Some remarks on the problem of ergodicity of the Standard Map

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Abstract We consider the problem of removing the islands of stability in the phase space of the standard map by means of tuning the parameter. A possible construction which gives a classification of periodic chains of islands in terms of a symbolic dynamics and predicts the values of the parameters for which the island with a given symbolic code exists is suggested and discussed.

1 Introduction

Recently O.Knill issued a preprint [7] where he proves that the metric entropy of the Standard map of 2-dimensional torus, $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, given (in our notations) by the equations $F_{g,M}(x,y) = (x_1, y_1)$ where

$$\begin{aligned} x_1 &= Mx + g\cos(2\pi x) - y \bmod 1\,, \\ y_1 &= x \bmod 1\,, \end{aligned} \tag{1}$$

is positive for any integer M and for all large enough values of the parameter g. Nevertheless the problem of finding the values of the parameter g for which the map (1) is ergodic (and isomorphic to a Bernoulli shift) remains still open. It seems probable that the method, used by Knill, which do not involve the study of the dynamics of the map itself, cannot be applied to solve this problem.

In this paper we try to formulate another approach, based on the study of the partition of the phase space created by drawing suitable arcs of the stable and unstable manifolds of some periodic points (see [10]).

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Before getting into the details, let us dwell on the basic notions. Numerical experiments show that there is a nonvoid open subset $\mathcal{E} = \mathcal{E}(M)$, of the real line, such that $F = F_{g,M}$ with $g \in \mathcal{E}$ possesses an *island* of stability. This means that there is an invariant subset of positive Lebesgue measure filled up with periodic chains of KAM circles (see [1], [2], [8], [13]) . Recall that a *periodic chain of KAM circles* is an invariant set, C, of the form $C = \bigcup_{k=0}^{p-1} F^k(C_0)$, where C_0 is a smoothly embedded one-dimensional torus, $\mathbf{T}^1 = \mathbf{R}/\mathbf{Z}$, and $F^p|_{C_0}$ is smoothly conjugated to a shift: $\xi \mapsto \xi + \omega \,\mathrm{mod} 1$, with an irrational (*e.g.* Diophantine) rotational number ω. The number p is the *period* of C, the circles $C_k = F^k(C_0)\,, k = 0, 1, \ldots, p-1$. are the *KAM circles,* or the *components* of C.

Of course, if the map possesses an island it cannot be ergodic. How large is $\mathcal{E}$? It has been proven [5] that it is dense in $[a, +\infty]$ for sufficiently large a. So the problem of ergodicity reduces essentially to the estimation of the $leb(\mathcal{E} \cap [n, n+1])$ for large values of $n \in \mathbf{N}$. Here *leb* stands for the Lebesgue measure on the real line.

In this paper we suggest a method which provides us with a classification of islands and, in some naïve way, gives a necessary condition of existing of island of a given type. An essential ingredient of the construction is the notion of a *Fundamental Locus of Ellipticity (FLE).* Let F be a diffeomorphism of a surface Z onto itself. Suppose that F preserves a measure μ with positive smooth density with respect to the Lebesgue 2-dimensional measure in each local chart. Then all definitions and assertions of KAM theory are true for such F.

A subset $L \subset Z$ is called FLE if it contains at least one component of each periodic chain of KAM circles.

We are going now to formulate a sufficient condition for a subset to be a FLE. Let Z be equipped with a Riemannian metric. Let $\alpha \in]0, 2\pi[$. A *cone* with an *opening* α at a point $z \in Z$ is a subset of the tangent space to Z at z of the form: $\{\vec{v} \neq 0 \, : \, |\angle(\vec{v}, \vec{v}_0)| < \alpha/2\}$, where $\vec{v}_0$ is a fixed nonzero vector. The image of a cone at z with respect to the tangent map to F is again a cone at $F(z)$ but possibly with another opening. Let $\Lambda \subset Z$ be a (not necessarily invariant) subset. A family of cones, $K_z \subset T_z Z\,, z \in \Lambda$, is called a *uniformly shrinking cone field* on Λ with respect to the map F if there exists a number $0 < q < 1$ such that the following is true:

for any point $z \in \Lambda$, such that $F(z) \in \Lambda$, the image of K_z with respect to the tangent map to F is contained strictly in $K_{F(z)}$, and the ratio of the openings of the image of K_z and $K_{F(z)}$ is less than q.

