A COMPLEX WKB METHOD FOR ADIABATIC PROBLEMS

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Abstract. This work is devoted to a new version of the complex WKB method suited for adiabatic perturbations of one-dimensional periodic Schrödinger operators. Therefore we introduce an additional parameter and it is this parameter (and not the variable of the equation) that will become complex. This naturally leads to canonical domains where we construct solutions of the Schrödinger equation with a standard asymptotic behavior. These can be used to compute the asymptotics of the exponentially small coefficients of transfer matrices (e.g. scattering matrices, monodromy matrices, etc). We give an example of such a computation.

Résumé. Ce travail est consacré à une nouvelle version de la méthode BKW complexe adaptée à l'étude des perturbations adiabatiques d'équations de Schrödinger périodiques en dimension 1. Nous introduisons un paramètre supplémentaire; c'est ce paramètre qui sera rendu complexe (au lieu de la variable de l'équation). Ceci nous conduit de façon naturelle à la définition des domaines canoniques dans lesquels on construit des solutions aux comportements asymptotiques connus. Ces développements asymptotiques peuvent être utilisés pour contrôler l'asymptotique des coefficients exponentiellement petits de matrices de transfert (par exemple, de matrices de diffusion, de matrices de monodromie, etc). On donne un exemple d'un tel calcul.

1. Introduction and the main result

Consider the Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x) + (V(x) + W(\varepsilon x))\psi(x) = \psi(x), \quad x \in \mathbb{R}. \tag{1.1}$$

Here V is a periodic function, $V(x+1) = V(x)$, $x \in \mathbb{R}$, and ε is a fixed positive number. This equation is one of the main models of Solid State Physics. It describes a Bloch electron in a cristal placed in an external field. The function V represents the electric field of the cristal and W is the external field. Usually, the external field can be considered as very regular and slowly varying with respect to the field of the cristal, and so, we shall assume here that

- $W(\xi)$ is analytic in ξ;
- ε is small i.e. we consider the adiabatic limit.

The internal field V can be rather irregular. It can even be a sum of Dirac δ-functions. For the sake of simplicity, we restrict ourselves to the case where V is L_2^{loc}.

In the problems coming from Solid State Physics, one has to study rather delicate objects sensitive to exponentially small tunneling effects. To control such exponentially small effects, one needs to have a good enough control of solutions of (1.1). The classical adiabatic asymptotic constructions (on the real axis) do not allow to get a control of exponentially small terms due to the complex tunneling (see [1]). The natural idea to get a hold of these terms would be to shift the variable x into the complex plane hoping that, there, they become dominant. Of course this is not possible in the present case, V not being analytic. To overcome this problem, we introduce an additional parameter φ into equation (1.1) so that it becomes

$$-\frac{d^2}{dx^2}\psi(x) + (V(x) + W(\varepsilon x + \varphi))\psi(x) = E\psi(x), \quad x \in \mathbb{R} \tag{1.2}$$

and consider solutions of (1.2) analytic in the parameter φ. It is then φ that gets shifted into the complex plane and that allows us to compute exponentially small quantities.

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We begin with some preliminary remarks concerning the standard version of the complex WKB method developed for ordinary differential equations of the form (see, for example, [4])

$$-\varepsilon^2 \frac{d^2}{dz^2}\psi(z) + W(z)\,\psi(z) = E\psi(z), \quad z \in \mathbb{C}. \tag{1.3}$$

Here, W is an analytic function of z, and ε is a small parameter. For (1.3), the complex WKB method consists in the description of certain *canonical domains* in the complex plane $z \in \mathbb{C}$ on which one constructs analytic solutions of (1.3) having a standard asymptotic behavior when $\varepsilon \to 0$. Then, to obtain a solution with global asymptotics, one uses transfer matrices to relate bases of solutions constructed for different canonical domains.

For a given canonical domain K, one constructs two linearly independent solutions $\psi_\pm$ with the asymptotics

$$\psi_\pm(z) = \frac{1}{\sqrt{p(z)}}\, e^{\pm\frac{i}{\varepsilon}\int_{z_0}^{z} p(\tilde z)\, d\tilde z + o(1)}, \quad z \in K, \quad \varepsilon \to 0.$$

Here z_0 is fixed point in K and p is the complex momentum. The complex momentum is the main analytic object of the method. It is defined by the relation

$$p^2 + W(z) = E,$$

i.e. by the symbol of equation (1.3). A canonical domain can be roughly characterized as a domain such that, through any point of it, one can draw a smooth curve along which the function $\operatorname{Im}\int_{z_0}^{z} p(z)\, dz$ is monotonically increasing ([4, 11]).

As said above, we introduce an additional parameter $\varphi \in \mathbb{C}$ into equation (1.1) so that it becomes (1.2). For equation (1.2) the role of the complex momentum is played by the function $\kappa(\varphi)$ satisfying the relation

$$\mathcal{E}(\kappa) + W(\varphi) = E, \tag{1.4}$$

where $\mathcal{E}(k)$ is the dispersion law of the periodic Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x) + V(x) = E\psi(x), \quad x \in \mathbb{R}. \tag{1.5}$$

In other words,

$$\kappa(\varphi) = k(E - W(\varphi)), \tag{1.6}$$

where $k(E)$ is the Bloch quasi-momentum of (1.5). The fact that the function κ plays a crucial role in adiabatically perturbed problems has been known for a long time (see [1]).

Our version of the complex WKB method requires four major steps:

1. to describe certain *canonical domains* on the complex plane $\varphi \in \mathbb{C}$ i.e. domains in the complex plane of φ where one can construct solutions of (1.2) with the simplest possible asymptotic behavior for $\varepsilon \to 0$,
2. on any given canonical domain, to construct a basis of solutions of (1.2) having the simplest asymptotic behavior when $\varepsilon \to 0$ (x being in a fixed interval $[-X, X] \subset \mathbb{R}$),
3. to relate bases of solutions obtained for different canonical domains to construct solutions of (1.1) for which the asymptotic behavior is known globally in φ,
4. to use this knowledge to recover the asymptotic behavior in x of solutions of (1.2).

