# Hawking–Unruh Temperature and Einstein Causality in Anti–de Sitter Space–Time

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If the vacuum is passive for uniformly accelerated observers in anti–de Sitter space–time (*i.e.* cannot be used by them to operate a *perpetuum mobile*), they will (a) register a unique value of the Hawking–Unruh temperature and (b) find that products of observables which are localized in complementary wedge–shaped regions necessarily commute with each other in the vacuum state. Thus the stability properties of the vacuum induce a "geodesic causal structure" on this space–time which seems incompatible with true interaction.

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Quantum field theory in anti–de Sitter space–time (AdS) has recently received considerable attention [1]. It seems therefore worthwhile to clarify in a model independent setting the universal properties of such theories which are implied by generally accepted and physically meaningful constraints. We report here on the results of an investigation of this question [2] which brought to light an intimate relation between stability properties of the vacuum, the value of the Hawking–Unruh temperature registered by uniformly accelerated observers, and locality properties of their respective observables. As will be explained, these results hold in any theory which allows one to specify the spacetime localization properties of observables, and they suggest that quantum field theories on AdS cannot describe true interaction.

We consider here AdS of any dimension $n \geq 3$. It can conveniently be described in terms of Cartesian coordinates in the ambient space $\mathbf{R}^{n+1}$ as the quadric

$$\mathrm{AdS}_n = \{x \in \mathbf{R}^{n+1} : x^2 \equiv x_0^2 - x_1^2 - \ldots - x_{n-1}^2 + x_n^2 = R^2\}$$

with metric $g = \mathrm{diag}(1, -1, \ldots, -1, 1)$ in diagonal form. Its isometry group is $\mathrm{O}(2, n-1)$ whose identity component will be denoted by $\mathrm{SO}_0(2, n-1)$.

Any quantum field theory on AdS has to determine the following structure, which is indispensible for its physical interpretation:
(i) A Hilbert space $\mathcal{H}$, describing the physical states, and a continuous unitary representation $U$ of the symmetry group $\mathrm{SO}_0(2, n-1)$, acting on $\mathcal{H}$ [3].
(ii) A set of operators which describe the observables of the theory. Since these operators can be added and multiplied, they generate an algebra $\mathcal{A}$ which will be called the algebra of observables [4].

It is a fundamental feature of quantum field theory and its modern ramifications that the spacetime localization properties of observables enter into the formalism in an essential way. One can tell whether an observable is localized about a spacetime point, a string, loop, disk etc. Thus, given any region $\mathcal{O} \subset \mathrm{AdS}_n$ , one can identify those observables which are localized within that region and define a corresponding subalgebra $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}$ which is generated by them. Since there exist no fewer observables if the region becomes larger, one obviously has

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2.$$

The localization of observables has to be compatible with the action of the spacetime symmetry group. If an observable $A$ is localized in the region $\mathcal{O}$, say, its image $U(\lambda)AU(\lambda)^{-1}$ under the action of $\lambda \in \mathrm{SO}_0(2, n-1)$ should be localized in the transformed region $\lambda\mathcal{O}$. The transformation properties of the observables under the action of the unitaries $U(\lambda)$ may be quite complicated, but fortunately this detailed information is not needed here. All that matters are the geometrical aspects of the transformations, which can be summarized as follows.
(iii) For each region $\mathcal{O} \subset \mathrm{AdS}_n$ and $\lambda \in \mathrm{SO}_0(2, n-1)$ one has the equality (in the set–theoretic sense)

$$U(\lambda)\mathcal{A}(\mathcal{O})U(\lambda)^{-1} = \mathcal{A}(\lambda\mathcal{O}).$$

We emphasize that we do not postulate here from the outset local commutation relations of the observables. For, in contrast to the case of globally hyperbolic space–times, the principle of Einstein causality does not provide any clues as to which observables on AdS should commute. Instead, we will derive such commutation relations from stability properties of the vacuum. Since this stability aspect sheds also new light on the phenomenon of the Hawking–Unruh temperature, we explain this point in somewhat more detail.