We say that Λ is *uniformly hyperbolic* if it possesses two families of cones: $K^u_z\,, z \in \Lambda$, the unstable one, which is uniformly shrinking with respect to F, and $K^s_z\,, z \in \Lambda$, the stable one, which is uniformly shrinking with respect to F^{-1}.

One can easily deduce the following sufficient condition for a subset L to be a FLE:

Statement 1 *Let $L \subset S$ possesses the properties:*

1. *if a KAM circle has a nonvoid intersection with L, it is wholly contained in L;*
2. *there is a natural r such that the complement to $\bigcup_{k=-r}^{r} F^k(L)$ is uniformly hyperbolic.*

Then L is a FLE.

Proof. If a periodic chain of KAM circles lies completely in Λ, it possesses a uniform hyperbolic structure (see [4], [14]). This contradicts to the quasiperiodicity of the dynamics induced by the iterates F on it. QED

Note that, if the boundary of an open set L consists of arcs of the stable and unstable manifolds of some periodic hyperbolic points of F, then L satisfies automatically to the condition 1 of the Statement 1.

Further in this paper we try to construct FLE and derive consequences concerning the existence of island of different types.

Suppose we succeeded in constructing a decreasing sequence, $L_1 \supset L_2 \supset \dots$, of FLE such that $\mu(L_n) \to 0$ as $n \to \infty$. Then, as it follows from the preceding discussion, the map F does not possess islands at all.

2 A model map

We will illustrate the notions defined in Introduction taking the map 1 as a model. To make it simpler we put $M = 0$. The map $F_{g,0}$ possesses the following periodic trajectory of period 4 (we use the representation of the torus by means of the square $[-0.5, 0.5] \times [-0.5, 0.5]$ with glued opposite sides) :

$$(-1/4, 1/4) \mapsto (-1/4, -1/4) \mapsto (1/4, -1/4) \mapsto (1/4, 1/4)\,. \tag{2}$$

This trajectory is hyperbolic provided $g > 0$. In our further constructions we will use the arcs of the stable and unstable manifolds of (2).

Since we are going to make in some sense a global analysis of the situation with islands of stability, it seems reasonable to reduce the phase space by means of a symmetry. Consider the group G of transformations of $\mathbf{T}^2$ generated by the maps $S_1 : (x, y) \mapsto (1/2 - x, 1/2 - y) \bmod 1$, $S_2 : (x, y) \mapsto (-x, y + 1/2) \bmod 1$, $S_3 : (x, y) \mapsto (x + 1/2, -y) \bmod 1$. It is a symmetry group of $F_{g,0}$ in the sense that $F_{g,0}$ maps orbits of G to orbits of G (see([11]) for the details). Note that each point of $\mathbf{T}^2$, different from those belonging to (2), lies on an orbit of G which consists of four points while the set (2) consists of two two-points-orbits : $\{(-1/4, 1/4), (1/4, -1/4)\}$ and $\{(-1/4, -1/4), (1/4, 1/4)\}$.

Denote by Z the space of the orbits of G. It can be obtained by gluing together the points of the torus belonging to the same orbit. The space Z can be represented by means of the fundamental domain which is the square $[-0.25, 0.25] \times [-0.25, 0.25]$ with edges glued by the maps

$$g_1 : (-0.25, y) \mapsto (0.25, -y) \quad \text{and} \quad g_2 : (x, -0.25) \mapsto (-x, 0.25)\,. \tag{3}$$

The space Z is not a smooth surface because of two singular orbits described above: they produce $\mathbf{Z}_2$ points $A = \{(-1/4, 1/4), (1/4, -1/4)\}$ and $B = \{(-1/4, -1/4), (1/4, 1/4)\}$ of the orbifold Z. As a topological space, Z is homeomorphic to the projective plane.