Note that there is no equation controlling the dependence of the solutions of (1.2) on the parameter φ. However, there is a natural condition which replaces such an equation. We say that $(\psi_\pm)$ two solutions of (1.1) form *a consistent basis* if their Wronskian is independent of φ and if these solutions satisfy the relation

$$\psi_\pm(x+1, \varphi) = \psi_\pm(x, \varphi + \varepsilon). \tag{1.7}$$

We call (1.7) the *consistency condition.* To clarify this condition, we pass to the variables $\phi = \varphi/\varepsilon$ and $t = x + \phi$. In terms of these variables, (1.2) takes the form

$$-\frac{d^2}{dt^2}\tilde\psi + (V(t-\phi) + W(\varepsilon t))\tilde\psi = E\tilde\psi,$$

and (1.7) means just the 1-periodicity of the solutions in ϕ

$$\tilde\psi_\pm(t, \phi + 1) = \tilde\psi_\pm(t, \phi).$$

This condition will enable us to realize the program described above.

In the first step (as will be seen from the proof), it is the consistency condition that leads to the definition of the canonical domains. These can be roughly characterized as domains such that, through any of their points, one can draw a smooth vertival curve along which the function $\mathrm{Im}\,\int_{\varphi_0}^{\varphi}\kappa\,d\varphi$ is monotonically increasing and the function $\mathrm{Im}\,\int_{\varphi_0}^{\varphi}(\kappa-\pi)\,d\varphi$ is monotonically decaying (see section 3 for precise definitions). To realize step 2, we need to consider the solutions of the underlying periodic problem

$$-\frac{d^2}{dx^2}\psi(x)+V(x)\,\psi(x)=\mathcal{E}\psi(x),\quad x\in\mathbb{R}, \tag{1.8}$$

where $\varphi\in\mathbb{C}$ is fixed and

$$\mathcal{E}=E-W(\varphi). \tag{1.9}$$

Then, the solutions $\psi_\pm$ having the simplest asymptotic behavior on a given canonical domain are described by

Theorem 1.1. *Fix $X>0$. Let K be a bounded canonical domain for the family of equations* (1.1)*. For sufficiently small ε, there exist a consistent basis $(f_\pm)$ defined for $x\in\mathbb{R}$ and $\varphi\in K$ and having the following properties:*

- *for any fixed $x\in\mathbb{R}$, the functions $f_\pm(x,\varphi)$ are analytic in $\varphi\in K$.*
- *for $-X\le x\le X$, the functions $f_\pm(x,\varphi)$ have the asymptotic representations*

$$f_\pm(x,\varphi)=e^{\pm\frac{i}{\varepsilon}\int_0^{\varphi}\kappa\,d\varphi}\left(\Psi_\pm(x,\varphi)+o(1)\right),\quad \varepsilon\to 0. \tag{1.10}$$

  Here $\Psi_\pm$ are the canonical Bloch solutions of equation (1.8) *corresponding to the domain K.*
- *the error estimates in (1.10) may differentiated once in x. Moreover they are uniform in x and locally uniform in φ in the interior of K.*

The Bloch solutions mentioned in the theorem have the form

$$\Psi_\pm(x,\varphi)=A_\pm(\varphi)\,\psi_\pm(x,\varphi).$$

Here $\psi_\pm$ are two linearly independent Bloch solutions of equation (1.8) defined for $\varphi\in K$. The factors $A_\pm$ are constructed in terms of these Bloch solutions so that $\Psi_\pm$ be analytic in φ. The reader will find precise definitions in section 3.

Remark 1.1. One can prove a similar theorem in the case of unbounded canonical domains. This requires some changes in the proof that we will not discuss in the present paper.

Let us now turn to third step of our plan. Pick two canonical domains $K^{1,2}$ and the associated basis of solutions constructed in Theorem 1.1 denoted by $(f_\pm^{1,2})$. Equation 1.2 being linear of second order, there exists a transfer matrix T_{12} depending only on φ such that

$$\begin{pmatrix} f_+^2 \\ f_-^2\end{pmatrix}=T_{12}\begin{pmatrix} f_+^1 \\ f_-^1\end{pmatrix}.$$

The consistency condition (1.7) implies that T_{12} is ε-periodic in φ. Moreover $(f_\pm^{1,2})$ being analytic in φ, T_{12} is also analytic.

If K^1 and K^2 overlap then, in $K^1\cap K^2$, we can use the asymptotics (1.10) for both basis and get the asymptotics of T_{12}. Using the ε periodicity of T_{12}, we get these asymptotics in a horizontal strip containing $K^1\cap K^2$. Now, if we know the asymptotics of T_{12} near the boundary of a complex strip containing $\mathbb{R}$, then, we can easily get the asymptotics of the Fourier coefficients of T_{12} and this up to an exponential precision of roughly the width of the strip. Hence, we have the same precision in the asymptotics of T_{12} on the real line.

At last, the consistency condition directly relates the behavior of the solution in x to its behavior in φ. This realizes the 4th step of our plan.

In the present paper, we will only give the details for steps 1 and 2. The actual form of the canonical domains, hence, step 3 depend crucially on the potential W. The computations of the transfer matrices remind of the computations done in the case of the classical complex WKB method (see [4]) though more technical. Similar computations can be found in [2]. Therefore, we will not describe them here but only discuss an example in section 5.

The paper is organized as follows. First, we recall well known facts on the one-dimension periodic Schrödinger operator. In section 3, we give precise definitions of the canonical domains and describe other important objects of the complex WKB method. Section 4 is devoted to the proof of Theorem 1.1. Finally, section 5 is devoted to the example $W(x) = \cos x$. We give the description of some canonical domains, of the transfer matrices.

Contents

2. Periodic Schrödinger operators

Here, we collect some well known facts (see, for example, [8, 9]) on one-dimensional periodic Schrödinger operators (1.5). We assume that V is a $L^2_{\rm loc}(\mathbb{R})$ real valued function such that

$$V(x+1) = V(x), \quad x \in \mathbb{R}.$$

2.1. **Bloch solutions.** Let ψ be a solution of (1.5) satisfying the relation

$$\psi(x+1) = \lambda\, \psi(x), \quad \forall x \in \mathbb{R}, \tag{2.1}$$

with λ independent of x. Such solution is called a *Bloch* solution, and the number λ is called the *Floquet multiplier*.

Consider the spectrum of equation (1.5) in $L^2(\mathbb{R})$. It consists of intervals $[E_1, E_2]$, ... $[E_{2n+1}, E_{2n+2}]$, ..., of the real axis such that

$$E_1 < E_2 \le E_3 < E_4 \dots E_{2n} \le E_{2n+1} < E_{2n+2} \le \dots \tag{2.2}$$

$$E_n \to +\infty, \quad n \to +\infty. \tag{2.3}$$

The intervals $[E_{2n-1}, E_{2n}]$ $(n \ge 1)$ are called the *spectral zones*, and the intervals (E_{2n}, E_{2n+1}), $n \ge 1$, are called the *spectral gaps*. Usually, the number of the gaps is infinite.