We begin by describing the observers in AdS which are of interest here. Let $x_O \in \mathrm{AdS}_n$ be any point and let $\lambda(t)$, $t \in \mathbf{R}$, be any one–parameter subgroup of $\mathrm{SO}_0(2, n-1)$ such that $t \to \lambda(t)x_O$ is an orthochronous curve [5]. We interpret this curve as the worldline of some observer (assumed to be male for concreteness). Depending on the initial data chosen, this observer will be geodesic or experience some constant acceleration. Points in the neighborhood of $x_O$ will in general also give rise to orthochronous curves under the action of the chosen subgroup of $\mathrm{SO}_0(2,n-1)$, and we denote by $\mathcal{W}$ the connected neighborhood of $x_O$ in $\mathrm{AdS}_n$ consisting of all such curves. The region $\mathcal{W}$, which can be all of AdS or a subset of it, is that part of the space–time where the observer can perform measurements. His observables are described by the self–adjoint elements of $\mathcal{A}(\mathcal{W})$ and his dynamics is given by $e^{\,itM} \equiv U(\lambda(t))$ with generator ("Hamiltonian") $M$ [6].

Let us discuss next what this observer can say about the properties of the vacuum, described by the unit vector $|0\rangle \in \mathcal{H}$. First, he should find that the vacuum is in a stationary state with respect to his dynamics, *i.e.*

$$M\,|0\rangle = 0 .$$

Second, he can check whether the vacuum is suitable to operate a *perpetuum mobile* of the second kind. To this end he would perform all kinds of operations in $\mathcal{W}$ which can be described by perturbations of the generator $M$ in that region. Time–dependent perturbation theory then tells us that the state of the vacuum changes under the influence of such perturbations and is described by a vector $V|0\rangle$ for some unitary operator $V \in \mathcal{A}(\mathcal{W})$ after the perturbation has been turned off. Since this challenge of the second law ought to fail, the observer should find that, no matter what he does, the energy of the final state is no less than that of the initial vacuum, *i.e.* he cannot extract energy from the vacuum. In formula form,

$$\langle 0|\, V^* M V\, |0\rangle \geq \langle 0|\, M\, |0\rangle = 0$$

for all *unitary* operators $V \in \mathcal{A}(\mathcal{W})$. This property of a state — not being able to perform work in a cyclic process — is called passivity [7].

Finally, the observer can test with the help of time averages of his observables (yielding order parameters) whether the vacuum is a mixture of different phases. Since the vacuum is the most elementary system, the answer should be negative, *i.e.* all order parameters should have sharp values. This is the content of the relation (mixing property)

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, \langle 0|A(t)B|0\rangle = \langle 0|A|0\rangle\, \langle 0|B|0\rangle \,,$$

for $A, B \in \mathcal{A}(\mathcal{W})$, where $A(t) \equiv e^{\,itM} A e^{\,-itM}$. These basic features of the vacuum can be summarized as follows. (iv) The vacuum $|0\rangle$ is, for all geodesic and all uniformly accelerated observers, a stationary, passive and mixing state.

Let us now turn to the analysis of the implications of these assumptions. Here we profit from a deep result of Pusz and Woronowicz for arbitrary quantum dynamical systems [7]. In the present context this result says that the vacuum vector $|0\rangle$ is, as a consequence of its passivity and mixing properties, either a ground state for the dynamics $M$ of the above observer, or it satisfies, for some *a priori* unknown $\beta \geq 0$, the Kubo–Martin–Schwinger (KMS) condition, which can be presented in the form [8]

$$\langle 0|B e^{\,-\beta M} A^*|0\rangle = \langle 0|A^* B|0\rangle ,$$

for all $A, B \in \mathcal{A}(\mathcal{W})$. In the latter case the respective observer would interpret the vacuum as a thermal equilibrium state at temperature $\beta^{-1}$.