Consider the map $F : Z \to Z$ induced by $F_{g,0}$ on Z. It will play a role of our model in the rest of this paper. The map F has a hyperbolic (if $g > 0$) periodic orbit $\{A, B\}$ of period 2. We will use the stable and unstable manifolds (separatrices) of $\{A, B\}$ as a building material for constructing FLE.

The map F has a reversor $R : (x, y) \mapsto (y, x)$, which satisfies the relations: $R^2 = Id$ and $RFR = F^{-1}$.

One has to notice that all the constructions and conclusions can be extended to more general families of maps, including those consisting of finite differentiable maps.

3 FLE(1)

Denote by W_0^u the arc of the unstable manifold of the point A which goes from A up to the first intersection, H, with the diagonal $x = y$ (see Fig.1). The symmetric arc, $W_0^s = R(W_0^u)$, is an arc of the stable manifold of the point A. These two arcs are seen on Fig.1 as lines which are running correspondingly close to the upper and right edges of the square $[-0.25, 0.25] \times [-0.25, 0.25]$, and meet at the point H, the latter being situated close to the right upper corner. We will use the following notations for the iterates of these beginning arcs: $W_k^u = F^k(W_0^u)$, $k = 1, 2 \dots$, and $W_k^s = F^{-k}(W_0^s)$, $k = 1, 2 \dots$.

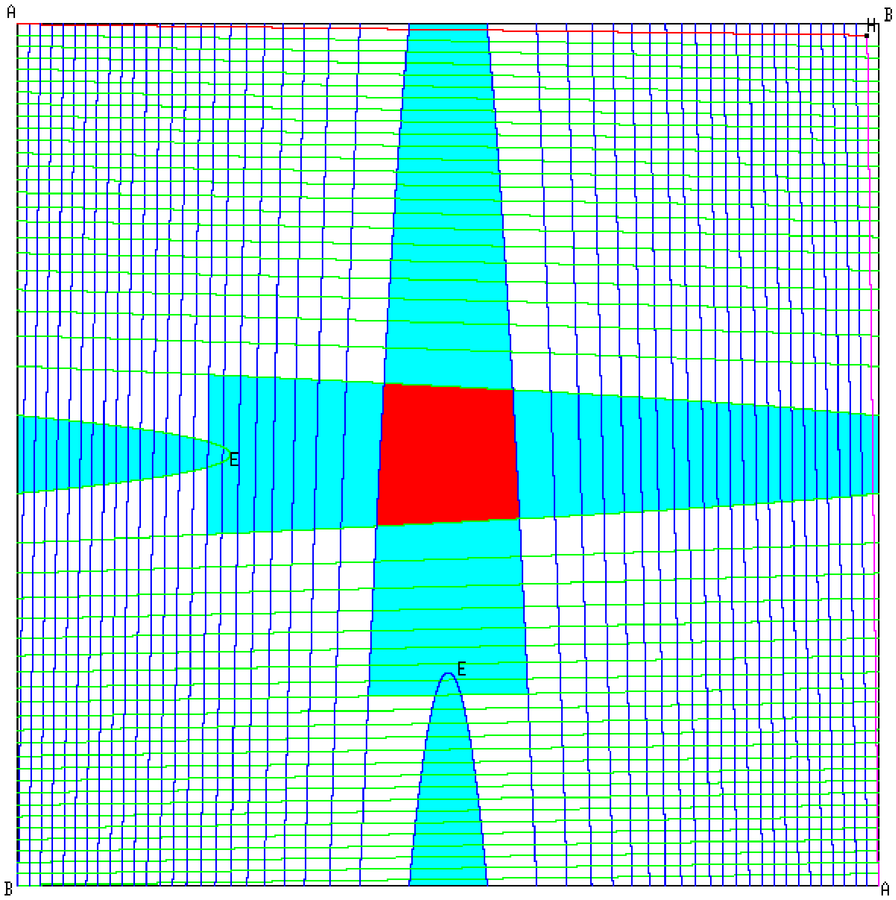

Figure 1: The picture of FLE(1) in the window $[-0.25, 0.25] \times [-0.25, 0.25]$ for $g = 12.12$.