Consider two copies $\Gamma_\pm$ of the complex plane $E \in \mathbb{C}$ cut along the spectral zones. Paste them together to get a Riemann surface with square root branching points. We call this Riemann surface Γ. Γ_+ (resp. Γ_-) will be refered to as the upper (resp. lower) sheet.

One can construct a Bloch solution $\psi(x, E)$ of the equation (1.5) meromorphic on this Riemann surface. The poles of this solution are located in the spectral gaps. More precisely, each spectral gap contains precisely one of these poles. All the poles are simple. They are located either on Γ_+ or on Γ_-. The position of a pole is independent of x. The Bloch solution $\psi(x, E)$ has the asymptotics

$$\psi(x, E) = e^{i\sqrt{E}\,x}(1 + o(1)), \quad |E| \to \infty, \qquad E \in \Gamma, \quad 0 < \arg E < 2\pi. \tag{2.4}$$

We define the branch of the square root so that, for $E < 0$, $-i\sqrt{E} > 0$ if $E \in \Gamma_+$.

Denote by $\psi_\pm$ the restrictions of the above defined Bloch solution on $\Gamma_\pm$. Outside the ends of the spectral zones, these functions are linearly independent solutions of (1.5). Along the cuts, we have

$$\psi_\pm(x, E - i0) = \psi_\mp(x, E + i0), \quad E_{2l+1} < E < E_{2l+2}, \quad l \in \mathbb{N}. \tag{2.5}$$

2.2. **Bloch quasi-momentum.**

2.2.1. Consider the Bloch solution $\psi(x, E)$ introduced in the previous subsection. The corresponding Floquet multiplier $\lambda(E)$ is analytic on Γ. Represent it in the form

$$\lambda(E) = \exp(ik(E)). \tag{2.6}$$

The function $k(E)$ is called *Bloch quasi-momentum.* It has the same branching points as $\psi(x, E)$, but the corresponding Riemann surface is more complicated.

To describe the main properties of k, consider the complex plane cut along the real line from E_1 to $+\infty$. Denote the cut plane by $\mathbb{C}_0$. One can fix there a single valued branch of the quasi-momentum by the condition

$$ik_0(E) < 0, \quad E < E_1. \tag{2.7}$$

The image of $\mathbb{C}_0$ by k_0 is located in the upper half of the complex plane,

$$\operatorname{Im} k_0(E) > 0, \quad E \in \mathbb{C}_0. \tag{2.8}$$

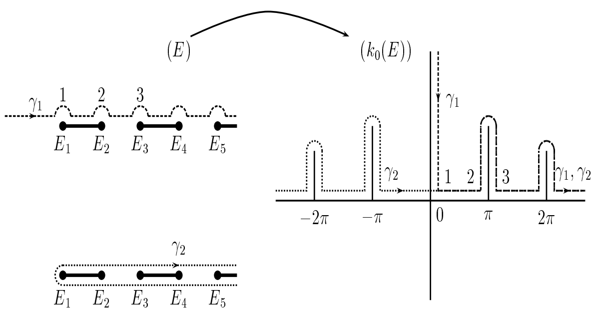

Figure 1: The mapping k_0

In Fig. 1, we drew several curves in $\mathbb{C}_0$ and their images under the transformation $E \mapsto k_0(E)$.
The quasi-momentum $k_0(E)$ is real along the spectral zones, and, along the spectral gaps, its real part is constant; in particular, we have

$$k_0(E_1) = 0, \quad k_0(E_{2l} \pm i0) = k_0(E_{2l+1} \pm i0) = \pm\pi l, \quad l \in \mathbb{N}. \tag{2.9}$$

All the branching point of k are of square root type. Let E_m be one of the branching points of k. Then, one has

$$k(E) - k(E_m) = c_m\sqrt{E - E_m} + O(E - E_m), \quad c_m \neq 0. \tag{2.10}$$

Finally note that

$$k_0(E) = \sqrt{E} + O(1/\sqrt{E}), \quad |E| \to \infty, \tag{2.11}$$

where $E \in \Gamma$ and $0 \leq \arg E < 2\pi$.

2.2.2. The values of the quasi-momentum k_0 on the different sides of the cut $E_1 < E < +\infty$ are related to each other by the formula

$$k_0(E + i0) = -\overline{k_0(E - i0)}, \quad E_1 \leq E. \tag{2.12}$$

Consider the spectral gap $E_{2l} < E < E_{2l+1}$, $l \in \mathbb{N}$. Let $\mathbb{C}_l$ be the complex plane cut from $-\infty$ to E_{2l} and from E_{2l+1} to $+\infty$. Denote by k_l the branch of the quasimomentum analytic on $\mathbb{C}_l$ and coinciding with k_0 for $\operatorname{Im} E > 0$. One has

$$k_l(E + i0) + k_l(E - i0) = 2\pi l, \quad \text{if } E < E_{2l} \text{ or } E > E_{2l+1}. \tag{2.13}$$

This relation follows from (2.9) and from the analyticity of the functions given by (2.10).

2.3. **Periodic components of the Bloch solution.** Let D be a simply connected domain that does not contain any branching point of k. On D, we fix an analytic branch of k. Consider two copies of D, denoted by $D_\pm$, corresponding to two sheets of Γ. As above we define solutions $\psi_\pm$ (restrictions of ψ on $D_\pm$). They can be represented in the form

$$\psi_\pm(x, E) = e^{\pm ik(E)\,x}\, p_\pm(x, E), \quad E \in D, \tag{2.14}$$

where $p^\pm(x, E)$ are functions periodic in x,

$$p_\pm(x + 1, E) = p_\pm(x, E), \quad \forall x \in \mathbb{R}. \tag{2.15}$$

2.4. **Useful formulae.** Let D, k and $\psi_\pm$ be as in the previous section. We shall use the following two integral relations.

Lemma 2.1. *Let $E \neq E_l$, $l \in \mathbb{N}$, and let $x \in \mathbb{R}$. Then*

$$\int_{x-1}^{x} \psi_+(x, E)\, \psi_-(x, E)\, dx = -ik'(E)\, w\,(E), \tag{2.16}$$

where $w(E) = w\,(\psi_+(x, E), \psi_-(x, E))$ and

$$\begin{aligned} &2\int_{x-1}^{x} \frac{\partial \psi_+}{\partial E}\,(x, E)\,\psi_-(x, E)\,dx \\ &= -(ik''(E) + (k'(E))^2)\,w\,(E) - 2ik'(E)\,w\left(\frac{\partial \psi_+}{\partial E}\,(x, E),\ \psi_-(x, E)\right). \end{aligned} \tag{2.17}$$

Here $w\,(f\,(x),\, g\,(x)) = f'(x)\, g\,(x) - g'(x)\, f\,(x)$.

