+(x_1)) = \gamma_{U_2}(f_{\rm pls}\,;\, x_1) f(x_1\,,\, w_-(x_1)) \\ &\qquad \textit{for almost all } -R < x_1 < R \Bigg\}. \end{aligned}$$

Proof (cf. Example1 in X.1 in [19]). Here we prove part (a) only.
Fix U_1. We define an operator T_{U_1} by

$$\begin{aligned} T_{U_1} &\overset{\rm def}{=} \frac{1}{i}\frac{\partial}{\partial x_1}, \\ D(T_{U_1}) &\overset{\rm def}{=} \Bigg\{ f \in AC^1_{loc}(\Omega_R) \,\Bigg|\, \int_{-R}^{R} dx_2 |f(w_\pm(x_2)\,,\, x_2)|^2 < \infty, \\ &\qquad \text{and there exists } f_{\rm pls} \in \mathrm{Ker}(p_1^* - i) \text{ such that} \\ &\qquad f(w_+(x_2)\,,\, x_2) = f_{\rm pls}(w_+(x_2)\,,\, x_2), \\ &\qquad f(w_+(x_2)\,,\, x_2) = \gamma_{U_1}(f_{\rm pls}\,;\, x_2) f(w_-(x_2)\,,\, x_2) \\ &\qquad \text{for almost all } -R < x_2 < R \Bigg\}. \end{aligned}$$

Here we have introduced a notation $f_{\rm pls}$. We will show importance of $f_{\rm pls}$ in the next section 3.

First, we show that $D(p_{U_1}) \subset D(T_{U_1})$. Let $\varphi = \varphi_0 + \varphi_+ + U_1\varphi_+ \in D(p_{U_1})$. By Lemma 2.7(c), we have $\int_{-R}^{R} dx_2 |\varphi(w_\pm(x_2)\,,\, x_2)|^2 < \infty$. Taking $\varphi_{\rm pls} \equiv \varphi_+$, it follows from Lemma 2.7(c) again that $\varphi(w_+(x_2)\,,\, x_2) = \varphi_{\rm pls}(w_+(x_2)\,,\, x_2)$, and

$\varphi(w_+(x_2)\,,\,x_2) = \gamma_{U_1}(\varphi_{\rm pls}\,;\,x_2)\varphi(w_-(x_2)\,,\,x_2)$ for $-R < x_2 \le R$. Thus, $\varphi \in D(T_{U_1})$, which implies that $D(p_{U_1}) \subset D(T_{U_1})$. Namely, $p_{U_1} \subset T_{U_1} \subset p_1^*$ by Proposition 2.4.

Next, we show that T_{U_1} is symmetric. Since $D(p_{U_1}) \subset D(T_{U_1})$, $D(T_{U_1})$ is dense in $L^2(\Omega_R)$. We note here that $f(w_+(x_2)\,,\,x_2) = f_{\rm pls}(w_+(x_2)\,,\,x_2)$ for $f \in D(T_{U_1})$ and $-R < x_2 < R$. So, for $f, g \in D(T_{U_1})$, in the same way as Remark 2.4, we have $\langle g\,,\,T_{U_1}f\rangle_{L^2(\Omega_R)} - \langle T_{U_1}g\,,\,f\rangle_{L^2(\Omega_R)} = 0$. Thus, T_{U_1} is a symmetric operator, namely, $T_{U_1} \subset T_{U_1}^*$.

Therefore, we have $p_{U_1} \subset T_{U_1} \subset p_1^*$ and $T_{U_1} \subset T_{U_1}^*$, which implies $p_{U_1} \subset T_{U_1} \subset T_{U_1}^* \subset p_{U_1}^* = p_{U_1}$, that is $p_{U_1} = T_{U_1}$. Similarly, we obtain part (b). Q.E.D.

Now that we obtain exactly the boundary conditions on ∂D_R depending on the domain of p_{U_j} $(j = 1, 2)$, we can comprehend the behavior of $\exp\left[itp_{U_j}\right]$, which implies that no pair of $\exp\left[itp_{U_1}\right]$ and $\exp\left[itp_{U_2}\right]$ satisfies Weyl's CCR. It will be shown in the next section.

3 Inequivalence between Heisenberg's CCR and Weyl's

In this section, we investigate behavior of $\exp\left[itp_{U_j}\right]$ $(j = 1, 2)$. Here arises difficulties. In fact, we shall find it turns out in this section that any pair of self-adjoint extensions of $-i\partial/\partial x_j$ $(j = 1, 2)$ on $L^2(\Omega_R)$ does not satisfy Weyl's CCR. This is not a case with $L^2(\Omega_0)$, which was studied by Reeh[20] and Arai[4].

Let $\chi_{B^+}(x_1\,,\,x_2) \stackrel{\rm def}{=} \chi_{S^+(x_2)}(x_1) = 1$ if $|x_2| \le R$ and $w_+(x_2) \le x_1$; 0 otherwise, and $\chi_{B^-}(x_1\,,\,x_2) \stackrel{\rm def}{=} \chi_{S^-(x_2)}(x_1)1$ if $|x_2| \le R$ and $x \le w_-(x_2)$; 0 otherwise. For $f \in AC^1_{loc}(\Omega_R)$ with $\displaystyle\int_{-R}^R dx_2|f(w_\pm(x_2)\,,\,x_2)|^2 < \infty$, we can give explicit construction of $f_{\rm pls}$ and $f_{\rm mns}$ by

$$\begin{aligned} f_{\rm pls}(x_1\,,\,x_2) &\stackrel{\rm def}{=} e^{-x_1}f(w_+(x_2)\,,\,x_2)e^{w_+(x_2)}\chi_{B^+}(x_1\,,\,x_2),\\ f_{\rm mns}(x_1\,,\,x_2) &\stackrel{\rm def}{=} e^{x_1}f(w_-(x_2)\,,\,x_2)e^{-w_-(x_2)}\chi_{B^-}(x_1\,,\,x_2). \end{aligned}$$

Then, we have $\displaystyle\int\!\!\int_{\Omega_R} dx_1dx_2|f_{\rm pls}(x_1\,,\,x_2)|^2 = \frac{1}{2}\int_{-R}^R dx_2|f(w_+(x_2)\,,\,x_2)|^2 < \infty$, So, $f_{\rm pls} \in L^2(\Omega_R)$. It is clear that $f_{\rm pls} \in {\rm Ker}(p_1^* - i)$ by Proposition 2.4. Similarly, $f_{\rm mns} \in {\rm Ker}(p_1^* + i)$. In the same way, we define for $f \in AC^2_{loc}(\Omega_R)$ with $\displaystyle\int_{-R}^R dx_2|f(x_1\,,\,w_\pm(x_1))|^2 < \infty$,

$$\begin{aligned} f_{\rm pls}(x_1\,,\,x_2) &\stackrel{\rm def}{=} e^{-x_2}f(x_1\,,\,w_+(x_1))e^{w_+(x_1)}\chi_{A^+}(x_1\,,\,x_2),\\ f_{\rm mns}(x_1\,,\,x_2) &\stackrel{\rm def}{=} e^{x_2}f(x_1\,,\,w_-(x_1))e^{-w_-(x_1)}\chi_{A^-}(x_1\,,\,x_2), \end{aligned}$$

where $\chi_{_{A^+}}(x_1\,,\,x_2) \stackrel{\rm def}{=} \chi_{_{S^+(x_1)}}(x_2) = 1$ if $|x_1| < R$ and $w_+(x_1) \leq x_2$; 0 otherwise, and $\chi_{_{A^-}}(x_1\,,\,x_2) \stackrel{\rm def}{=} \chi_{_{S^-(x_1)}}(x_2) = 1$ if $|x_1| < R$ and $x_2 \leq w_-(x_1)$; 0 otherwise.

Remember Lemma 2.7(c). Then, for $f \in D(p_{_{U_j}})$ $(j = 1,2)$, we define f_0 by

$$f_0(x_1\,,\,x_2) \stackrel{\rm def}{=} f(x_1\,,\,x_2) - f_{\rm pls}(x_1\,,\,x_2) - f_{\rm mns}(x_1\,,\,x_2).$$

Then, it is follows from Proposition 2.4 that $f_0 \in D(p_j)$.

We will clarify the meaning of the functions $f_{\rm pls}$ and $f_{\rm mns}$, which gives an important decomposition.

LEMMA 3.1. *Fix U_j $(j = 1,2)$. Then, $U_j f_{\rm pls} = f_{\rm mns}$ for every $f \in D(p_{_{U_j}})$, and*

$$f = f_0 + f_{\rm pls} + U_j f_{\rm pls} \quad \in D(p_j) + {\rm Ker}(p_j^* - i) + {\rm Ker}(p_j^* + i)$$

is a unique decomposition.