Fig.1 represents the web $W_0^u \cup W_1^u \cup W_0^s \cup (W_1^s)$ which creates a partition of Z into *cells.* Some cells of this partition are filled by grey. They constitute FLE(1). The reason why we selected these cells is following. Let c be a white cell, that is, not belonging to FLE(1). Then $F(c)$ is a straight thin "parallelogram" which goes around Z horizontally and intersects all lines of $F_{-1}(W_0^s)$ *transversally,* its vertical sides are mapped by F into W_0^s, while $F(c)$ approaches W_0^u from *both* of its sides. The analogous assertion is true with respect to $F^{-1}(c)$ and W_0^u, the former being thin "parallelograms" going around Z horizontally. The behaviour of grey cells with respect to the application of F and F^{-1} is quite different. Let c be a typical cell which belongs to the vertical string of the grey cells. Then $F(c)$ is a "horseshoe", the images of the vertical edges of c still belong to W_0^s, while $F(c)$ approaches W_0^u *only from one* side. Analogous assertion is true with respect to the backward images of the typical cells belonging to the horizontal string. There are four exceptional cells which are not "typical": their positions on Fig.1 are marked by the letter E. The exceptional cells have either two or six edges. One can glue each pair of adjacent exceptional cells to obtain a normal cell, let us call it a *section,* which has two vertical edges made of W_1^s and two horizontal edges made of W_1^u. Then we can operate with this section in the same manner as with typical cells.

Statement 2 *The union of white cells in the preceding description is hyperbolic.*

Proof. All white cells, with one exception, are small quadrangles which are close to parallelograms, while the restrictions of the maps F and F^{-1} can be approximated by linear maps. The only exception is the cell in the right upper corner of Fig.1, but in this case we can prolong W_0^u and W_0^s a little bit up to the first intersection with $F^{-1}(W_0^s)$ and $F(W_0^u)$ correspondingly, and divide this "bad" cell into three "good" ones. Let c be a "good" white cell. Draw its two diagonals. The point D of the intersection of the diagonals belongs, evidently, to c and the vectors tangent to the diagonals at point D create two complementary cones in the tangents space at D. By means of the parallel translation we spread these pair of cones onto the whole cell c. The horizontal cones constitute the field of the unstable cones, the vertical cones are the stable ones. The description of white cells given above results in deduction that these cones fields are stretching ones with respect to F and F^{-1} correspondingly, the stretching constants being estimated from above by const$/\sqrt{g}$. To make this proof effective we have added to FLE(1)two additional pairs of "geometrically good" cells in the vicinity of the points close to homoclinic tangency (marked by letter E on Fig.1). QED

One can give an equivalent local description of cells which belong to FLE(1). Note first that all cells which possess at least one edge contained in $W_0^u \cup W_0^s$ is "good", that is it has no intersection with FLE(1). Consider cells with all edges made from $W_1^u \cup W_1^s$. Let us supply the curves W_1^u and W_1^s with the orientation in the sense that a local observer living at a point belonging to a curve knows which side of surrounding surface is "right" and which one is "left". Note that, due to the character of the gluing functions (3) the sides change their sense each time when the curve crosses the border of the fundamental square $[-0.25, 0.25] \times [-0.25, 0.25]$ and appears at the opposite side of it. Then the " geometric" test for a cell c to be "good" is that c must have exactly four edges: two made from W_1^u and another two from W_1^s, and the following orientation rule must be fulfilled: the cell c acquires the opposite orientation senses from the edges of W_1^u and the same rule is for the edges of W_1^s. This means, for example, that c is at the "right hand side" with respect to one of two unstable edges and at the "left hand side" with respect to another. All god cells satisfy this geometric test.

There is another description of cells belonging to FLE(1). Consider the horizontal vector field, $\frac{\partial}{\partial x}$, on the torus. It cannot be projected onto Z as a vector field, but it can be done with the corresponding field of projective directions. It is not difficult to deform slightly the horizontal field so that W_0^u becomes an integral line of the projected field of projected directions, denote the latter by X^u, and its image with respect to the reversor R by X^s. Consider the function Δ_1 whose value at a point z is the square of the sinus of the angle between $F_*(X^u)$ and $F_*^{-1}(X^s)$ both evaluated at z. The gray cells are those which contain zeroes of Δ_1 (cf.[6] where the corresponding zeroes constituted the lines of folds). More precisely, the line of zeroes of Δ_1 goes close to the axes $x = 0$ and $y = 0$. It passes trough each grey cell and has a quasi-intersection at the central cell.