Proof. Though these two formulae are known, we describe their simple proofs here. Begin with the formula (2.16). Derivating (1.5) in E, we get:

$$-\frac{\partial^3 \psi_+}{\partial x^2\, \partial E}\,(x, E) + V\,(x)\,\frac{\partial \psi_+}{\partial E}\,(x, E) = E\,\frac{\partial \psi_+}{\partial E}\,(x, E) + \psi_+(x, E). \tag{2.18}$$

Multiplying this equality by $\psi_-(x, E)$, integrating from $x-1$ to x, then integrating twice by parts, we obtain

$$-w\left(\frac{\partial \psi_+}{\partial E}\,(x', E),\ \psi_-(x', E)\right)\Bigg|_{x'=x-1}^{x'=x} = \int_{x-1}^{x} \psi_+(x', E)\,\psi_-(x', E)\,dx'. \tag{2.19}$$

Now, by means of the formulae

$$\psi_\pm(x - 1, E) = e^{\mp ik\,(E)}\,\psi_\pm(x, E),$$

we can express all the functions in the left hand side of (2.19) in terms of functions of the argument $x' = x$. This leads to

$$w\left(\frac{\partial \psi_+}{\partial E}\,(x', E),\ \psi_-(x', E)\right)\Bigg|_{x'=x-1}^{x'=x} = ik'(E)\,w\,(E),$$

which implies (2.16). The formula (2.17) can be obtained similarly. But now, instead of (2.18), we start with the relation

$$-\frac{\partial^4 \psi_+}{\partial x^2\,\partial E^2}\,(x, E) + V\,(x)\,\frac{\partial^2 \psi_+}{\partial E^2}\,(x, E) = E\,\frac{\partial^2 \psi_+}{\partial E^2}\,(x, E) + 2\,\frac{\partial \psi_+}{\partial E}\,(x, E).$$

We omit the elementary details. □

In section 4.4, we shall also need the

Corollary 2.1. *Under the assumptions of Lemma 2.1,*

$$
\begin{aligned}
&-w\left(\psi_+(x,E),\frac{\partial\psi_-}{\partial E}(x,E)\right)+\int_{x-1}^{x}\psi_+(x',E)\,\psi_-(x',E)\,x'\,dx'=\\
&\qquad=-w'(E)+w\,(E)\left(-\frac{1}{2}\frac{d}{dE}\ln\,k'(E)+\frac{i}{2}\,k'(E)-\omega^+(E)\right),
\end{aligned}
\tag{2.20}
$$

where

$$
\omega^+(E)=-\int_0^1\frac{\partial p_+}{\partial E}\,(x,E)\,p_-(x,E)\,dx\,\left(\int_0^1 p_+\,(x,E)\,p_-(x,E)\,dx\right)^{-1}.
$$

Note that this corollary easily follows from formulae (2.16) and (2.17).

3. Main notions of the complex WKB method

In this section, we consider the family of equations (1.2), define the corresponding canonical domains.

3.1. **Canonical domains.** The canonical domain is the main geometric notion of the complex WKB method. Its origin is tightly connected with the construction of analytic solutions having standard a-symptotic behavior in the parameter $\varphi\in\mathbb{C}$.

3.1.1. The canonical domains can be described in terms of the complex momentum $\kappa\,(\varphi)$. Remind that it is defined by formula (1.6) in terms of the Bloch quasi-momentum $k\,(E)$ of the operator $H_0=-\frac{d^2}{dx^2}+V\,(x)$. The complex momentum κ is a multi-valued analytic function. Its branching points are related to the branching points of the quasi-momentum by the relations

$$
E_l=E-W\,(\varphi),\quad l\in\mathbb{N},
\tag{3.1}
$$

where E_l are the ends of the spectral zones of the operator H_0. The conformal properties of κ and the structure of its Riemann surface depend on the analytic properties of the function W and on the value of the spectral parameter E. For $W(x)=\cos x$, the analytic properties of the complex momentum are described in subsection 5.1.

3.1.2. In what follows, ζ_1 and ζ_2 are two points in $\overline{\mathbb{C}}$ such that

$$
\operatorname{Im}\zeta_1<\operatorname{Im}\zeta_2.
$$

These points can either be situated in the complex plane $\mathbb{C}$ or be the points $\pm i\infty$. We shall denote by γ a smooth curve going from ζ_1 to ζ_2 (i.e. $\gamma\in\mathcal{C}^1([0,1],\mathbb{C})$ if $\zeta_{1,2}$ are finite). When we will integrate along such a curve, the curve will always be oriented from ζ_1 to ζ_2. We now state the definitions.

3.1.3. We call a smooth curve $\gamma\subset\mathbb{C}$ *vertical* if it intersects the lines $\operatorname{Im}z=\text{Const}$ at non-zero angles θ,

$$
0<\theta<\pi.
\tag{3.2}
$$

3.1.4. Let γ be a vertical curve. On γ, fix a continuous branch of the momentum κ. We call γ *canonical* if, along γ, $\operatorname{Im}\int^{\varphi}\kappa\,d\varphi$ is strictly monotonically increasing, and $Im\int^{\varphi}(\kappa-\pi)\,d\varphi$ is strictly monotonically decreasing that is if, along γ, one has

$$
\frac{d}{ds}\left(\operatorname{Im}\int^{\varphi}\kappa\,d\varphi\right)>0\text{ and }\frac{d}{ds}\left(\operatorname{Im}\int^{\varphi}(\kappa-\pi)\,d\varphi\right)<0
$$

where ds is the length element of the curve γ.

3.1.5. Let K a simply connected domain containing no branching points of the complex momentum κ. On K, fix a continuous branch κ. We call K a *canonical domain* if it is the union of canonical curves connecting some two points ζ_1 and ζ_2 fixed as above. To be precise, we have to exclude these two points from K.

Remark 3.1. We will describe the other classical geometric objects and constructions usually related to the complex WKB method (for example, Stokes lines) only in the case when $W(x)=\cos x$ (see section 5). These constructions will not be needed neither to formulate nor to prove our main theorem. However they are essential to understand the geometry of canonical domains for a given equation.

3.2. **Canonical Bloch solutions.** To describe the asymptotic formulae of the complex WKB method for equation (1.2), one needs specially normalized Bloch solutions of the equation (1.8).