Proof. Let $(g_1\,,\,g_2)_{p_j} \stackrel{\rm def}{=} \langle g_1\,,\,g_2\rangle_{L^2(\Omega_R)} + \langle p_j^* g_1\,,\,p_j^* g_2\rangle_{L^2(\Omega_R)}$ for every $g_1, g_2 \in D(p_j^*)$. For $f \in D(p_{_{U_j}})$, we have a decomposition $f = f_0 + f_{\rm pls} + f_{\rm mns}$, where $f_0 \in D(p_j)$, $f_{\rm pls} \in {\rm Ker}(p_j^* - i)$, and $f_{\rm mns} \in {\rm Ker}(p_j^* + i)$. Furthermore, by the corollary of Theorem 2.3, there exist $\varphi_0 \in D(p_j)$ and $\varphi_+ \in {\rm Ker}(p_j^* - i)$ such that $f = \varphi_0 + \varphi_+ + U_j\varphi_+$. So, we have $0 = (f_0 - \varphi_0) + (f_{\rm pls} - \varphi_+) + (f_{\rm mns} - U_j\varphi_+)$, $f_0 - \varphi_0 \in D(p_j)$, $f_{\rm pls} - \varphi_+ \in {\rm Ker}(p_j^* - i)$, and $f_{\rm mns} - U_j\varphi_+ \in {\rm Ker}(p_j^* + i)$. By using p_j-orthogonality (see Lemma on p138 in [19]), we have $(f_0 - \varphi_0\,,\,f_0 - \varphi_0)_{p_j} = 0$; $(f_{\rm pls} - \varphi_+\,,\,f_{\rm pls} - \varphi_+)_{p_j} = 0$; and $(f_{\rm mns} - U_j\varphi_+\,,\,f_{\rm mns} - U_j\varphi_+)_{p_j} = 0$. Thus, by the definition of $(\cdot\,,\,\cdot)_{p_j}$, we have $f_0 = \varphi_0$, $f_{\rm pls} = \varphi_+$, and $f_{\rm mns} = U_j\varphi_+$. Uniqueness of decomposition also follows from this p_j-orthogonality. Q.E.D.

Fix U_j $(j = 1,2)$ and $t \in \mathbb{R}$. Since $p_{_{U_j}}$ is self-adjoint, there exists a dense set $\mathcal{A}(p_{_{U_j}})$ of analytic vectors of $p_{_{U_j}}$ in $C^\infty(p_{_{U_j}})$ (see Corollary1 on p203 in [19]).

We can give behaviors of $\exp\left[itp_{_{U_j}}\right]$ $(j = 1,2)$ as the following proposition, which tell us that any pair of $\exp\left[itp_{_{U_1}}\right]$ and $\exp\left[itp_{_{U_2}}\right]$ destroy Weyl's CCR:

PROPOSITION 3.2.

(a) *For any $f \in \mathcal{A}(p_{_{U_1}})$,*

(a-1) *if $(x_1\,,\,x_2)$ satisfies one of the following conditions at least, (i) $|x_2| > R$; (ii) $|x_2| \leq R$ with $x_1, x_1 + t \leq w_-(x_2)$; or (iii) $|x_2| \leq R$ with $w_+(x_2) \leq x_1, x_1 + t$, then*

$$\left(e^{itp_{U_1}} f\right)(x_1\,,\,x_2) = f(x_1 + t\,,\,x_2),$$

(a-2) *if $t > 0$ and $|x_2| \leq R$, then for $x_1 = w_-(x_2)$*

$$\left(e^{itp_{U_1}} f\right)(w_-(x_2)\,,\,x_2) = \sum_{n=0}^{\infty} \gamma_{_{U_1}}(f_{\rm pls}^{(n)}\,;\,x_2)^{-1}\frac{1}{n!}\left(\frac{d^n f}{dt^n}(w_+(x_2) + t\,,\,x_2)\lceil_{t=0}\right) t^n,$$

where $g^{(n)}$ denotes $\partial^n g/\partial x^n$, (a-3) *if $t > 0$, and $|x_2| \leq R$ with $x_1 < w_-(x_2) < x_1 + t$, then*

$$\begin{aligned}
&\left(e^{itp_{U_1}} f\right)(x_1\,,\, x_2) \\
= \ &\sum_{n=0}^{\infty} \gamma_{U_1}(f^{(n)}_{\mathrm{pls}}\,;\, x_2)^{-1} \frac{1}{n!} \left(\frac{d^n f}{dt^n}(x_1 + t + 2w_+(x_2)\,,\, x_2)\lceil_{t=-x_1-w_+(x_2)} \right) (t + x_1 + w_+(x_2))^n,
\end{aligned}$$

(a-4) *if $t < 0$ and $|x_2| \leq R$, then for $x_1 = w_+(x_2)$*

$$\left(e^{itp_{U_1}} f\right)(w_+(x_2)\,,\, x_2) = \sum_{n=0}^{\infty} \gamma_{U_1}(f^{(n)}_{\mathrm{pls}}\,;\, x_2) \frac{1}{n!} \left(\frac{d^n f}{dt^n}(w_-(x_2) + t\,,\, x_2)\lceil_{t=0} \right) t^n,$$

(a-5) *if $t < 0$, and $|x_2| \leq R$ with $x_1 + t < w_+(x_2) < x_1$, then*

$$\begin{aligned}
&\left(e^{itp_{U_1}} f\right)(x_1\,,\, x_2) \\
= \ &\sum_{n=0}^{\infty} \gamma_{U_1}(f^{(n)}_{\mathrm{pls}}\,;\, x_2) \frac{1}{n!} \left(\frac{d^n f}{dt^n}(x_1 + t + 2w_-(x_2)\,,\, x_2)\lceil_{t=-x_1-w_-(x_2)} \right) (t + x_1 + w_-(x_2))^n.
\end{aligned}$$

(b) *For any $f \in \mathcal{A}(p_{U_2})$,*

(b-1) *if $(x_1\,,\, x_2)$ satisfies one of the following conditions at least, (i) $|x_1| > R$; (ii) $|x_1| \leq R$ with $x_2, x_2 + t \leq w_-(x_1)$; or (iii) $|x_1| \leq R$ with $w_+(x_1) \leq x_2, x_2 + t$, then*

$$\left(e^{itp_{U_2}} f\right)(x_1\,,\, x_2) = f(x_1\,,\, x_2 + t),$$

(b-2) *if $t > 0$ and $|x_1| \leq R$, then for $x_2 = w_-(x_1)$*

$$\left(e^{itp_{U_2}} f\right)(x_1\,,\, w_-(x_1)) = \sum_{n=0}^{\infty} \gamma_{U_2}(f^{(n)}_{\mathrm{pls}}\,;\, x_1)^{-1} \frac{1}{n!} \left(\frac{d^n f}{dt^n}(x_1\,,\, w_+(x_1) + t)\lceil_{t=0} \right) t^n,$$

(b-3) *if $t > 0$, and $|x_1| \leq R$ with $x_2 < w_-(x_1) < x_2 + t$, then*

$$\begin{aligned}
&\left(e^{itp_{U_2}} f\right)(x_1\,,\, x_2) \\
= \ &\sum_{n=0}^{\infty} \gamma_{U_2}(f^{(n)}_{\mathrm{pls}}\,;\, x_1)^{-1} \frac{1}{n!} \left(\frac{d^n f}{dt^n}(x_1\,,\, x_2 + t + 2w_+(x_1))\lceil_{t=-x_2-w_+(x_1)} \right) (t + x_2 + w_+(x_1))^n,
\end{aligned}$$

(b-4) *if $t < 0$ and $|x_1| \leq R$, then for $x_2 = w_+(x_1)$*

$$\left(e^{itp_{U_2}} f\right)(x_1\,,\, w_+(x_1)) = \sum_{n=0}^{\infty} \gamma_{U_2}(f^{(n)}_{\mathrm{pls}}\,;\, x_1) \frac{1}{n!} \left(\frac{d^n f}{dt^n}(x_1\,,\, w_-(x_1) + t)\lceil_{t=0} \right) t^n,$$

(b-5) *if $t < 0$, and $|x_1| \le R$ with $x_2 + t < w_+(x_1) < x_2$, then*

$$\begin{aligned}
&\left(e^{itp_{U_2}} f\right)(x_1\,,\, x_2) \\
= \;&\sum_{n=0}^{\infty} \gamma_{U_2}(f^{(n)}_{\rm pls}\,;\, x_1) \frac{1}{n!} \left(\frac{d^n f}{dt^n}(x_1\,,\, x_2 + t + 2w_-(x_1)) \lceil_{t=-x_2-w_-(x_1)} \right) (t + x_2 + w_-(x_1))^n.
\end{aligned}$$

Proof. For $f \in \mathcal{A}(p_{U_1})$,

$$\left(e^{itp_{U_1}} f\right)(x_1\,,\, x_2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f^{(n)}(x_1\,,\, x_2). \tag{3.1}$$

Since $f \in \mathcal{A}(p_{U_1})$, $f^{(n)} \in D(p_{U_1})$. Thus, by Lemma 2.7, Theorem 2.8 and Lemma 3.1, we have

$$f^{(n)}(w_+(x_2)\,,\, x_2) = \gamma_{U_1}(f^{(n)}_{\rm pls}\,;\, x_2) f^{(n)}(w_-(x_2)\,,\, x_2) \tag{3.2}$$

for $|x_2| \le R$. Parts (a-1) and (a-2) follows from (3.1) and (3.2). Let $t > 0$, $x_1 < w_-(x_2) < x_1 + t$. We set $t_0 \stackrel{\rm def}{=} w_-(x_2) - x_1$, and $t_1 \stackrel{\rm def}{=} t - t_0$. Then, since $e^{itp_{U_1}} = e^{it_0 p_{U_1}} e^{it_1 p_{U_1}}$ and $w_-(x_2) = -w_+(x_2)$, defining $F_{t_1} \stackrel{\rm def}{=} e^{it_1 p_{U_1}} f$, we have

$$\begin{aligned}
&\left(e^{itp_{U_1}} f\right)(x_1\,,\, x_2) = \left(e^{it_0 p_{U_1}} F_{t_1}\right)(x_1\,,\, x_2) = F_{t_1}(x_1 + t_0\,,\, x_2) \\
= \;& F_{t_1}(w_-(x_2)\,,\, x_2) = \sum_{n=0}^{\infty} \gamma_{U_1}(f^{(n)}_{\rm pls}\,;\, x_2)^{-1} \frac{1}{n!} \left(\frac{d^n f}{dt_1^n}(w_+(x_2) + t_1\,,\, x_2) \lceil_{t_1=0} \right) (x_1 + t + w_+(x_2))^n \\
= \;& \sum_{n=0}^{\infty} \gamma_{U_1}(f^{(n)}_{\rm pls}\,;\, x_2)^{-1} \frac{1}{n!} \left(\frac{d^n f}{dt^n}(x_1 + t + 2w_+(x_2)\,,\, x_2) \lceil_{t=-x_1-w_+(x_2)} \right) (x_1 + t + w_+(x_2))^n
\end{aligned}$$

by (a-1) and (a-2). Similarly, we have parts (a-4), (a-5) and (b-1)-(b-5). Q.E.D.