4 Islands inside FLE(1)

Here we consider the islands whose iterates never quit FLE(1). The *central island* is the collection of all KAM circles which are contained, together with all their iterates, in the central cell which is painted by dark grey in Fig.1. Of course, the KAM circles of the central island cannot intersect with the boundary of the cell. Neither they do intersect with additional lines if we draw some W_k^u and W_k^s . For small values of k , say $k = 2, 3$ the intersection of these additional lines with the central cell look like straight lines or parabolas of the form $x = C_1 - 2\pi^2 gy^2$ and $y = C_2 - 2\pi^2 gx^2$ (the coefficient at the square comes from the equation 1 where *cos* is replaced with its Taylor quadratic approximation). Consider a more general situation where we have two families of parabolas of the form

$$x = \frac{1}{2}K_1 y^2 + C_1 , \qquad y = \frac{1}{2}K_2 x^2 + C_1 , \tag{4}$$

where C_1 and C_2 are the parameters of the families, and K_1 and K_2 are the curvatures which are supposed to be large constants. One can make a change of variables of the form

$$x = A\xi , \qquad y = B\eta \tag{5}$$

where A and B are constants so that in new coordinates the parabolas of both families have maximal curvatures equal to 1. An easy calculation gives the following simple expressions for A and B :

$$A = K_1^{-1/3} K_2^{-2/3} , \qquad B = K_1^{-2/3} K_2^{-1/3} . \tag{6}$$

In our case $K_1 = K_2 = 4\pi^2 g$, and, using the above scaling, we can expect that, after adding few iterates, the size of the central window which contains the central island reduces to some amount which is of order 1 in the scaled variables ξ, η, or of the order $1/g$ in the old variables x, y . Denote this reduced central window by w^* . Consider the image, $F(w^*)$, of the reduced window which looks like a horseshoe. It lies somewhere in the horizontal string of FLE(1), the x-coordinate near $g \mathrm{mod} \frac{1}{2}$, and moves to the right with velocity close to 1 if one increases the parameter g . Since the $sin(2\pi x)$ is close to zero in w^* , the map does not change strongly the sizes of the cell, and we conclude that the width and the height of the horseshoe, $F(w^*)$, is also of the order $1/g$. The necessary condition for the existing of the central island reads now as:

$$F(w^*) \cap w^* \neq \emptyset \tag{7}$$

In view of the above description of the position, the sizes, and the velocity of $F(w^*)$, one obtains easily that (7) holds if the parameter g runs trough an interval of the width of the order $1/g$, placed in the vicinity of $N/2$, where N is an integer, and the sizes of the central island in its maximal phase are also of the order $1/g$ in both coordinates. Both numerical experiments and analytic calculations (see [12]) show that this result is in a very good accordance with the reality. It also follows from [12] that the condition (7) is close to the sufficient one.

Consider now the question whether there exist islands which are inside other cells of FLE(1) and such that their iterates never quit FLE(1). It is convenient to supply them with some labels, or codes. To this end, let us construct a symbolic representation for the dynamical system (F, Z) taking a partition $\mathcal{P}$ of Z created by the cells of the web $W_0^u \cup W_0^s \cup W_1^s$. These lines together with all its images with respect to $F^k, k = 0, \pm 1, \pm 2, \ldots$, constitute a set of Lebesgue measure zero. So we can restrict ourselves with considering points whose trajectories never fall on the boundaries of the elements of $\mathcal{P}$. Denote by N the number of the elements of $\mathcal{P}$. The following asymptotic estimate is true for large values of $g : N \sim 4g$. Let us attach numbers from 1 to N to the elements of the partition $\mathcal{P}$ in some order: $\mathcal{P} = \bigcup_{i=1}^N A_i$. Then almost each point z of Z acquires an infinite code $(\ldots, n_{-2}(z), n_{-1}(z), n_0(z), n_1(z), n_2(z), \ldots)$ by the rule:

$$F^i(z) \in A_{n_i(z)}\,, i \in \mathbf{Z}\,. \tag{8}$$

The equality (8) defines a sequence of functions $n_i : Z \to \{1, 2, \ldots, N\}, i \in \mathbf{Z}$.