3.2.1. Assume that D is a simply connected domain in the plane $\varphi \in \mathbb{C}$ containing no branching points of κ. Consider (1.8) as an equation with spectral parameter $\mathcal{E}$ and consider the associated Riemann surface Γ. Let $\psi(x, \mathcal{E})$ be the Bloch solution of equation (1.8) meromorphic on Γ.

The mapping $\varphi \to E - W(\varphi)$ maps D onto a domain $\hat{D} \subset \mathbb{C}$. Clearly, $\hat{D}$ is simply connected and does not contain any branching point of $\psi(x, \mathcal{E})$. So, since Γ is a two sheeted Riemann surface, we can also consider two different copies $\hat{D}_+$ and $\hat{D}_-$ of the domain $\hat{D}$.

Denote the restrictions of $\psi(x, \mathcal{E})$ on $\hat{D}_\pm$ by $\psi_\pm$. These functions are two Bloch solutions of (1.8). Fix on $\hat{D}_+$ a continuous branch of the Bloch quasi-momentum $k(\mathcal{E})$. Clearly,

$$\psi_\pm(x, \mathcal{E}) = e^{\pm ik(\mathcal{E})\,u}\, p_\pm(x, \mathcal{E}),$$

where $p_\pm$ are 1-periodic in u,

$$p_\pm(x+1, \mathcal{E}) = p_\pm(x, \mathcal{E}).$$

So, $\pm k(\mathcal{E})$ are the quasi-momenta of $\psi_\pm(x, \mathcal{E})$. Note also, that $p_\pm(x, \mathcal{E})$ are meromorphic in $\mathcal{E} \in \hat{D}$.

3.2.2. Let

$$g_\pm(\mathcal{E}) = -\frac{\int_0^1 p_\mp(x, \mathcal{E})\,\frac{\partial p_\pm}{\partial \mathcal{E}}(x, \mathcal{E})dx}{\int_0^1 p_+(x, \mathcal{E})\,p_-(x, \mathcal{E})dx}, \quad \mathcal{E} \in \hat{D}. \tag{3.3}$$

In view of (2.16), $g_\pm$ are meromorphic in $\hat{D}$.

Define

$$f_\pm(x, \mathcal{E}) = \sqrt{k'_E(\mathcal{E})}\; e^{\int^{\mathcal{E}} g_\pm(e)de}\, \psi^0_\pm(x, \mathcal{E}), \quad \mathcal{E} \in \hat{D}. \tag{3.4}$$

We call the functions

$$\Psi_\pm(x, \varphi) = f_\pm(x, \mathcal{E}), \quad \mathcal{E} = E - W(\varphi), \tag{3.5}$$

canonical Bloch solutions associated to the domain D. Note that they are independent of the choice of the branch k. One has

Lemma 3.1. *The Bloch solutions $\Psi_\pm(x, \varphi)$ are analytic in $\varphi \in D$.*

Proof. It suffices to prove that the functions $f_\pm(x, \mathcal{E})$ are analytic in $\hat{D}$. Let us prove it for f_+. The case of f_- can be dealt with in the same way. In what follows, "$\sim$" denotes equality up to terms analytic in $\mathcal{E} \in \hat{D}$.

Obviously, the formulae from Lemma 2.1 remain valid for $f_\pm(x, \mathcal{E})$ when $\mathcal{E} \in \hat{D}$. They imply that

$$\begin{aligned} g_\pm(\mathcal{E}) \sim -\frac{1}{2}\frac{d}{d\mathcal{E}}\ln k'(\mathcal{E}) - \frac{1}{w(\psi_+, \psi_-)}\, w\left(\frac{\partial \psi_+}{\partial \mathcal{E}}(x, \mathcal{E}),\, \psi_-(x, \mathcal{E})\right)\Big|_{u=1} \\ - \frac{1}{w(\psi_+, \psi_-)} \int_0^1 \psi_+(x, \mathcal{E})\psi_-(x, \mathcal{E})x dx. \end{aligned} \tag{3.6}$$

We have omitted the term $\frac{i}{2}\, k'(\mathcal{E})$ since it is analytic outside of the branching points.

Consider the second term in (3.6). Remember that the poles of $\psi_\pm(x, \mathcal{E})$ are independent of x. Let $\mathcal{E}_0 \in \hat{D}$ be a pole of $\psi_\pm(x, \mathcal{E})$. In a neighborhood of $\mathcal{E}_0$, we have

$$\psi_\pm(x, \mathcal{E}) = \frac{1}{(\mathcal{E} - \mathcal{E}_0)^{\nu_\pm}}\, \tilde{\psi}_\pm(x, \mathcal{E}),$$

where $\nu_\pm$ is the multiplicity of $\mathcal{E}_0$ as a pole of $\psi_\pm(x, \mathcal{E})$, and $\tilde{\psi}_\pm(x, \mathcal{E})$ is a solution of (1.8) analytic in a neighborhood of $\mathcal{E}_0$.

The functions $\tilde{\psi}_\pm(x, \mathcal{E})$ are Bloch solutions with the same Bloch quasi-momentum $\pm k(\mathcal{E})$ as $\psi_\pm(x, \mathcal{E})$. Since $k(\mathcal{E}) \in \pi\mathbb{Z}$ only at branching points, the solutions $\tilde{\psi}_\pm(x, \mathcal{E})$ are linearly independent. These arguments imply that, in a vicinity of $\mathcal{E}_0$, the third term in (3.6) is analytic and, that

$$\frac{1}{w(\psi_+, \psi_-)}\, w\left(\frac{\partial \psi_+}{\partial \mathcal{E}}(x, \mathcal{E}),\, \psi_-(x, \mathcal{E})\right)\Big|_{u=1} \sim -\frac{\nu_+}{\mathcal{E} - \mathcal{E}_0},$$

and, thus, (3.6) takes the form

$$g_\pm(\mathcal{E}) \sim -\frac{1}{2}\frac{d}{d\mathcal{E}}\ln k'(\mathcal{E}) + \frac{\nu_+}{\mathcal{E}-\mathcal{E}_0}.$$

This implies the analyticity of $f_\pm$ in a vicinity of any point $\mathcal{E}_0 \in \hat{D}$. $\square$

In the sequel, we shall use the notation

$$\omega_\pm(\varphi) = -W'(\varphi)\cdot g_\pm(E - W(\varphi)). \tag{3.7}$$

4. Proof of Theorem 1.1

This section is devoted to the proof of this theorem. For the sake of brevity, we do not check explicitly the uniformity of errors nor the differentiability of the asymptotics: this does not require any additional estimates or discussions, and the reader will easily fill this gap.