Since $\mathcal{A}(p_{U_j})$ $(j = 1, 2)$ is dense in $L^2(\Omega_R)$, for arbitrary $f \in L^2(\Omega_R)$ we can approximate elements in $\mathcal{A}(p_{U_j})$ to f in the sense of the almost everywhere convergence on Ω_R. So, Proposition 3.2 means that, for $f \in L^2(\Omega_R)$, $\exp\left[itp_{U_1}\right] f$ jumps at the boundary of the hole D_R in a moment. When this f goes across the hole D_R, the equality between $\left(\exp\left[isp_{U_1}\right] \exp\left[itp_{U_2}\right] f\right)(x_1\,,\, x_2)$ and $\left(\exp\left[itp_{U_2}\right] \exp\left[isp_{U_1}\right] f\right)(x_1\,,\, x_2)$, which would normally be valid, must be destroyed. Roughly speaking, for instance, let $-2R < x_j < -R$ $(j = 1, 2)$, $s = R$, and $t = 3R$. Defining $F^{U_2}_s \stackrel{\rm def}{=} e^{isp_{U_2}} f$, since $|x_2| > R$,

$$\left(e^{itp_{U_1}} F^{U_2}_s\right)(x_1\,,\, x_2) = F^{U_2}_s(x_1 + t\,,\, x_2) = \left(e^{isp_{U_2}} f\right)(x_1 + t\,,\, x_2)$$

holds by Proposition 3.2 (a-1). Since $R < x_1 + t$,

$$\left(e^{isp_{U_2}} f\right)(x_1 + t\,,\, x_2) = f(x_1 + t\,, x_2 + s)$$

holds by Proposition 3.2 (b-1). Thus, we have

$$\left(e^{itp_{U_1}}e^{itp_{U_2}}f\right)(x_1\,,\,x_2) = f(x_1+t\,,x_2+s)$$

for almost all $(x_1\,,\,x_2) \in (-2R\,,\,-R)\times(-2R\,,\,-R)$.

Now, defining $G_t^{U_1} \stackrel{\text{def}}{\equiv} e^{itp_{U_1}}f$, since $|x_1| > R$,

$$\left(e^{isp_{U_2}}G_t^{U_1}\right)(x_1\,,\,x_2) = G_t^{U_1}(x_1\,,\,x_2+s) = \left(e^{itp_{U_1}}f\right)(x_1\,,\,x_2+s)$$

holds by Proposition 3.2 (b-1). Since $-R < x_2+s < 0$, and $x_1 < w_-(x_2) < w_+(x_2) < x_1+t$ for all $|x_2| \le R$,

$$\begin{aligned}&\left(e^{itp_{U_1}}f\right)(x_1\,,\,x_2+s)\\ =\ &\sum_{n=0}^{\infty}\gamma_{U_1}(f_{\text{pls}}^{(n)}\,;\,x_2+s)^{-1}\frac{1}{n!}\\ &\qquad\times\left(\frac{d^nf}{dt^n}(x_1+t+2w_+(x_2+s)\,,\,x_2+s)\lceil_{t=-x_1-w_+(x_2+s)}\right)(t+x_1+w_+(x_2+s))^n\end{aligned}$$

by Proposition 3.2 (a-3). Thus, we have

$$\begin{aligned}&\left(e^{isp_{U_2}}e^{itp_{U_1}}f\right)(x_1\,,\,X_2)\\ =\ &\sum_{n=0}^{\infty}\gamma_{U_1}(f_{\text{pls}}^{(n)}\,;\,X_2+s)^{-1}\frac{1}{n!}\\ &\qquad\times\left(\frac{d^nf}{dt^n}(x_1+t+2w_+(X_2+s)\,,\,X_2+s)\lceil_{t=-x_1-w_+(X_2+s)}\right)(t+x_1+w_+(x_2+s))^n\end{aligned}$$

for almost all $(x_1\,,\,x_2) \in (-2R\,,\,-R)\times(-2R\,,\,-R)$.

Therefore, we realize that

$$\left(e^{itp_{U_1}}e^{isp_{U_2}}f\right)(x_1\,,\,x_2) \ne \left(e^{isp_{U_2}}e^{itp_{U_1}}f\right)(x_1\,,\,x_2)$$

for almost all $(x_1\,,\,x_2) \in (-2R\,,\,-R)\times(-2R\,,\,-R)$. So, Weyl's CCR is destroyed but remember that Heisenberg's holds for p_{U_j} (see Proposition 2.1).

Remember the case of $\Omega_0 \equiv \mathbb{R}^2 \setminus \{(0,0)\}$. In Reeh's and Arai's case, they essentially used the hole, the origin, has the 0-Lebesgue measure.

Since Weyl's CCR is destroyed in our case, the momentum operators defined by using $-i\partial/\partial x_j$ $(j=1,2)$ as (2.1) cannot give any representation equivalent to the Schrödinger one even if we consider any boundary condition on ∂D_R. Thus, we redefine the momentum and the position operators such that they are equivalent to the Schrödinger representation.

4 The Definition of the Schrödinger Representation Using Streamlines

In order that the momentum operators generate shift in a space Ω_R having a hole of the shape of a disc of radius R, we introduce streamlines.

We take a coordinate given by a velocity potential $\xi \stackrel{\mathrm{def}}{=} \phi(x\,,\,y)$ and a flow function $\eta \stackrel{\mathrm{def}}{=} \psi(x\,,\,y)$ for $(x\,,\,y) \in \mathbb{R}^2$. Then we first define $\phi(x\,,\,y)$ and $\psi(x\,,\,y)$ by the Joukowski transformation $\zeta(z)$: For $z \stackrel{\mathrm{def}}{=} x+iy$, we set $\zeta(z) \stackrel{\mathrm{def}}{=} z + R^2/z$ and $\zeta \stackrel{\mathrm{def}}{=} \xi + i\eta$. So $\phi(x\,,\,y)$ and $\psi(x\,,\,y)$ are determined as

$$\xi = \phi(x\,,\,y) \stackrel{\mathrm{def}}{=} x\left(1+\frac{R^2}{x^2+y^2}\right), \qquad \eta = \psi(x\,,\,y) \stackrel{\mathrm{def}}{=} y\left(1-\frac{R^2}{x^2+y^2}\right).$$

By ϕ^{-1} and ψ^{-1}, we denote functions satisfying $x = \phi^{-1}(\xi\,,\,\eta)$ and $y = \psi^{-1}(\xi\,,\,\eta)$.

By the Joukowski transformation $\zeta(z)$, we have two conformal mappings J_{int} : $\mathrm{Int}D_R \stackrel{1-1}{\longrightarrow} \mathbb{R}^2_{2R}$ and $J_{out} : \Omega_R \stackrel{1-1}{\longrightarrow} \mathbb{R}^2_{2R}$, where $\mathbb{R}^2_{2R} \stackrel{\mathrm{def}}{=} \mathbb{R}^2 \backslash \{(\xi\,,\,0)\,|\,-2R \le \xi \le 2R\}$, and $\mathrm{Int}D_R \stackrel{\mathrm{def}}{=} \{(x\,,\,y)\,|\,x^2+y^2 < R^2\}$.

We note here the Cauchy-Riemann relations:

$$\frac{\partial\phi}{\partial x} = a(x,y) \stackrel{\mathrm{def}}{=} 1 - R^2\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\partial\psi}{\partial y} \tag{4.1}$$

$$\frac{\partial\phi}{\partial y} = b(x,y) \stackrel{\mathrm{def}}{=} -2R^2\frac{xy}{(x^2+y^2)^2} = -\frac{\partial\psi}{\partial x} \tag{4.2}$$

By the change of variables, we have

$$\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} = \begin{pmatrix} a^u & -b^u \\ b^u & a^u \end{pmatrix} \begin{pmatrix} \partial/\partial \xi \\ \partial/\partial \eta \end{pmatrix},$$

where $a^u(\xi\,,\,\eta) \stackrel{\mathrm{def}}{=} a(\phi^{-1}(\xi\,,\,\eta), \psi^{-1}(\xi\,,\,\eta))$, and $b^u(\xi\,,\,\eta) \stackrel{\mathrm{def}}{=} b(\phi^{-1}(\xi\,,\,\eta)\,,\,\psi^{-1}(\xi\,,\,\eta))$.