We supply a non-central cell, c, of the horizontal part of FLE(1) with a *code* which consists of one number equal to $n_0(z)$, $z \in c$, and a non-central cell, c', belonging to the vertical part of FLE(1) with the 1-letter code equal to $n_{-1}(z)$, $z \in c'$. It is easy to verify that the cells symmetric with respect to the diagonal possess the same code. Note that the exceptional cells marked by letter E on Fig.1 acquire a same code which is in the accordance with the mentioned above gluing of the adjacent couples of the exceptional cells together.

Suppose now that a non-central cell c belonging to the horizontal part of FLE(1) with code k contains an island whose images never quit FLE(1). Then the first forward image of the island should be inside the non-central cell c' belonging to the vertical part of FLE(1) with the same code as c, that is symmetric to c. As well as in the case of the central island, the the spot containing the island appears to be much smaller than the cell c. Let us apply the same heuristic analysis as in the case of the central island by use of families of parabolas. The horizontal family of the parabolas in our case is the same as in the case of the central island, and their maximal curvatures have the same order, that is $K_1 \sim g$. To obtain the vertical family of parabolas in the cell c one has to take the preimage of that for the central island via the map F. Since the linear part of the map F is close to the diagonal linear map with the matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda \sim g$, the maximal curvature of the preimage of a parabola from the initial vertical family acquires a multiplier of the order λ^3. So, the vertical family of parabolas passing trough the cell c has the maximal curvature, K_2, of the order g^4. Substituting these estimates for K_1 and K_2 into (6), we obtain the following estimates for the sizes of the window, denote it by w_k^* where k is the code of c, where the island may reside if it exists in the cell c (recall that in this section we study only the islands whose iterates never quit FLE(1)):

$$\text{the width of } w_k^* \sim g^{-3}\,, \quad \text{the height of } w_k^* \sim g^{-2}\,, \tag{9}$$

Let now determine the position and the size of the interval in the real line of the parameter g where an island of such kind can exist. The image of w_k^* under the map F coincides essentially with the symmetric window, $w_k'^*$, on the vertical part of FLE(1). The second image of w_k^*, that is $w_k''^* = F(w_k'^*) = F^2(w_k^*)$, is a small horseshoe which lies again in the horizontal part of FLE(1). This can be done starting with each cell c in the horizontal part of FLE(1). Therefore, on this horizontal part we have plenty of windows $\{w_k^*\}$ and horseshoes $\{w_k''^*\}$ the index k ranging all but one possible values. They create two more or less regular sequences of spots where island can exist. If there exists an island in $w_k''^*$ it must be contained in some initial window, say w_l^*. This gives a necessary condition for the existing of an island of the considered type:

$$w_k''^* \cap w_l^* \neq \emptyset \,. \tag{10}$$

If $k = l$ in this relation, we expect an island of period 2. This means that all KAM circles which contain in w_k^* move under the first iterate to $w_k'^*$, and then, after applying the next iterate return to w_k^*. The periods of KAM curves in this case have to be even. If $k \neq l$, we obtain an island of period 4.

To estimate the width of the interval in the real axis of the parameter g containing the values of g for which the relation (10) takes place, one has to take into account that the string of initial small windows, $\{w_k^*\}$, undergoes little changes when one starts to change g, while the string of horseshoes, $\{w_k''^*\}$, starts to move along FLE(1) with a velocity close to 1. So, the width of the interval in g-axis where the collision (10) occurs is of the order g^{-3}. This estimate together with the estimates (9) for the sizes of the window containing the island both are in an excellent agreement with analytic an numerical analysis made for such kind of islands in [12]. Let us call the considered islands *the islands of the order 1.* Denote by $\mathcal{E}_1$ the set on the axis of g consisting of the values with the property that the corresponding to this value map has an island of the order 1. Since the number of the collisions of the form (10) when g ranges the interval $[n/2, (n+1)/2]$, n integer, equals to the product of the number of the windows and the number of the horseshoes, we obtain that the total estimate of the Lebesgue measure of $\mathcal{E}_1 \cap [n/2, (n+1)/2]$:

$$leb(\mathcal{E}_1 \cap [n/2, (n+1)/2]) \leq \text{const}\, n^{-1} \,. \tag{11}$$