It is convenient here to proceed to another formulation of the above result. Making use of the fact that $\langle 0|A^*B|0\rangle = \overline{\langle 0|B^*A|0\rangle}$ and that the set of vectors $\mathcal{A}(\mathcal{W})\,|0\rangle$ is dense in $\mathcal{H}$ [2], the above relation allows one to define an anti–unitary operator $J$ on $\mathcal{H}$ by setting [9]

$$JA|0\rangle \equiv e^{\,-(\beta/2)M} A^*|0\rangle \quad \text{for} \quad A \in \mathcal{A}(\mathcal{W}).$$

Our main task will be to determine the specific properties of this operator. In order to simplify the necessary computations, we consider the particular choice of region

$$\mathcal{W} = \{x \in \mathrm{AdS}_n \,:\, x_1 > |x_0|\,, x_n > 0\}$$

on which the one–parameter subgroup of boosts $\lambda_{01}(t)$, $t \in \mathbf{R}$, in the 0–1–plane acts in an orthochronous manner. For any $x_O \in \mathcal{W}$, the curve $t \rightarrow \lambda_{01}(t)x_O$ is the worldline of some observer, as described above.

We shall determine the temperature felt by this observer. Since the generator $M_{01}$ of his dynamics is transfomed into $-M_{01}$ by the adjoint action of the rotation $e^{\,i\pi M_{12}}$ in the 1–2–plane, it cannot be a positive operator. Hence $|0\rangle$ is not a ground state for this observer; it must therefore satisfy the KMS–condition for some $\beta \geq 0$. So what is the value of $\beta$ ?

To answer this question, we apply the methods in [11]. We pick a region $\mathcal{O}$ such that $\lambda_{02}(s)\,\mathcal{O} \subset \mathcal{W}$ for the boosts $\lambda_{02}(s)$ in the 0–2–plane with sufficiently small parameter $s$. By the group law in $\mathrm{SO}_0(2,n-1)$ we have

$$e^{\,itM_{01}}\, e^{\,isM_{02}} = e^{\,is(\cosh(t)M_{02} + \sinh(t)M_{12})}\, e^{\,itM_{01}} .$$

Taking into account that the boost $e^{\,isM_{02}}$ is the dynamics of some other observer, we infer from (iv) that $|0\rangle$ is invariant under its action. Thus we get for any vector $|\,\phi\rangle \in \mathcal{H}$ and operator $A^* \in \mathcal{A}(\mathcal{O})$

$$\begin{aligned} &\langle \phi|\, e^{\,itM_{01}}\, e^{\,isM_{02}} A^* e^{\,-isM_{02}}|0\rangle \\ &= \langle \phi|\, e^{\,is(\cosh(t)M_{02} + \sinh(t)M_{12})}\, e^{\,itM_{01}} A^*|0\rangle . \end{aligned}$$

At this point we are in the position to combine the KMS–condition with a basic result in the representation theory of Lie groups, due to Nelson [10]. In the present situation this result can be stated as follows: There exists a dense set of vectors $|\,\phi\rangle \in \mathcal{H}$ such that the corresponding vector functions

$$u, v \rightarrow e^{\,i(uM_{02} + vM_{12})}\, |\,\phi\rangle$$

are analytic in a fixed neighborhood of the origin of $\mathbf{C}$.

Returning to the analysis of the above equality, we note that according to condition (iii) the operator $e^{isM_{02}}A^*e^{-isM_{02}}$ appearing in the matrix element on the left hand side is an element of $\mathcal{A}(\lambda_{02}(s)\mathcal{O}) \subset \mathcal{A}(\mathcal{W})$. So, in view of the KMS–condition, we can continue this term analytically in $t$ to $i\beta/2$. In the expression on the right hand side there appears a product of two unitary operators which both depend on $t$. If we choose for $|\,\phi\rangle$ a Nelson vector, as described above, the first operator, standing next to $\langle\phi|$, can be analytically continued in $t$ to $i\beta/2$, provided $s$ is sufficiently small. The second operator, standing next to $A^*|0\rangle$, can likewise be continued to $i\beta/2$ by the KMS–condition. So we arrive at