Here we set $c(x\,,\,y) \stackrel{\mathrm{def}}{=} \sqrt{a(x\,,\,y)^2 + b(x\,,\,y)^2}$. We define two operators, $p_{[j]}$ $(j=1,2)$, acting in $L^2(\Omega_R)$ by

$$p_{[1]} \stackrel{\mathrm{def}}{=} \frac{1}{ic}\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\frac{1}{c}, \qquad D(p_{[1]}) \stackrel{\mathrm{def}}{=} C_0^\infty(\Omega_R),$$

$$p_{[2]} \stackrel{\mathrm{def}}{=} \frac{1}{ic}\left(-b\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}\right)\frac{1}{c}, \qquad D(p_{[2]}) \stackrel{\mathrm{def}}{=} C_0^\infty(\Omega_R).$$

We can define two self-adjoint operators, $q^S_{[j]}$ $(j=1,2)$, by

$$q^S_{[1]} \stackrel{\mathrm{def}}{=} c\phi\frac{1}{c}, \qquad D(q^S_{[1]}) \stackrel{\mathrm{def}}{=} \left\{ f \in L^2(\Omega_R) \,\middle|\, \int\!\!\int_{\Omega_R} dxdy |\phi(x\,,\,y)f(x\,,\,y)|^2 < \infty \right\},$$

$$q^S_{[2]} \stackrel{\mathrm{def}}{=} c\psi\frac{1}{c}, \qquad D(q^S_{[2]}) \stackrel{\mathrm{def}}{=} \left\{ f \in L^2(\Omega_R) \,\middle|\, \int\!\!\int_{\Omega_R} dxdy |\psi(x\,,\,y)f(x\,,\,y)|^2 < \infty \right\}$$

(see Example 5.11 and its Remark 1 in [8] or Proposition 1 and the proof of Proposition 3 in §VIII.3 of [18]).

For functions f of $(x\,,\,y) \in \Omega_R$, we define functions f^u of $(\xi\,,\,\eta) \in \mathbb{R}^2_{2R}$ by $f^u(\xi\,,\,\eta) \stackrel{\mathrm{def}}{=} f(\phi^{-1}(\xi\,,\,\eta)\,,\,\psi^{-1}(\xi\,,\,\eta))$. So, $f(x\,,\,y) = f^u(\phi(x\,,\,y)\,,\,\psi(x\,,\,y))$.

We define a Hilbert space $L^2_c(\mathbb{R}^2_{2R})$ by

$$L^2_c(\mathbb{R}^2_{2R}) \stackrel{\mathrm{def}}{=} \left\{ v\,:\,\text{measurable functions of }(\xi\,,\,\eta) \,\middle|\, \int\int_{\mathbb{R}^2_{2R}} d\xi d\eta \frac{|v(\xi\,,\,\eta)|^2}{c^u(\xi\,,\,\eta)^2} < \infty \right\}$$

with an inner product $\langle v\,,\,w\rangle_{L^2_c(\mathbb{R}^2_{2R})} \stackrel{\mathrm{def}}{=} \int\int_{\mathbb{R}^2_{2R}} d\xi d\eta \frac{\overline{v(\xi\,,\,\eta)}w(\xi\,,\,\eta)}{c^u(\xi\,,\,\eta)^2}$. And we define a linear operator $U\,:\,L^2(\Omega_R) \longrightarrow L^2_c(\mathbb{R}^2_{2R})$ by $(Uf)\,(\xi\,,\,\eta) \stackrel{\mathrm{def}}{=} f^u(\xi\,,\,\eta)$ for every $f \in L^2(\Omega_R)$. Then, it is clear that U is a unitary operator, and

$$\begin{aligned}
&Up_{[1]}U^{-1} = \frac{1}{i}c^u\frac{\partial}{\partial\xi}\frac{1}{c^u}, \qquad D(Up_{[1]}U^{-1}) = C^\infty_0(\mathbb{R}^2_{2R}),\\
&Up_{[2]}U^{-1} = \frac{1}{i}c^u\frac{\partial}{\partial\eta}\frac{1}{c^u}, \qquad D(Up_{[2]}U^{-1}) = C^\infty_0(\mathbb{R}^2_{2R}),\\
&Uq_{[1]}U^{-1} = c^u\xi\frac{1}{c^u}, \qquad D(Uq_{[1]}U^{-1}) = \left\{ v \in L^2_c(\mathbb{R}^2_{2R}) \,\middle|\, \int\int_{\mathbb{R}^2_{2R}} d\xi d\eta \frac{|\xi|^2|v(\xi\,,\,\eta)|^2}{c^u(\xi\,,\,\eta)^2} < \infty \right\},\\
&Uq_{[2]}U^{-1} = c^u\eta\frac{1}{c^u}, \qquad D(Uq_{[2]}U^{-1}) = \left\{ v \in L^2_c(\mathbb{R}^2_{2R}) \,\middle|\, \int\int_{\mathbb{R}^2_{2R}} d\xi d\eta \frac{|\eta|^2|v(\xi\,,\,\eta)|^2}{c^u(\xi\,,\,\eta)^2} < \infty \right\}.
\end{aligned}$$

Of course, we have

Lemma 4.1. *The operator $p_{[j]}$ $(j = 1, 2)$ is symmetric.*

Thus, since $p_{[j]}$ $(j = 1, 2)$ is closable, we can denote by $\overline{p_{[j]}}$ $(j = 1, 2)$ the closure of $p_{[j]}$. As we expect, we have

Proposition 4.2. $\left\{\overline{p_{[j]}}\,,\,q_{[j]}\right\}_{j=1,2}$ *satisfies Heisenberg's CCR on $C^\infty_0(\Omega_R)$.*

Since the hole $\{(\xi\,,\,0)\,|\,-2R \le \xi \le 2R\}$ intercepts the differential $-i\partial/\partial\xi$ at only $\eta = 0$, we can easily get the following proposition by investigating the deficiency index of $-i\partial/\partial\xi$:

Proposition 4.3. $\overline{p_{[1]}}$ is self-adjoint.

Proof: Let $g_\pm \in \mathrm{Ker}\left(p^*_{[1]} \mp i\right)$. Then, for all $f \in D(p_{[1]}) \equiv C^\infty_0(\Omega_R)$, we get

$$\begin{aligned}
0 &= \langle \left(p^*_{[1]} \mp i\right) g_\pm\,, cf\rangle_{L^2(\Omega_R)} = \langle g_\pm\,,\,\left(p_{[1]} \pm i\right) cf\rangle_{L^2(\Omega_R)}\\
&= \frac{1}{i}\int\int_{\mathbb{R}^2_{2R}} d\xi d\eta \overline{\left(-\left(D_\xi\frac{(g^u_\pm)}{c^u}\right)(\xi\,,\,\eta) \mp \frac{(g^u_\pm)}{c^u}(\xi\,,\,\eta)\right)} f^u(\xi\,,\,\eta),
\end{aligned}$$

where D_ξ is the derivative in the sense of distribution with test functions in $C^\infty_0(\mathbb{R}^2_{2R})$. So, since f^u runs all over $C^\infty_0(\mathbb{R}^2_{2R})$, $v = g^u_\pm$ is a solution of $(D_\xi \pm 1)\,v = 0$ in the sense of the distributions with test functions, $C^\infty_0(\mathbb{R}^2_{2R})$. On the other hand, if the solutions of the differential equation $(D_\xi \pm 1)\,\dfrac{g^u_\pm}{c^u} = 0$ are considered, $g^u_\pm$ has the form of $g^u_\pm(\xi\,,\,\eta) =$ (a function of η)$c^u(\xi\,,\,\eta)e^{\mp\xi}$, which can not live in $L^2_c(\mathbb{R}^2_{2R})$. Thus, we obtain $\mathrm{dimKer}(p^*_{[1]} - i) = 0 = \mathrm{dimKer}(p^*_{[1]} + i)$. Hence our desired statement follows our proposition from Corollary on p141 in [19]. Q.E.D.

¿From now on, we denote $\overline{p_{[1]}}$ by $p^S_{[1]}$. In order to get self-adjoint extensions of $p_{[2]}$ and exact descriptions of their domains, we will prepare some lemmas for a while.

In the same way as Theorem 2.3 and its corollary, we have the following lemma:

LEMMA 4.4. *$U\overline{p_{[2]}}U^{-1}$ has uncountably many different self-adjoint extensions in $L^2_c(\mathbb{R}^2_{2R})$. And let $U_{[2]} : \mathrm{Ker}\left(\left(U\overline{p_{[2]}}U^{-1}\right)^* - i\right) \subset L^2_c(\mathbb{R}^2_{2R}) \to \mathrm{Ker}\left(\left(U\overline{p_{[2]}}U^{-1}\right)^* + i\right) \subset L^2_c(\mathbb{R}^2_{2R})$ be an arbitrary unitary operator, and $Up_{U_{[2]}}U^{-1}$ be the self-adjoint extension of $U\overline{p_{[2]}}U^{-1}$ corresponding to $U_{[2]}$. Then,*

$$D(Up_{U_{[2]}}U^{-1}) = \left\{\varphi_0 + \varphi_+ + U_{[2]}\varphi_+ \,\middle|\, \varphi_0 \in D(U\overline{p_{[2]}}U^{-1}),\ \varphi_+ \in \mathrm{Ker}\left(\left(U\overline{p_{[2]}}U^{-1}\right)^* - i\right)\right\},$$
$$Up_{U_{[2]}}U^{-1}(\varphi_0 + \varphi_+ + U_{[2]}\varphi_+) = U\overline{p_{[2]}}U^{-1}\varphi_0 + i\varphi_+ - iU_{[2]}\varphi_+.$$