5 FLE's and islands of higher orders

We would like to construct a decreasing sequence FLE(n), n=1,2,3,..., such that 1)FLE(n) would consist of the cells created by the web $\bigcup_{k=0}^{n} (W_k^u \cup W_k^s)$; 2)it would as simple as possible, in particularly, it would consist of two parts: vertical an horizontal trunks, and the cells of each trunk would be ordered in a linear way following to a more or less smooth curve.

We can try to construct the sequence of FLE's inductively. Suppose all FLE(k) with $k < n$ are constructed. To build FLE(n) let us first subdivide the cells of FLE(n-1) by the all possible arcs of W_n^u and W_n^s. The second step is to subject the new smaller cells inside FLE(n-1) to a selection by the "geometric test" described in the Section 3. The union of the remaining cells is still too complicated and form a kind of a "tree" and a more complicated structure. To select the "main trunk" one should "cut off" the "lateral branches". The motivation is that these lateral branches can be filled by the images of parts of the main trunk and some smaller good cells. The following hint can be useful in selecting the cells belonging to FLE(n). Consider the n-th iterates of the fields of the stable and the unstable directions defined at the end of Section 3: $X_n^u = F_*^n(X^u)$ and $X_n^s = F_*^{-n}(X^s)$ and denote by $D_n(z)$ the square of the sinus of the angle between these two directions at a point z. Then the above "geometric" selection is the same as to retain the cells containing zeroes of $D_n(z)$. The latter constitute a smooth curve. Let us consider the *curvatures* of the lines of the fields X_n^u and X_n^s passing through a point of this curve and take the maximal of these two. The curvature so defined can help to select lateral branches from the main trunk: at points of the lateral line the value of the curvature is much bigger than at points of the zero line passing trough the main trunk. Unfortunately, at the present moment, there is no rigorous description of the construction in question,as well as a rigorous proof that it provides a FLE. This will be a subject of future investigations.

Suppose we have constructed FLE(n). Then the same analysis as in preceding section can be carried out to determine if there exist an island inside a cell c of FLE(n). Suppose that c belongs to the horizontal trunk of FLE(n). As well as in the case of FLE(1), the cell c acquires a code k which is now a string of n symbols: $k = (k_1, k_2, \ldots, k_n)$ where $k_i = n_{i-1}(z)\,, z \in c$ (recall that the functions $n_k(z)$ are defined by the relation (8)). If the cell c contains an island then its $n-1$ iterates are outside of FLE(n) and the n-th iterate falls into a cell, c', belonging to the vertical trunk of FLE(n). The code of c' is reverse to k, it is $\overline{k} = (k_n, k_{n-1}, \ldots, k_1)$. Then the same analysis yields the following necessary condition for the existence of an island inside c. This island is confined in a much smaller window $w_k^* \subset c$. Denote $w_{\overline{k}}^{\prime *} = F^n(w_k^*$ and $w_{\overline{k}}^{\prime\prime *} = F(w_{\overline{k}}^{\prime *})$ (a horseshoe). Then the condition is

$$w_{\overline{k}}^{\prime\prime *} \cap w_l^* \neq \emptyset \tag{12}$$

for some window w_l^* belonging to the horizontal trunk of FLE(n). To calculate sizes of a typical window w_k^* one should take $K_1 \sim g$ and $K_2 \sim g^{1+3n}$ in (6) since the second (vertical) family of parabolas is a reverse image of the initial vertical family with maximal curvature of order g via the n-th iterate of F, each iterates supplying a multiplier of the order g^3. Substituting this into (6) gives the estimates