4.1. **Sketch of the proof.** Before beginning the actual proof of Theorem 1.1, let us outline the main ideas of this proof.

4.1.1. Let us assume that φ belongs to a bounded domain $D \subset \mathbb{C}$ and that x is in a fixed interval $[-X, X] \subset \mathbb{R}$. Clearly,

$$W(\varepsilon x + \varphi) = W(\varphi) + O(\varepsilon), \quad \varepsilon \to 0.$$

So, we may "replace" (1.2) by equation (1.8) with $\mathcal{E}$ given by (1.9). Assuming that the domain D does not contain any branching points of the complex momentum $\kappa(\varphi) = k(E - W(\varphi))$, we introduce two linearly independent Bloch solutions $\psi^0_\pm(x,\varphi)$ of (1.8) – (1.9). Assuming that these two solutions have no singularities in $\overline{D}$, we prove that the initial equation (1.2) has two linearly independent solutions $\psi^1_\pm$ close to $\psi^0_\pm$ when $\varepsilon \to 0$.

4.1.2. Though the constructed solutions $\psi^1_\pm(x,\varphi)$ are linearly independent and analytic in $\varphi \in D$, they need not be consistent. If there exists any solution $f(x,\varphi)$ of (1.2) satisfying the consistency condition (1.7), it certainly can be represented in the form

$$f(x,\varphi) = A(\varphi)\,\psi^1_+(x,\varphi) + B(\varphi)\,\psi^1_-(x,\varphi)$$

where the coefficients A and B are independent of u. The consistency condition is equivalent to a difference equation of the form

$$\begin{pmatrix} A(\varphi+\varepsilon) \\ B(\varphi+\varepsilon) \end{pmatrix} = T(\varphi)\begin{pmatrix} A(\varphi) \\ B(\varphi) \end{pmatrix}, \quad \varphi, \varphi+\varepsilon \in D. \tag{4.1}$$

The coefficient of the matrix T can be calculated in terms of the Wronskians of the solutions $\psi^1_\pm(x-1,\varphi+\varepsilon)$ and $\psi^1_\pm(x,\varphi)$.

4.1.3. We then compute the asymptotics of the matrix T to find out that it is close to a diagonal one i.e.

$$T = \begin{pmatrix} e^{i\kappa(\varphi)+O(\varepsilon)} & O(\varepsilon) \\ O(\varepsilon) & e^{-i\kappa(\varphi)+O(\varepsilon)} \end{pmatrix}, \quad \varepsilon \to 0,$$

Using a transformation of the form

$$A_1(\varphi) = e^{\frac{i}{\varepsilon}\int_{\varphi_0}^{\varphi}\kappa(\varphi)\,d\varphi + O(\varepsilon)}\,A(\varphi), \quad B_1(\varphi) = e^{-\frac{i}{\varepsilon}\int_{\varphi_0}^{\varphi}\kappa(\varphi)\,d\varphi+O(\varepsilon)}\,B(\varphi),$$

we obtain the equation

$$\begin{pmatrix} A_1(\varphi+\varepsilon) \\ B_1(\varphi+\varepsilon) \end{pmatrix} - \begin{pmatrix} A(\varphi) \\ B(\varphi) \end{pmatrix} = T_1(\varphi)\begin{pmatrix} A(\varphi) \\ B(\varphi) \end{pmatrix}, \quad \varphi, \varphi+\varepsilon \in D. \tag{4.2}$$

4.1.4. The operator in the left-hand side of (4.2) can be easily inverted (see, for example [3]); so that, finally, we come to the equation

$$\begin{pmatrix} A_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + K \begin{pmatrix} A_1 \\ B_2 \end{pmatrix}, \tag{4.3}$$

where K is a singular integral operator acting in a space of Hölder functions defined on a vertical curve $\gamma \subset D$. If the curve γ is canonical then the norm of the operator is small. So, we can construct a solution of (4.3). This solution can be analytically continued in the domain D, and, as a result, one gets an analytic solution satisfying the condition (1.7).

4.2. **Solutions of equation (1.2) close to Bloch solutions of equation (1.8).** Let D be a bounded domain in the complex plane of φ such that $\overline{D}$ does not contain any branching points of the complex momentum. Define the functions $\psi_{\pm}(x, \mathcal{E})$ as in section 3.2, and let

$$\psi_{\pm}^0(x,\varphi) = \psi_{\pm}(x, E - W(\varphi)), \quad \varphi \in D. \tag{4.4}$$

In the sequel, we shall assume that $\overline{D}$ does not contain any pole of $\psi_{\pm}^0(x,\varphi)$[1].

The functions $\psi_{\pm}^0(x,\varphi)$ are two linearly independent Bloch solutions of the equation (1.8) (with $\mathcal{E}$ given by (1.9)) with the Bloch quasi-momenta $\pm\kappa(\varphi)$ related to the quasi-momenta of $\psi_{\pm}(x,\mathcal{E})$ by the formula

$$\kappa(\varphi) = k(E - W(\varphi)).$$

We denote the Wronskian of the solutions $\psi_{\pm}^0(x,\varphi)$ by $w_0(\varphi)$ i.e.

$$w_0(\varphi) = w\left(\psi_+^0(.,\varphi), \psi_-^0(.,\varphi)\right).$$

Clearly, $w_0(\varphi)$ is analytic and has no zeros in $\overline{D}$.

Fix $X > 0$. Let us construct two solutions $\psi_{\pm}^1$ of equation (1.2) close to $\psi_{\pm}^0$ for $\varepsilon \to 0$ uniformly in $\varphi \in \overline{D}$ and $x \in [-X, X]$.

4.2.1. Fix a positive X_1 so that $0 < X_1 < X$. Define an integral operator acting on $C([-X,X])$ by the formula

$$R f(x,\varphi) = \psi_+^0(x,\varphi) \int_{-X_1}^{x} \psi_-^0(t,\varphi) f(t)\, dt - \psi_-^0(x,\varphi) \int_{X_1}^{x} \psi_+^0(t,\varphi) f(t)\, dt. \tag{4.5}$$

This operator coincides with a Green's function of equation (1.8) – (1.9). Indeed, letting

$$H_0 = -\frac{d^2}{dx^2} + V(x) + W(\varphi) - E,$$

we get

$$H_0 R f = -w_0 f.$$

Clearly, under our assumptions on V and W and on the domain D, the operator R is bounded,

$$\sup_{\varphi \in D} \|R\| \le C. \tag{4.6}$$

4.2.2. For $\psi_{\pm}^1$, take the ansatz

$$\psi_{\pm}^1 = \psi_{\pm}^0 + \varepsilon R g_{\pm} \tag{4.7}$$

Substituting this ansatz into (1.2) and assuming that $\psi_{\pm}^1$ are solutions of this equation, we get

$$-\Delta \cdot \psi_{\pm}^0 = -w_0 g_{\pm} + \varepsilon \Delta \cdot R g_{\pm}, \tag{4.8}$$

where

$$\Delta \equiv \Delta(\varepsilon x, \varphi) = \frac{W(\varepsilon x + \varphi) - W(\varphi)}{\varepsilon}. \tag{4.9}$$

[1] From the very beginning, we could have considered the canonical Bloch solutions associated to the domain D (since they are analytic in D). However, we did not want to introduce these objects as an "ansatz" but show how they arise naturally in the constructions of the complex WKB method. This will neither complicate nor enlarge our analysis.