We define a set $H(\xi)$ for $\xi \in \mathbb{R}$ by $H(\xi) \stackrel{\text{def}}{=} \{\eta \in \mathbb{R} \,|\, \text{there exists a } \xi_\eta \text{ such that } (\xi_\eta\,,\,\eta) \in \mathbb{R}^2_{2R}\}$. Then we define a set $AC^{[2]}_{loc}(\mathbb{R}^2_{2R})$ of functions on $\mathbb{R}^2_{2R}$ by

$$AC^{[2]}_{loc}(\mathbb{R}^2_{2R}) \stackrel{\text{def}}{=} \left\{ f \in L^2_c(\mathbb{R}^2_{2R}) \,\middle|\, \begin{array}{l}\text{for almost all } \xi \in \mathbb{R},\ f(\xi\,,\,\cdot) \text{ is absolutely continuous}\\ \text{on arbitrary closed interval } [c\,,\,c'] \text{ contained in } H(\xi) \text{ such that}\\ \dfrac{\partial f}{\partial \eta} \in L^2_c(\mathbb{R}^2_{2R})\end{array}\right\}.$$

Let $\{e_n\}_{n\in\mathbb{N}}$ be a complete orthonormal basis of $L^2((-2R\,,\,2R))$. We define functions $h^\pm_n$ $(n \in \mathbb{N})$ on $\mathbb{R}^2_{2R}$ by $h^\pm_n(\xi\,,\,\eta) \stackrel{\text{def}}{=} \sqrt{2}e^{\mp\eta}c^u(\xi\,,\,\eta)\chi_{_{H\pm(\xi)}}(\eta)e_n(\xi)$, where $\chi_{_{H+(\xi)}}(\eta) \stackrel{\text{def}}{=} 1$ if $|\xi| < 2R$ and $0 < \eta$; 0 otherwise, and $\chi_{_{H-(\xi)}}(\eta) \stackrel{\text{def}}{=} 1$ if $|\xi| < 2R$ and $\eta < 0$; 0 otherwise.

In the same way as Lemma 2.7, we have

LEMMA 4.5. *$\{h^\pm_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis of $\mathrm{Ker}\left(\left(U\overline{p_{[2]}}U^{-1}\right)^* \mp i\right) \subset L^2_c(\mathbb{R}^2_{2R})$. For every $\varphi \in D\left(Up_{U_{[2]}}U^{-1}\right)$,*

$$\lim_{\eta\downarrow 0}\varphi(\xi\,,\,\eta) = \gamma_{_{U_{[2]}}}(\varphi_+\,;\,\xi)\lim_{\eta\uparrow 0}\varphi(\xi\,,\,\eta). \tag{4.3}$$

where

$$\gamma_{U_{[2]}}(\varphi_+\,;\,\xi) = \frac{\displaystyle\sum_{m=1}^{\infty}\langle h_m^+\,,\,\varphi_+\rangle_{L_c^2(\mathbb{R}^2_{2R})}e_m(\xi)}{\displaystyle\sum_{n=1}^{\infty}\langle h_n^-\,,\,U_\eta\varphi_+\rangle_{L_c^2(\mathbb{R}^2_{2R})}e_n(\xi)}.$$

And

$$\begin{aligned}&\sum_{m=1}^{\infty}\overline{\langle h_m^+\,,\,g_+\rangle_{L_c^2(\mathbb{R}^2_{2R})}}\langle h_m^+\,,\,f_+\rangle_{L_c^2(\mathbb{R}^2_{2R})}\\ =\ &\sum_{m=1}^{\infty}\overline{\langle h_m^-\,,\,U_{[2]}g_+\rangle_{L_c^2(\mathbb{R}^2_{2R})}}\langle h_m^-\,,\,U_{[2]}f_+\rangle_{L_c^2(\mathbb{R}^2_{2R})}\end{aligned}$$

holds for every $f_+, g_+ \in \mathrm{Ker}\left(\left(U\overline{p_{[2]}}U^{-1}\right)^* - i\right)$.

We denote by $U_{[2]}^I$ the unitary operator $U_{[2]}$ satisfying $\gamma_{U_{[2]}}(\varphi_+\,;\,\xi) = 1$ for all $\varphi_+ \in \mathrm{Ker}\left(\left(U\overline{p_{[2]}}U^{-1}\right)^* - i\right)$ and $\xi \in (-2R\,,\,2R)$. And let $p_{[2]}^S$ be $p_{U_{[2]}^I} \equiv U^{-1}(Up_{U_{[2]}^I}U^{-1})U$.

Now that we have Lemmas 4.5 and 4.6, we can describe exactly the domain of $p_{[2]}^S$. In the same way as Theorem 2.8, we can prove the following lemma:

Lemma 4.6. $\quad Up_{[2]}^S U^{-1} = \dfrac{1}{i}c^u\dfrac{\partial}{\partial\eta}\dfrac{1}{c^u}$ *with*

$$\begin{aligned}D(Up_{[2]}^S U^{-1}) \;=\; \Bigg\{ f \in AC_{loc}^{[2]}(\mathbb{R}^2_{2R}) \;\Bigg|\; &\int_{|\xi|<2R} \left|\frac{f(\xi\,,\,0)}{c^u(\xi\,,\,0)}\right|^2 < \infty,\\ &\textit{and } \lim_{\eta\downarrow 0} f(\xi\,,\,\eta) = \lim_{\eta\uparrow 0} f(\xi\,,\,\eta)\\ &\textit{for almost all } \xi \in [-2R\,,\,2R]\Bigg\}.\end{aligned}$$

For the self-adjoint operator $p_{[1]}^S$ and the selected self-adjoint extension $p_{[2]}^S$, the behavior of $\exp\left[itp_{[j]}^S\right]$ is given by the following lemma in the same way as Proposition 3.2,

Lemma 4.7. *For all* $f^u \in L_c^2(\mathbb{R}^2_{2R})$ *and* $t \in \mathbb{R}$,

$$\begin{aligned}\left(Ue^{itp_{[1]}^S}U^{-1}f^u\right)(\xi\,,\,\eta) &= \left(e^{itUp_{[1]}^S U^{-1}}f\right)(\xi\,,\,\eta) = f(\xi+t\,,\,\eta)\frac{c^u(\xi\,,\,\eta)}{c^u(\xi+t\,,\,\eta)},\\ \left(Ue^{itp_{[2]}^S}U^{-1}f^u\right)(\xi\,,\,\eta) &= \left(e^{itUp_{[2]}^S U^{-1}}f^u\right)(\xi\,,\,\eta) = f^u(\xi\,,\,\eta+t)\frac{c^u(\xi\,,\,\eta)}{c^u(\xi\,,\,\eta+t)}.\end{aligned}$$

Proof. First, we here note $L^2(\mathbb{R}^2_{2R}\,;\,d\xi d\eta) = L^2(\mathbb{R}^2\,;\,d\xi d\eta)$. Let

$$\mathcal{A} \stackrel{\mathrm{def}}{=} \left\{ f \in L^2(\mathbb{R}^2_{2R}\,;\,d\xi d\eta) \mid f \text{ is an analytic vector for } \frac{1}{i}\frac{\partial}{\partial \xi} \text{ acting in } L^2(\mathbb{R}^2\,;\,d\xi d\eta) \right\},$$

$$c^u\mathcal{A} \stackrel{\mathrm{def}}{=} \{c^u f \mid f \in \mathcal{A}\}.$$

It is evident that $c^u\mathcal{A} \subset C^\infty(p^S_{[1]})$. We can give the following equality by simple calculation: $\| \left(U p^S_{[1]} U^{-1}\right)^n c^u f \|_{L^2_c(\mathbb{R}^2_{2R})} = \left\| \left(\frac{1}{i}\right)^n \frac{\partial^n}{\partial \xi^n} f \right\|_{L^2(\mathbb{R}^2;d\xi d\eta)}$, which implies that $c^u\mathcal{A}$ is a set of analytic vectors for $p^S_{[1]}$.

For arbitrary $f \in c^u\mathcal{A}$ and $g \in L^2_c(\mathbb{R}^2_{2R})$, we have

$$\begin{aligned}
&\int\!\!\int_{\mathbb{R}^2_{2R}} d\xi d\eta\, \frac{\overline{g(\xi\,,\,\eta)}}{c^u(\xi\,,\,\eta)^2} \left(e^{itUp^S_{[1]}U^{-1}} f\right)(\xi\,,\,\eta) \\
=\ &\int\!\!\int_{\mathbb{R}^2_{2R}} d\xi d\eta\, \frac{\overline{g(\xi\,,\,\eta)}}{c^u(\xi\,,\,\eta)^2} \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\left(c^u \frac{\partial}{\partial \xi} \frac{1}{c^u} \right)^n f \right)(\xi\,,\,\eta) \\
=\ &\int\!\!\int_{\mathbb{R}^2_{2R}} d\xi d\eta\, \frac{\overline{g(\xi\,,\,\eta)}}{c^u(\xi\,,\,\eta)^2} c^u(\xi\,,\,\eta) \sum_{n=0}^{\infty} \frac{s^n}{n!} \left(\frac{\partial^n}{\partial \xi^n} \frac{f}{c^u} \right)(\xi\,,\,\eta) \\
=\ &\int\!\!\int_{\mathbb{R}^2_{2R}} d\xi d\eta\, \frac{\overline{g(\xi\,,\,\eta)}}{c^u(\xi\,,\,\eta)^2} f(\xi + t\,,\,\eta) \frac{c^u(\xi\,,\,\eta)}{c^u(\xi + t\,,\,\eta)}.
\end{aligned}$$

Since f and g are arbitrary elements in $c^u\mathcal{A}$ and $L^2_c(\mathbb{R}^2_{2R})$ respectively, we have for $f \in L^2_c(\mathbb{R}^2_{2R})$

$$\left(e^{itUp^S_{[1]}U^{-1}} f\right)(\xi\,,\,\eta) = f(\xi + t\,,\,\eta) \frac{c^u(\xi\,,\,\eta)}{c^u(\xi + t\,,\,\eta)}.$$

Similarly, we have the second equality in our lemma. Q.E.D.