$$\begin{aligned}&\langle\phi|\,e^{-(\beta/2)M_{01}}\,e^{isM_{02}}A^*e^{-isM_{02}}|0\rangle\\&=\langle\phi|\,e^{is(\cos(\beta/2)M_{01}+i\sin(\beta/2)M_{12})}e^{-(\beta/2)M_{01}}A^*|0\rangle.\end{aligned}$$

Bearing in mind the definition of $J$ and the invariance of the vacuum under the boosts $e^{-isM_{02}}$, we can bring this relation into the more transparent form

$$\langle\phi|\,Je^{isM_{02}}A|0\rangle=\langle\phi|\,e^{is(\cos(\beta/2)M_{02}+i\sin(\beta/2)M_{12})}JA|0\rangle.$$

As this holds for the dense sets of vectors $|\,\phi\rangle$ and $A|0\rangle$, respectively, we conclude that

$$Je^{isM_{02}}=e^{is(\cos(\beta/2)M_{02}+i\sin(\beta/2)M_{12})}J.$$

The operator on the left hand side is anti–unitary, so the same must be true for the operator on the right hand side. This is only possible if $\beta$ is an *integer* multiple of $2\pi$, for otherwise the operator appearing in the exponential function is not skew-adjoint. By a more refined functional analytic argument one can restrict $\beta$ even further and show that its only possible value is $\beta = 2\pi$ [2].

Proceeding to the proper time scale of the observer [6], we conclude that he is exposed to the Hawking–Unruh temperature $(1/2\pi)((\dot{\lambda}_{01}(0)\,x_O)^2)^{-1/2}$, in accordance with the value found in model computations [12]. By similar arguments one can compute the temperature felt by other observers, such as geodesic ones, for whom it is zero [2]. Thus we have established the following general fact:

*Each geodesic or uniformly accelerated observer testing the vacuum state in AdS finds a universal value of the Hawking–Unruh temperature which depends only on his particular orbit.*

This result is entirely a consequence of the passivity of the vacuum. Hence this state is the only one which is passive for all observers.

Having computed the value of $\beta$, let us return now to the analysis of $J$. Plugging $\beta = 2\pi$ into the above equality for $J$, we see that for small $s$

$$Je^{isM_{02}}=e^{-isM_{02}}J.$$

This relation can be extended to arbitrary $s$ by iteration, if one decomposes $e^{isM_{02}}$ into an $m$–fold product $(e^{i(s/m)M_{02}})^m$ for sufficiently large $m$.

In a similar manner one can determine the intertwining properties of $J$ with other one–parameter subgroups of $\mathrm{SO}_0(2, n-1)$, and thereby with all unitaries $U(\lambda)$. We skip these computations and only state the result [2]:

$$JU(\lambda)=U(\theta\lambda\theta)J,$$

where $\theta$ is the reflection which changes the sign of the 0–1–coordinates of the points in $\mathrm{AdS}_n$. This result also allows us to show explicitly that $J^2 = 1$ [9]: For the intertwining relation and the anti–unitarity of $J$ imply

$$Je^{-(\beta/2)M_{01}}=e^{(\beta/2)M_{01}}J,$$

hence we have, in view of the defining equation for $J$,

$$\begin{aligned}J^2A\Omega&=Je^{-(\beta/2)M_{01}}A^*\Omega=e^{(\beta/2)M_{01}}JA^*\Omega\\&=e^{(\beta/2)M_{01}}e^{-(\beta/2)M_{01}}A^{**}\Omega=A\Omega,\end{aligned}$$

for all $A\in\mathcal{A}(\mathcal{W})$. Summing up, we have established the following analogue of the TCP–theorem:

*The unitary representation $U$ of $SO_0(2, n-1)$ extends to a representation of $SO(2, n-1)$, in which the reflection $\theta$ is implemented by the anti–unitary operator $J$.*

This result is a purely group–theoretic statement which does not say anything about the adjoint action of $J$ on the observables. In order to gain insight into the nature of this latter action, we consider the observables which are localized in the region

$$\mathcal{W}'\equiv\{x\in\mathrm{AdS}_n : -x_1>|x_0|, x_n>0\}.$$

Since the regions $\mathcal{W}$ and $\mathcal{W}'$ are obtained by intersecting opposite wedge–shaped regions in the ambient space with $\mathrm{AdS}_n$, we call them opposite wedges.

By a rotation in the 1–2–plane about the angle $\pi$, the region $\mathcal{W}$ is transformed into $\mathcal{W}'$, and *vice versa.* Thus, if $A\in\mathcal{A}(\mathcal{W})$ one has $A'\equiv e^{i\pi M_{12}}Ae^{-i\pi M_{12}}\in\mathcal{A}(\mathcal{W}')$, cf. (iii). By the intertwining relation for $J$, we have $Je^{i\pi M_{12}}=e^{-i\pi M_{12}}J=e^{i\pi M_{12}}J$, where we used the fact that $e^{i2\pi M_{12}}=1$ [3]. Finally, the group structure implies that $e^{i\pi M_{12}}e^{-(\beta/2)M_{01}}=e^{(\beta/2)M_{01}}e^{i\pi M_{12}}$. Combing these facts, we can compute

$$\begin{aligned}JA'\Omega&=Je^{i\pi M_{12}}A\Omega=e^{i\pi M_{12}}JA\Omega\\&=e^{i\pi M_{12}}e^{-(\beta/2)M_{01}}A^*\Omega=e^{(\beta/2)M_{01}}e^{i\pi M_{12}}A^*\Omega\\&=e^{(\beta/2)M_{01}}A'^*\Omega.\end{aligned}$$

Picking any $A'\in\mathcal{A}(\mathcal{W}')$, $B\in\mathcal{A}(\mathcal{W})$, we obtain with the help of this result

$$\begin{aligned}&\langle 0|A'^*B|0\rangle=\overline{\langle 0|B^*A'|0\rangle}=\langle 0|B^*J^*JA'|0\rangle\\&=\langle 0|B^{**}e^{-(\beta/2)M_{01}}e^{(\beta/2)M_{01}}A'^*|0\rangle=\langle 0|BA'^*|0\rangle.\end{aligned}$$

Thus we find that the elements of $\mathcal{A}(\mathcal{W})$ and $\mathcal{A}(\mathcal{W}')$ commute in the vacuum state. It is apparent that the same statement holds also for the algebras corresponding to all transformed opposite wedges $\lambda\mathcal{W}$ and $\lambda\mathcal{W}'$. This commutativity, in the vacuum state, of observables which are localized in certain complementary regions is sometimes called *weak locality.* So we can state:

*Observables which are localized in opposite wedges of AdS are weakly local with respect to each other.*

This establishes the asserted general relation between passivity properties of the vacuum and commensurability properties of observables. In theories where the basic fields satisfy c–number commutation relations, it follows from this result that observables localized in opposite wedges commute also in the usual (operator) sense, in accord with the structure found in concrete models [13].

Before we turn to the discussion of our results, let us mention another property of $J$ which holds in all theories complying with the following stronger locality condition:
(v) Observables which are localized in opposite wedges of AdS commute with each other.

This situation prevails in the models of current interest [15]. It implies that the operator $J$ implements the geometric action of the reflection $\theta$ also on the observables,

$$J\mathcal{A}(\mathcal{W})J^* = \mathcal{A}(\theta\mathcal{W}).$$

As a matter of fact, the validity of this relation is equivalent to the strong locality condition. The proof of this result relies on methods of Tomita–Takesaki theory and will be given elsewhere [2]. There it will also be shown that mild constraints on the mulitplicities of the eigenvalues $2\pi\mathbf{N}_0$ of the generator $M_{0n}$ (existence of the partition function $\mathrm{Tr}\, e^{-\beta M_{0n}}$) imply that observables in opposite wedges are statistically (causally) independent [14].