4.2.3. Since the Wronskian $w_0(\varphi)$ has no zeros in $\overline{D}$, then

$$\left|\frac{\Delta}{w_0}\right| \le C, \quad -X \le x \le X, \quad \varphi \in D,$$

where C is clearly independent of ε, x and φ. Therefore, for sufficiently small ε, equation (4.8) has a unique solution in $C([-X, X])$; this solution satisfies

$$\sup_{\varphi \in D} \|g_\pm\|_\infty \le C. \tag{4.10}$$

The formula (4.8) implies that

$$g_\pm = \frac{\Delta}{w_0}\psi^0_\pm + O(\varepsilon), \quad \varphi \in D, \quad -X \le x \le X \tag{4.11}$$

in the sense of the supremum norm.

4.2.4. By formula (4.7), we reconstruct the function $\psi^1_\pm$ in terms of $g_\pm$. This leads to the following results:

1. for any fixed $\varphi \in D$, $\psi^1_\pm$ satisfies (1.2) in $C([-X, X])$;
2. for any fixed $x \in [-X, X]$, $\psi^1_\pm$ is analytic in $\varphi \in D$;
3. if $-X \le x \le X$ and $\varphi \in D$, then

$$\psi^1_\pm(x, \varphi) = \psi^0_\pm(x, \varphi) + O(\varepsilon), \tag{4.12}$$

the error estimate being uniform in φ and x.

4.2.5. We shall need the asymptotics for the Wronskian $w_1(\varphi)$ of the solutions $\psi^1_+(x, \varphi)$ and $\psi^1_-(x, \varphi)$. Therefore we prove the

Lemma 4.1. *For $\varepsilon \to 0$,*

$$w_1(\varphi) = w_0(\varphi) + \varepsilon W'(\varphi) \int_{-X_1}^{X_1} \psi^0_+(x, \varphi)\, \psi^0_-(x, \varphi)\, x\, dx + O(\varepsilon^2) \tag{4.13}$$

uniformly in $\varphi \in D$.

Proof. By (4.7),

$$w_1 = w(\psi^0_+ + \varepsilon R g_+, \psi^0_- + \varepsilon R g_-) = w_0 + \varepsilon\left(w(\psi^0_+, R g_-) + w(R g_+, \psi^0_-)\right) + O(\varepsilon^2). \tag{4.14}$$

Now, using the definition (4.5) of R, we obtain

$$w(\psi^0_+, R g_-) + w(R g_+, \psi^0_-) = -w_0 \int_{X_1}^{x} \psi^0_+ g_-\, dt + w_0 \int_{-X_1}^{x} \psi^0_- g_+\, dt.$$

Substituting the expressions given by (4.11) for $g_\pm$ in this formula, we get

$$w(\psi^0_+, R g_-) + w(R g_+, \psi^0_-) = \int_{-X_1}^{X_1} \Delta\, \psi^0_+\, \psi^0_-\, dx + O(\varepsilon).$$

This formula and formula (4.14) imply the desired result as $\Delta(\varepsilon x, \varphi) = W'(\varphi)\, x + O(\varepsilon)$. □

Note that, since w_0 has no zeros in $\overline{D}$, Lemma 4.1 implies in particular that, for sufficiently small ε, the newly constructed solutions are linearly independent for any $\varphi \in D$.

4.3. **Consistency condition.** The functions $\psi^1_\pm(x, \varphi)$ satisfy (1.2), but they need not form a consistent basis of solutions for the family of equations (1.2). However, in terms of these two solutions, we will construct two consistent ones.

Let $A(\varphi)$ and $B(\varphi)$ be analytic functions of φ. Define

$$\psi(x, \varphi) = A(\varphi)\, \psi^1_+(x, \varphi) + B(\varphi)\, \psi^1_-(x, \varphi), \quad \varphi \in D, \tag{4.15}$$

Clearly, $\psi(x, \varphi)$ verifies (1.2). This solution satisfies the consistency condition (1.7) if and only if, for $\varphi, \varphi + \varepsilon \in D$, $x \in \mathbb{R}$, A and B satisfy

$$A(\varphi + \varepsilon)\, \psi^1_+(x - 1, \varphi + \varepsilon) + B(\varphi + \varepsilon)\, \psi^1_-(x - 1, \varphi + \varepsilon) = A(\varphi)\, \psi^1_+(x, \varphi) + B(\varphi)\, \psi^1_-(x, \varphi). \tag{4.16}$$

This relation implies that

$$\left(\begin{array}{c} A(\varphi+\varepsilon) \\ B(\varphi+\varepsilon) \end{array}\right) = T(\varphi) \left(\begin{array}{c} A(\varphi) \\ B(\varphi) \end{array}\right), \quad \varphi,\ \varphi+\varepsilon \in D, \tag{4.17}$$

where T is a 2×2-matrix with coefficients

$$\begin{aligned} T_{11} &= \frac{w\,(\psi_+^1(\,.\,,\varphi),\, \psi_-^1(\,.\,-1,\varphi+\varepsilon))}{w_1(\varphi+\varepsilon)}, \ T_{12} = \frac{w\,(\psi_-^1(\,.\,,\varphi), \psi_-^1(\,.\,-1,\varphi+\varepsilon))}{w_1(\varphi+\varepsilon)}, \\ T_{21} &= \frac{w\,(\psi_+^1(\,.\,-1,\varphi+\varepsilon), \psi_+^1(\,.\,,\varphi))}{w_1(\varphi+\varepsilon)}, \ T_{22} = \frac{w\,(\psi_+^1(\,.\,-1,\varphi+\varepsilon),\, \psi_-^1(\,.\,,\varphi))}{w_1(\varphi+\varepsilon)}. \end{aligned} \tag{4.18}$$

Now, our aim is to construct two analytic solutions of the difference equation (4.17) to get a consistent basis for the family of equations (1.2). Therefore, we shall first study the matrix T.