THEOREM 4.8. $\left\{p^S_{[j]}\,,\,q^S_{[j']}\right\}_{j,j'=1,2}$ *satisfies Weyl's' CCR.*

Proof. We have only to set $\mathsf{q} = 0$ in Theorem 5.7 below. Q.E.D.

5 Aharonov-Bohm Phase Appearing in Weyl's CCR

To discuss the Aharonov-Bohm phase in Weyl's CCR, we consider a spinless charged particle with the charge $\mathsf{q} \neq 0$ moving in the plane Ω_R under the influence of a magnetic field which goes through D_R perpendicularly to the plane and vanishes outside. Let $\mathbf{A}(x\,,\,y) \stackrel{\mathrm{def}}{=} (A_1(x\,,\,y)\,,\,A_2(x\,,\,y))$ be a gauge potential of the magnetic field. A_j $(j = 1, 2)$ may be singular at points inside D_R, but we assume that

$$A_j \in C^\infty(\overline{\Omega_R}), \qquad j = 1, 2, \tag{5.1}$$

$$B(x\,,\,y) \stackrel{\mathrm{def}}{=} \frac{\partial}{\partial x} A_2(x\,,\,y) - \frac{\partial}{\partial y} A_1(x\,,\,y) = 0, \qquad (x\,,\,y) \in \Omega_R, \tag{5.2}$$

where $\overline{\Omega_R}$ denote the closure of Ω_R.

We give a gauge potential $\widetilde{\mathbf{A}}(x\,,\,y) \stackrel{\mathrm{def}}{=} \Big(A_{[1]}(x\,,\,y)\,,\,A_{[2]}(x\,,\,y)\Big)$ by

$$A_{[1]}\,(x\,,\,y) \stackrel{\mathrm{def}}{=} \frac{1}{c(x\,,\,y)^2}\,(a\,(x\,,\,y)\,A_1\,(x\,,\,y)+b\,(x\,,\,y)\,A_2\,(x\,,\,y))\,, \qquad (5.3)$$

$$A_{[2]}\,(x\,,\,y) \stackrel{\mathrm{def}}{=} \frac{1}{c(x\,,\,y)^2}\,(-b\,(x\,,\,y)\,A_1\,(x\,,\,y)+a\,(x\,,\,y)\,A_2\,(x\,,\,y))\,. \qquad (5.4)$$

We denote $(A^u_{[1]}(\xi\,,\,\eta)\,,\,A^u_{[2]}(\xi\,,\,\eta))$ by $\widetilde{\mathbf{A}}^u(\xi\,,\,\eta)$.

Using $\partial a/\partial x=-\partial b/\partial y$ and $\partial a/\partial y=\partial b/\partial x$ by the Cauchy-Riemann relations (4.1) and (4.2), we know that the magnetic field determined by the new vector potentials vanishes on Ω_R:

LEMMA 5.1. *For* $(\xi,\eta)=(\phi(x\,,\,y)\,,\,\psi(x\,,\,y))\in\mathbb{R}^2_{2R}$ *(*$(x\,,\,y)\in\Omega_R$*),*

$$0=\frac{1}{c^2(x\,,\,y)}\left(\frac{\partial A_2}{\partial x}(x\,,\,y)-\frac{\partial A_1}{\partial y}(x\,,\,y)\right)=\frac{\partial A^u_{[2]}}{\partial\xi}(\xi\,,\,\eta)-\frac{\partial A^u_{[1]}}{\partial\eta}(\xi\,,\,\eta).$$

And, on $C_0^\infty(\mathbb{R}^2_{2R})$*,*

$$\Big[Up^S_{[1]}U^{-1}-\mathsf{q}A^u_{[1]}\,,\,Up^S_{[2]}U^{-1}-\mathsf{q}A^u_{[2]}\Big]=i\mathsf{q}\left(\frac{\partial A^u_{[2]}}{\partial\xi}-\frac{\partial A^u_{[1]}}{\partial\eta}\right)=0.$$

Remark 5.1. We defined $(A_{[1]}\,,\,A_{[2]})$ so that 1-form A_1dx+A_2dy is equal to $A^u_{[1]}d\xi+A^u_{[2]}d\eta$, i.e., $A_1dx+A_2dy=A^u_{[1]}d\xi+A^u_{[2]}d\eta$.

Functions, A_j $(j=1,2)$ and $A^u_{[j]}$ $(j=1,2)$, are real-valued measurable with respect to measure spaces, $(\Omega_R\,,\,dxdy)$ and $(\mathbb{R}^2_{2R}\,,\,d\xi d\eta/c^u(\xi\,,\,\eta)^2)$ respectively. And besides, A_j $(j=1,2)$ and $A^u_{[j]}$ $(j=1,2)$ are finite almost everywhere with respect to $dxdy$ and $d\xi d\eta/c^u(\xi\,,\,\eta)^2$ respectively. So, we can define self-adjoint operators acting in $L^2(\Omega_R)$ and $L^2_c(\mathbb{R}^2_{2R})$ as multiplication operators (see [8, Example 5.11 and its remark 1] or [18, §VIII.3, Proposition 1 and the proof of Proposition 3]). We denote them by the same symbols, i.e., for $j=1,2$

$$(A_jf)\,(x\,,\,y)\stackrel{\mathrm{def}}{=}A_j(x\,,\,y)f(x\,,\,y),$$

$$D(A_j)\stackrel{\mathrm{def}}{=}\left\{f\in L^2(\Omega_R)\,\middle|\,\int\!\!\int_{\Omega_R}|A_j(x\,,\,y)f(x\,,\,y)|^2<\infty\right\},$$

$$\Big(A^u_{[j]}f^u\Big)\,(\xi\,,\,\eta)\stackrel{\mathrm{def}}{=}A^u_{[j]}(\xi\,,\,\eta)f^u(\xi\,,\,\eta),$$

$$D(A^u_{[j]})\stackrel{\mathrm{def}}{=}\left\{f^u\in L^2_c(\mathbb{R}^2_{2R})\,\middle|\,\int\!\!\int_{\mathbb{R}^2_{2R}}\frac{|A^u_{[j]}(\xi\,,\,\eta)f^u(\xi\,,\,\eta)|^2}{c^u(\xi\,,\,\eta)^2}<\infty\right\}.$$

Here we modify Ω_R. We set $\Omega_1^{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \Omega_R \setminus \{(x\,,0)\,|\,R \le x\}$, and $\Omega_2^{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \Omega_R \setminus \{(0\,,y)\,|\,R \le y\}$. We also modify $\mathbb{R}^2_{2R}$ as $\mathbb{R}^{2,\mathrm{mod}}_{[1]} \stackrel{\mathrm{def}}{=} \mathbb{R}^2_{2R} \setminus \{(\xi\,,0)\,|\,2R < \xi\}$, and $\mathbb{R}^{2,\mathrm{mod}}_{[2]} \stackrel{\mathrm{def}}{=} \mathbb{R}^2_{2R} \setminus \{(0\,,\eta)\,|\,0 < \eta\}$.

Remark 5.2. $J_{out}\,(\Omega_1^{\mathrm{mod}}) = \mathbb{R}^{2,\mathrm{mod}}_{[1]}$, and $J_{out}\,(\Omega_2^{\mathrm{mod}}) = \mathbb{R}^{2,\mathrm{mod}}_{[2]}$.

Then, since $\mathbb{R}^{2,\mathrm{mod}}_{[1]}$ and $\mathbb{R}^{2,\mathrm{mod}}_{[2]}$ are simply connected, and Lemma 5.1 holds, we can use Poincaré's lemma, and there exist two functions $\Lambda^u_{[j]} \in C_0^\infty\left(\mathbb{R}^{2,\mathrm{mod}}_{[j]}\right)$ $(j = 1, 2)$ such that $\widetilde{\mathbf{A}}^u = \nabla\Lambda^u_{[j]}$ on $\mathbb{R}^{2,\mathrm{mod}}_{[j]}$ for $j = 1, 2$.

It is easy to prove the following lemmas, Lemmas 5.2 and 5.3, by simple calculations with well-known methods, but they are technical ones for our purpose that we show the Aharonov-Bohm phase in Weyl's CCR caused by inequivalence between Weyl's CCR and Heisenberg's.

LEMMA 5.2.
(a) $C_0^\infty(\Omega_j^{mod})$ *$(j = 1, 2)$ is dense in $L^2(\Omega_R)$.*
(b) $C_0^\infty(\mathbb{R}^{2,mod}_{[j]})$ *$(j = 1, 2)$ is dense in $L^2_c(\mathbb{R}^2_{2R})$.*

LEMMA 5.3.
(a) *All poles of $1/c(x\,,y)$ on Ω_R are $(\pm R\,,0)$, namely all poles of $1/c^u(\xi\,,\eta)$ on $\mathbb{R}^2$ are $(\pm 2R\,,0)$.*
(b) $C_0^\infty(\mathbb{R}^{2,mod}_{[1]})$ *is a core for $Up^S_{[1]}U^{-1}$ in $L^2_c(\mathbb{R}^2_{2R})$.*

Remark 5.3. Lemma 5.3 holds in the case when $j = 1$ only.