$$\text{the width of } w_k^* \sim g^{-1-2n}\,, \quad \text{the height of } w_k^* \sim g^{-1-n}\,, \tag{13}$$

and in its turn the estimate of the order g^{-1-2n} for the width of the interval of the values of the parameter g for which the collision (12) takes place. Putting together the contributions coming from the collisions between all possible horseshoes, $w_{\overline{k}}^{\prime\prime *}$, and targets, w_l^*, one gets the following estimate for the density of the set $\mathcal{E}_n$ of the values of the parameter g for which there exists an island of the considered type (let us call them islands of the order n):

$$leb(\mathcal{E}_n \cap [p/2, (p+1)/2]) \leq \text{const}\, p^{-1}\,. \tag{14}$$

This looks desperating: by throwing out a relatively small set we can kill islands of any finite order, but cannot kill all of them! But in fact the situation is a little bit better. In the case of the central island and the islands of the order 1 *every* collision of a "horseshoe" with a "target window" resulted in the creating of an island, as it follows from [12]. The situation is different for the island of higher orders. Indeed, the relation (12) is necessary for the the existing of an island inside w_k^*. It means that the island inside w_k^* comes after $(n+1)$-th iterate to another window, w_l^*. But the same condition should be fulfilled also for w_l^* : otherwise it could not contain an island. In fact, the trajectory of an island should be periodic, and hence this chain of collisions should be closed. So the analysis which counts *all* collisions of the form (12) is superfluous. Let us call the collision (12) "good" if the corresponding horseshoe of the target, namely $w_{\overline{l}}^{\prime\prime *}$, has no collision with some w_s^*. If a collision is good, then it cannot produce an island. This classification can reduce strongly the possible cases of the birth of islands. We have a string of targets, $\{w_l^*\}$ and another string of horseshoes, $\{w_{\overline{k}}^{\prime\prime *}\}$ the latter moving along the former with velocity of the order 1 when one moves g. A kind of interferention of these two (in some sense regular) strings will produce islands, and a quasiperiodicity could help to avoid the most double collisions (cf. [9]).

Nevertheless there are codes for which the collision of the form (12) always produces an island. There are two cases of special codes:

I. Both k and l are *symmetric,* that is $k = \overline{k}$ and $l = \overline{l}$ where the dash over a string denotes the inversion of the string: $\overline{(k_1, k_2, \ldots, k_n)} = (k_n, k_{n-1}, \ldots, k_1)$. In this case the island in the window w_k^* comes after n-th iterates to a symmetric window $w_k'^*$, then, after the next iterate it falls, due to (12), in w_l^*, and after additional n iterates it comes to a symmetric window $w_l'^*$ on the vertical trunk. Since all figure is symmetric with respect to diagonal (or the same: with respect to the action of the reversor R), the window $w_l'^*$ meets a horseshoe which is symmetric to $w_k''^*$ and, again by the symmetry (recall that the reversor conjugates F with F^{-1}), this new horseshoe is the preimage of the initial window w_k. Therefore the trajectory of the island trough the windows has period $2n+2$ if $k \neq l$. The number of such "symmetric" collisions is estimated from above by const g^n.

II. The case $k = l$. This cases produces an island of period $n+1$. The number of such collisions is again of the order g^n.

We conclude the paper by formulating the following

Conjecture 1 *In the case of the standard map there exist only the islands listed above in the cases I and II.*

If this conjecture is true then the right hand side in the estimate (14) can be replaced by $(\mathrm{const}\, p)^{-n-1}$. Since all the estimates for the individual windows come from some kind of renormalization procedure, self-similarity reasons involve that the constant in this estimate can be chosen independent of n and p. By our construction, all island belong to some cells in FLE's and hence $\mathcal{E} = \bigcup_{k=0}^{\infty} \mathcal{E}_k$ where $\mathcal{E}_0$ is the set of values of the parameter g for which the central islands exists, $\mathcal{E}_n$, $n \geq 1$, is the set of parameters corresponding to existing islands of the order n. Putting all these estimate together gives us the final estimate for the density of $\mathcal{E}$:

$$leb(\mathcal{E} \cap [p, p+1]) \leq \mathrm{const}\, p^{-1}.$$

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