We have established in this letter a tight relation between stability properties of the vacuum and commensurabilty properties of observables on AdS. Such relations exist also in other space–times, but they are of particular interest in the case of AdS. For they reveal that observables in opposite wedges $\mathcal{W}$ and $\mathcal{W}'$ necessarily commute with each other, either weakly or strongly.

This result appears to be very strange at first sight, since any point in $\mathcal{W}$ can be connected with any other point in $\mathcal{W}'$ by some timelike curve. Hence measurements at these points should affect each other and not be commensurable (let alone be statistically independent).

After a moment's reflection, however, one sees that the regions $\mathcal{W}$ and $\mathcal{W}'$, although causally connected in the usual sense, cannot be connected by timelike geodesics. So the following picture emerges, which explains the unexpected commensurability properties of the observables: the perturbations which are produced by observers in the region $\mathcal{W}$ move, after having left the region, along geodesics and therefore do not reach $\mathcal{W}'$. In noninteracting theories this picture is reasonable, and free field models show that it is also consistent. But it seems to be in conflict with the patterns of interaction processes.

Imagine an experimenter who produces unstable particles in the region $\mathcal{W}$. By preparing carefully the initial momenta of the particles, he can determine to some extent the spacetime point on their respective orbits where they will decay. As the decay products no longer move along the original geodesics, some can reach the region $\mathcal{W}'$ and perturb measurements there. But also the quantum effects of measurements would produce these unstable particles with a certain probability. It seems therefore impossible that all observables in $\mathcal{W}$ and $\mathcal{W}'$ are commensurable in this case. So the inevitable "geodesic" causal commutation relations of observables in AdS seem to exclude true interaction. The only way out of this clash between geometry and interaction seems to be to proceed from AdS to its covering space.

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[1] We refrain from giving references and refer the interested reader to the HEP database, where a comprehensive list of articles on this topic may be retrieved.
[2] D. Buchholz, M. Florig and S.J. Summers, (unpublished)
[3] It is sufficient here to consider the subspace of "bosonic" states, so we do not need to proceed to the covering group of the space–time symmetry group.
[4] We work in the framework of von Neumann algebras without dwelling on this point any further. For more information on the mathematical background, cf.
R. Haag, *Local Quantum Physics*, Springer 1992.
[5] It is meaningless to talk about past and future in AdS, but one can still assign a time direction to causal curves.
[6] We choose a fixed parametrization of the pertinent subgroups of $\mathrm{SO}_0(2, n-1)$. The proper time of the observer is obtained by rescaling $t$ with $((\dot{\lambda}(0)\, x_O)^2)^{1/2}$.
[7] W. Pusz and S.L. Woronowicz, Commun. Math. Phys. **58**, 273 (1978).
[8] Since $e^{-\beta M}$ is in general an unbounded operator, this relation is to be understood in the sense of quadratic forms.
[9] The operator $J$ is called modular conjugation in Tomita–Takesaki theory. Some of our explicit computations could be abbreviated by using general results from this theory.
[10] E. Nelson, Ann. Math. **70** 572 (1959).
[11] H.J. Borchers and D. Buchholz, Ann. Inst. H. Poincaré **A 70** 23 (1999).
[12] S. Deser and O. Levin, Class. Quant. Grav. **14** L163 (1997).
[13] C. Fronsdal, Phys. Rev. **D 10** 589 (1974).
[14] D. Buchholz and E.H. Wichmann, Commun. Math. Phys. **106** 321 (1986).
[15] K.H. Rehren, *Algebraic Holography*, Göttingen preprint (1999), hep-th/9905nnn