Now, we consider $\Lambda_j(x\,,y) \stackrel{\mathrm{def}}{=} \Lambda^u_{[j]}(\phi(x\,,y)\,,\psi(x\,,y))$ $(j = 1, 2)$. It is clear that Λ_j is measurable (actually $\Lambda_j \in C^\infty(\Omega_j^{\mathrm{mod}})$). So, by well-known fact, we can define the following self-adjoint operators, Λ_j $(j = 1, 2)$: for $j = 1, 2$

$$
\begin{aligned}
&(\Lambda_j f)\,(x\,,y) \stackrel{\mathrm{def}}{=} \Lambda_j(x\,,y) f(x\,,y),\\
&D(\Lambda_j) \stackrel{\mathrm{def}}{=} \left\{ f \in L^2(\Omega_R) \,\middle|\, \int\!\!\int_{\Omega_R} |\Lambda_j(x\,,y) f(x\,,y)|^2 < \infty \right\}.
\end{aligned}
$$

(see [8, Example 5.11 and its remark 1] or [18, §VIII.3, Proposition 1 and the proof of Proposition 3]).

The following theorem means that the mv-momentum w.r.t. ξ-axis is realized as a self-adjoint operator in the same way as Arai's proof[4].

THEOREM 5.4. *$P_{[1]} \stackrel{\mathrm{def}}{=} p^S_{[1]} - \mathsf{q}A_{[1]}$ is essentially self adjoint on $C_0^\infty(\Omega_1^{mod})$.*

The following theorem means that the mv-momentum w.r.t. η-axis is realized as a self-adjoint operator with a small difference from Arai's proof [4].

THEOREM 5.5. *$P_{[2]} \stackrel{\mathrm{def}}{=} p^S_{[2]} - \mathsf{q}A_{[2]}$ is essentially self adjoint on $e^{i\mathsf{q}\Lambda_{[2]}}D(p^S_{[2]})$.*

Proof. Since $e^{i\mathfrak{q}\Lambda_{[2]}}$ is a unitary operator, $p^S_{[2],A} \stackrel{\text{def}}{=} e^{i\mathfrak{q}\Lambda_{[2]}} p^S_{[2]} e^{-i\mathfrak{q}\Lambda_{[2]}}$ is a self-adjoint operator, whose domain is $D(p^S_{[2],A}) = e^{i\mathfrak{q}\Lambda_{[2]}} D(p^S_{[2]})$. For $f \in D(p^S_{[2],A})$, $p^S_{[2]} e^{-i\mathfrak{q}\Lambda_{[2]}} f$ is well-defined. So, we get by Lemma 4.6

$$\begin{aligned} & \int\int_{\Omega_R} dxdy\, \overline{g(x\,,\,y)} e^{i\mathfrak{q}\Lambda_{[2]}(x\,,y)} \left(p^S_{[2]} e^{-i\mathfrak{q}\Lambda_{[2]}} f \right)(x\,,\,y) \\ = \ & \frac{1}{i} \int\int_{\mathbb{R}^2_{2R}} d\xi d\eta\, \frac{\overline{(Ug)(\xi\,,\,\eta)}}{c^u(\xi\,,\,\eta)^2} e^{i\mathfrak{q}\Lambda^u_{[2]}(\xi\,,\eta)} c^u(\xi\,,\,\eta) \frac{\partial e^{-i\mathfrak{q}\Lambda^u_{[2]}} f^u/c^u}{\partial \eta}(\xi\,,\,\eta) \\ = \ & \int\int_{\Omega_R} dxdy\, \overline{g(x\,,\,y)} \left(P_{[2]} f \right)(x\,,\,y), \end{aligned}$$

which implies that $P_{[2]} = e^{i\mathfrak{q}\Lambda_{[2]}} p^S_{[2]} e^{-i\mathfrak{q}\Lambda_{[2]}}$ on $e^{i\mathfrak{q}\Lambda_{[2]}} D(p^S_{[2]})$. Therefore, our theorem follows from this equality. Q.E.D.

For $\xi, \eta,\ s,t \in \mathbb{R}$, we define two curves $C_\pm(\xi, \eta\,;\ s,t)$ in $\mathbb{R}^2$ from $(\xi\,,\,\eta) \in \mathbb{R}^2_{2R}$ to $(\xi{+}s\,,\,\eta{+}t)$ by $C_-(\xi, \eta\,;\ s,t) \stackrel{\text{def}}{=} \{(\xi + \theta s\,,\,\eta) \,|\, 0 \le \theta \le 1\} \cup \{(\xi + s\,,\,\eta + \theta t) \,|\, 0 \le \theta \le 1\}$, and $C_+(\xi, \eta\,;\ s,t) \stackrel{\text{def}}{=} \{(\xi\,,\,\eta + \theta t) \,|\, 0 \le \theta \le 1\} \cup \{(\xi + \theta s\,,\,\eta + t) \,|\, 0 \le \theta \le 1\}$. Then we define a rectangle $C(\xi, \eta\,;\ s,t)$ by $C(\xi, \eta\,;\ s,t) \stackrel{\text{def}}{=} C_-(\xi, \eta\,;\ s,t) - C_+(\xi, \eta\,;\ s,t)$, which is the rectangular closed curve: $(\xi\,,\,\eta) \to (\xi + s\,,\,\eta) \to (\xi + s\,,\,\eta + t) \to (\xi\,,\,\eta + t) \to (\xi\,,\,\eta)$.

In the same way as Arai's, for every fixed $s, t \in \mathbb{R}$, we define a function $\Phi^u_{s,t}$ on $\mathbb{R}^2_{2R}$ by

$$\Phi^u_{s,t}(\xi\,,\,\eta) \stackrel{\text{def}}{=} \int_{C(\xi,\eta\,;\,s,t)} \widetilde{\mathbf{A}}^u(\widetilde{\mathbf{r}}) \cdot d\widetilde{\mathbf{r}},$$

where $\widetilde{\mathbf{r}} \stackrel{\text{def}}{=} (\xi\,,\,\eta)$. Here we have to pay our attentions to the following: for $(\xi_0\,,\,\eta_0) \in \Delta_{s,t} \equiv (\mathbb{R} \setminus \{\pm 2R\,,\,\pm 2R - s\}) \times (\mathbb{R} \setminus \{0\,,\,-t\})$, if $C(\xi_0, \eta_0\,;\ s,t) \subsetneq \mathbb{R}^2_{2R}$, i.e., $C(\xi_0, \eta_0\,;\ s,t) \cap [-2R\,,\,2R] = \phi$, then it is evident that $\displaystyle\int_{C(\xi_0,\eta_0\,;\,s,t)} \widetilde{\mathbf{A}}^u(\widetilde{\mathbf{r}}) \cdot d\widetilde{\mathbf{r}}$ is finite. For the other cases, namely $C(\xi_0, \eta_0\,;\ s,t) \cap [-2R\,,\,2R] = \{(\xi_0\,,\,\eta_0 + \theta_0 t)\}$, $\{(\xi_0 + s\,,\,\eta_0 + \theta_0 t)\}$ or $\{(\xi_0\,,\,\eta_0 + \theta_0 t)\,,\,(\xi_0 + s\,,\,\eta_0 + \theta_0 t)\}$ for some $0 < \theta_0 < 1$, $A^u_{[2]}(\xi_0\,,\,\eta')$ and $A^u_{[2]}(\xi_0 + s\,,\,\eta')$ have discontinuous points, $(\xi_0\,,\,\eta_0 + \theta_0 t)$ and $(\xi_0 + s\,,\,\eta_0 + \theta_0 t)$ respectively. However $A^u_{[2]}(\xi_0\,,\,\eta')$ and $A^u_{[2]}(\xi_0 + s\,,\,\eta')$ are integrable functions of η' on $[\eta_0\,,\,\eta_0 + t]$ under our assumption (5.1). So we know that $\displaystyle\int_{C(\xi_0,\eta_0\,;\,s,t)} \widetilde{\mathbf{A}}^u(\widetilde{\mathbf{r}}) \cdot d\widetilde{\mathbf{r}}$ is also finite. Thus, the function $\Phi^u_{s,t}(\xi\,,\,\eta)$ is defined on $\Delta_{s,t}$. Then, $\Phi^u_{s,t}(\xi\,,\,\eta)$ is a real-valued measurable function with respect to the measure space $(\mathbb{R}^2_{2R}\,,\,d\xi d\eta/c^u(\xi\,,\,\eta)^2)$. And besides, $\Phi^u_{s,t}(\xi\,,\,\eta)$ is finite almost everywhere with respect to $d\xi d\eta/c^u(\xi\,,\,\eta)^2$. Here we note that the poles, $(\pm 2R, 0)$, of $1/c^u$ are outside $\Delta_{s,t}$. So, we can define a self-adjoint operator $\Phi^u_{s,t}$ as multiplication operator on $L^2_c(\mathbb{R}^2_{2R})$, i.e.,

$$\begin{aligned} D(\Phi^u_{s,t}) &\stackrel{\text{def}}{=} \left\{ f^u \in L^2_c(\mathbb{R}^2_{2R}) \,|\, \Phi^u_{s,t} f^u \in L^2_c(\mathbb{R}^2_{2R}) \right\}, \\ \left(\Phi^u_{s,t} f^u \right)(\xi\,,\,\eta) &\stackrel{\text{def}}{=} \Phi^u_{s,t}(\xi\,,\,\eta) f^u(\xi\,,\,\eta). \end{aligned}$$

(see [8, Example 5.11 and its Remark 1] or [18, §VIII.3, Proposition 1 and the proof of Proposition 3]).

Remark 5.4. If $C(\xi,\eta;\, s,t) \subsetneq \mathbb{R}^2_{2R}$ $((\xi\,,\,\eta) \in \Delta_{s,t})$, then it is clear that $J_{out}^{-1}C(\xi,\eta;\, s,t) \subsetneq \Omega_R$. Thus, $U^{-1}\Phi^u_{s,t}U$ means the flux going through the interior of $J_{out}^{-1}C(\xi,\eta;\, s,t)$ because the Stokes theorem holds. If $C(\xi,\eta;\, s,t) \cap [-2R\,,\,2R] \neq \phi$ $((\xi\,,\,\eta) \in \Delta_{s,t})$, then, by the definition of $\Delta_{s,t}$, the points in $C(\xi,\eta;\, s,t) \cap [-2R\,,\,2R]$ are not poles of $A^u_j(\xi\,,\,\eta)$. So, in this case, although $J_{out}^{-1}C(\xi,\eta;\, s,t)$ is not a loop in Ω_R, we can make a loop C in $\mathbb{R}^2$ by adding a curve in D_R to $J_{out}^{-1}C(\xi,\eta;\, s,t)$. So, $U^{-1}\Phi^u_{s,t}U$ can be extended an operator acting in $L^2(\mathbb{R}^2)$, and it then means the flux going through the interior C.

Set $\Phi^{\widetilde{\mathbf{A}}}_{s,t}(x\,,\,y) \stackrel{\text{def}}{=} \Phi^u_{s,t}(\phi(x\,,\,y)\,,\,\psi(x\,,\,y))$, in the sense of operator, $\Phi^{\widetilde{\mathbf{A}}}_{s,t} \stackrel{\text{def}}{=} U^{-1}\Phi^u_{s,t}U$. Then we obtain our desired theorem. By this theorem, we can show that the Aharonov-Bohm phase appears in Weyl's CCR for just $\exp\left[itP_{[j]}\right]$ $(j=1,2)$:

THEOREM 5.6. *For all* $s,t \in \mathbb{R}$,

$$e^{is\overline{P_{[1]}}}e^{it\overline{P_{[2]}}} = \exp\left[-i\mathsf{q}\Phi^{\widetilde{\mathbf{A}}}_{s,t}\right]e^{it\overline{P_{[2]}}}e^{is\overline{P_{[1]}}}.$$

Proof. By Lemma 4.7 for all $f \in L^2_c(\mathbb{R}^2_{2R})$,

$$\begin{aligned}
&\left(\left(e^{isUp^S_{[1]}U^{-1}/n}e^{-is\mathsf{q}A^u_{[1]}/n}\right)^n f\right)(\xi\,,\,\eta)\\
=\;& \left(e^{isUp^S_{[1]}U^{-1}/n}e^{-is\mathsf{q}A^u_{[1]}/n}\right)^{n-\nu}\exp\left[-i\mathsf{q}\sum_{k=1}^{\nu}\frac{s}{n}A^u_{[1]}\left(\xi+\frac{ks}{n}\,,\,\eta\right)\right]f(\xi+\frac{\nu s}{n}\,,\,\eta)\\
&\times\frac{c^u(\xi+(\nu-1)s/n\,,\,\eta)}{c^u(\xi+\nu s/n\,,\,\eta)}\frac{c^u(\xi+(\nu-2)s/n\,,\,\eta)}{c^u(\xi+(\nu-1)s/n\,,\,\eta)}\cdots\\
&\cdots\frac{c^u(\xi\,,\,\eta)}{c^u(\xi+s/n\,,\,\eta)}\\
=\;&\cdots\\
=\;&\exp\left[-i\mathsf{q}\sum_{k=1}^{n}\frac{s}{n}A^u_{[1]}\left(\xi+\frac{ks}{n}\,,\,\eta\right)\right]f(\xi+s\,,\,\eta)\frac{c^u(\xi\,,\,\eta)}{c^u(\xi+s\,,\,\eta)}.
\end{aligned}$$

So, by applying Trotter's formula to the equality above,

$$\begin{aligned}
&\left(e^{isU\overline{P_{[1]}}U^{-1}}f\right)(\xi\,,\,\eta)\\
=\;&\exp\left[-i\mathsf{q}\int_0^s A^u_{[1]}(\xi+\xi'\,,\,\eta)\,d\xi'\right]f(\xi+s\,,\,\eta)\frac{c^u(\xi\,,\,\eta)}{c^u(\xi+s\,,\,\eta)}.
\end{aligned}\tag{5.5}$$

In the same way, by using Lemma 4.7 again, for $f \in L^2_c(\mathbb{R}^2_{2R})$ and $t \in \mathbb{R}$ we have

$$\begin{aligned}
&\left(e^{itU\overline{P_{[2]}}U^{-1}}f\right)(\xi\,,\,\eta)\\
=\;&\exp\left[-i\mathsf{q}\int_0^t A^u_{[2]}(\xi\,,\,\eta+\eta')\,d\eta'\right]f(\xi\,,\,\eta+t)\frac{c^u(\xi\,,\,\eta)}{c^u(\xi\,,\,\eta+t)}.
\end{aligned}\tag{5.6}$$

By (5.5) and (5.6), we have

$$
\begin{aligned}
& \left(e^{isU\overline{P_{[1]}}U^{-1}}e^{itU\overline{P_{[2]}}U^{-1}}f\right)(\xi\,,\,\eta) \\
= \;& \exp\left[-i\mathsf{q}\int_0^s A^u_{[1]}(\xi+\xi'\,,\,\eta)d\xi'\right] \\
& \quad \left(\exp\left[-i\mathsf{q}\int_0^t A^u_{[1]}(\xi+s\,,\,\eta+\eta')d\eta'\right]\right. \\
& \qquad\qquad \left.\times f(\xi+s\,,\,\eta+t)\frac{c^u(\xi+s\,,\,\eta)}{c^u(\xi+s\,,\,\eta+t)}\right)\frac{c^u(\xi\,,\,\eta)}{c^u(\xi+s\,,\,\eta)} \\
= \;& \exp\left[-i\mathsf{q}\int_{C_-(\xi,\eta\,;\,s,t)}\widetilde{\mathbf{A}}^u(\widetilde{\mathbf{r}})\cdot d\widetilde{\mathbf{r}}\right]\,f(\xi+s\,,\,\eta+t)\frac{c^u(\xi\,,\,\eta)}{c^u(\xi+s\,,\,\eta+t)},
\end{aligned}
\tag{5.7}
$$

and, similarly,

$$
\begin{aligned}
& \left(e^{itU\overline{P_{[2]}}U^{-1}}e^{isU\overline{P_{[1]}}U^{-1}}f\right)(\xi\,,\,\eta) \\
= \;& \exp\left[-i\mathsf{q}\int_{C_+(\xi,\eta\,;\,s,t)}\widetilde{\mathbf{A}}^u(\widetilde{\mathbf{r}})\cdot d\widetilde{\mathbf{r}}\right]\,f(\xi+s\,,\,\eta+t)\frac{c^u(\xi\,,\,\eta)}{c^u(\xi+s\,,\,\eta+t)}.
\end{aligned}
\tag{5.8}
$$

Our theorem follows from (5.7) and (5.8). Q.E.D.

As a conclusion of our assertion in this paper,

THEOREM 5.7. *The system $\{\overline{P}_j, q^S_{j'}\}_{j,j'=\xi,\eta}$ of self-adjoint operators fulfills Heisenberg's CCR on a certain dense domain in $L^2(\Omega_R)$, on the other hand $\{\overline{P}_j, q^S_{j'}\}_{j,j'=\xi,\eta}$ fulfills Weyl's CCR if and only if $\Phi^{\widetilde{\mathbf{A}}}_{s,t}(x\,,\,y)$ is a function having a value of an integer multiple of $2\pi/\mathsf{q}$ for every $C(x,y\,;\,s,t)\equiv J_{out}^{-1}C(\xi,\eta\,;\,s,t)$ in Ω_R.*

Proof. The first part of this theorem is clear by Lemma 5.1. We can prove the last part of this theorem in the same way as Theorems 5.4 and 5.5. Q.E.D.

6 Open Problem

We constructed the Scrödinger representation on Ω_R, and extended Reeh's and Arai's results to the case where the "hole" has a finite radius. We used a disc for the hole, but the disc is not essential for our argument. Our method may be valid over Riemann surfaces conformally equivalent to $\mathbb{R}^2_{2R}$ with a conformal map which plays a role of J_{out}.

For some general holes, we realize that our method will be valid over Riemann surfaces conformally equivalent to the space removed finite segments parallel to ξ-axis and finite points from $\mathbb{R}^2$, which reminds us the following theorem: *Let $\mathcal{R}$ be a non-compact Riemann surface of planar character. Then, there exists a biholomorphic function $J:\mathcal{R}\to J(\mathcal{R})$ and $x_0\in\mathcal{R}$ such that $f(x_0)=\infty$, and $F=(\mathbb{C}\cup\{\infty\})\backslash J(\mathcal{R})$ is a bounded closed set in $\mathbb{C}$ and each connected components* *of F is a segment parallel to the real axis or a point* (see [14, Theorem 7.3]). Can we construct quantum mechanics on the Riemann surface $\mathcal{R}$ in the light of the canonical quantization? This is an open problem.

In material physics, physicists are interested in several physical phenomena on the surfaces of materials with some defects, and find very interesting and important phenomena on such surfaces [9, 10, 11, 15, 27]. We can regard the defects on a surface as "holes" of the surface [16, §9]. Thus, our open problem is the first step to construct fundamental and mathematical theories for quantum mechanics (quantum physics) on Riemann surfaces such that they are useful in explaining the above experimental phenomena.

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