4.4. **Asymptotics for the coefficients of the matrix T.** When constructing the solutions of the difference equation (4.17), we shall need to know the diagonal elements of the matrix T up to an error of size $O(\varepsilon^2)$, and the anti-diagonal elements up to an error of size $O(\varepsilon)$.

One has

Proposition 4.1. *For sufficiently small ε,*

$$T_{11} = \exp\left(i\kappa\,(\varphi+\varepsilon/2) + \frac{\varepsilon}{2}\frac{d}{d\varphi}\ln\, k_E + \varepsilon\omega_+(\varphi)\right) + O(\varepsilon^2),$$
$$T_{22} = \exp\left(-i\kappa\,(\varphi+\varepsilon/2) + \frac{\varepsilon}{2}\frac{d}{d\varphi}\ln\, k_E + \varepsilon\omega_-(\varphi)\right) + O(\varepsilon^2),$$

$$T_{12} = O(\varepsilon), \qquad T_{21} = O(\varepsilon),$$

where the error estimates are uniform in $\varphi \in D$. Here, the functions $\omega_\pm$ are defined by formulae (3.7), and $k_E = k'(E - W(\varphi)) = \left(\frac{d}{dE}k\right)(E - W(\varphi))$.

Proof. The estimates of T_{12} and T_{21} are elementary. For example, as $\psi_\pm^1$ are analytic in $\varphi \in D$,

$$T_{12} = \frac{w\,(\psi_-^1(\,.\,,\varphi),\, \psi_-^1(\,.\,-1,\varphi+\varepsilon))}{w_1(\varphi+\varepsilon)} = \frac{w\,(\psi_-^1(\,.\,,\varphi),\, \psi_-^1(\,.\,-1,\varphi))}{w_1(\varphi)} + O(\varepsilon), \tag{4.19}$$

and since

$$\psi_-^1(x-1,\varphi)) = e^{i\kappa\,(\varphi)}\,\psi_-^1(x,\varphi), \quad x\in\mathbb{R}, \quad \varphi\in D,$$

the leading term in (4.19) is zero.

To derive the asymptotics of T_{11} and T_{22}, we shall use Lemma 2.1. We shall check only the formula for T_{11}. The second one is obtained in a similar way.

In view of (4.7), for φ, $\varphi+\varepsilon\in D$, we have

$$\begin{aligned} \psi_-^1(x-1,\varphi+\varepsilon) &= \psi_-^0(x-1,\varphi+\varepsilon) + \varepsilon\,(R\,g_-)\,(x-1,\varphi+\varepsilon) \\ &= e^{i\kappa\,(\varphi+\varepsilon)}\psi_-^0(x,\varphi+\varepsilon) + \varepsilon\,(R\,g_-)\,(x-1,\varphi+\varepsilon) \\ &= e^{i\kappa\,(\varphi)}\psi_-^0(x,\varphi) + \varepsilon\,\Big(i\kappa'(\varphi)\,e^{i\kappa\,(\varphi)}\psi_-^0(x,\varphi) + \\ &\qquad + e^{i\kappa\,(\varphi)}\frac{\partial\psi_-^0}{\partial\varphi}(x,\varphi) + (R\,g_-)\,(x-1,\varphi)\Big) + O(\varepsilon^2). \end{aligned}$$

Now, again using (4.7) for ψ_+^1 and the result of the above calculation, we get

$$\begin{aligned} w\,(\psi_+^1(\,.\,,\varphi),&\psi_-^1(\,.\,-1,\varphi+\varepsilon)) = e^{i\kappa}\,w_0 + \varepsilon\,\left(i\kappa'\,e^{i\kappa}\,w_0 + e^{i\kappa}\,w\left(\psi_+^0(x),\, \frac{\partial\psi_-^0}{\partial\varphi}(x)\right) + \right. \\ &\left. + w\,(Rg_+(x),\, e^{i\kappa}\psi_-^0(x)) + w\,(\psi_+^0(x),\, Rg_-(x-1))\right) + O(\varepsilon^2). \end{aligned} \tag{4.20}$$

Here, for the sake of brevity, we do not write the argument φ anymore. Elementary calculations show that

$$w\,(Rg_+(x),\,\psi_-^0(x)) = w_0\int_{-X_1}^{x}\psi_-^0(t)\,g_+(t)\,dt,$$
$$w\,(\psi_+^0(x),\,Rg_-(x-1)) = -w_0e^{i\kappa}\int_{X_1}^{x-1}\psi_+^0(t)\,g_-(t)\,dt,$$

and, using formula (4.11) as when proving Lemma 4.1, and the analyticity of W, we get

$$w\,(Rg_+(x),e^{i\kappa}\psi_-^0(x)) + w\,(\psi_+^0(x),\,Rg_-(x-1)) =$$
$$= e^{i\kappa}\,W'\left(\int_{-X_1}^{X_1}\psi_+^0(t)\,\psi_-^0(t)\,t\,dt + \int_{x-1}^{x}\psi_+^0(t)\,\psi_-^0(t)\,t\,dt\right) + O\,(\varepsilon).$$

Substituting this expression into (4.20) and taking into account Lemma 4.1, we obtain

$$\begin{aligned} w\,(\psi_+^1(\,.\,,\varphi),\psi_-^1(\,.\,-1,\varphi+\varepsilon)) &= e^{i\kappa}\,w_1 + \varepsilon\,e^{i\kappa}\Bigg(i\kappa'\,w_0 + \\ &+ w\,(\psi_+^0(x),\,\frac{\partial\psi_-^0}{\partial\varphi}\,(x)) + W'\int_{x-1}^{x}\psi_+^0(t)\,\psi_-^0(t)\,t\,dt\Bigg) + O\,(\varepsilon^2). \end{aligned} \tag{4.21}$$

But, in view of Corollary 2.1, we have

$$w\,(\psi_+^0(x),\,\frac{\partial\psi_-^0}{\partial\varphi}\,(x)) + W'\int_{x-1}^{x}\psi_+^0(t)\,\psi_-^0(t)\,t\,dt = w_0' + w_0\left(\frac{1}{2}\frac{d}{d\varphi}\ln\,k_E - \frac{i}{2}\,\kappa'(\varphi) + \omega_+\right).$$

Substituting this result in (4.21), one easily gets

$$\begin{aligned} w\,(\psi_+^1(\,.\,,\varphi),\,\psi_-^1(\,.\,-1,\varphi+\varepsilon)) &= e^{i\kappa}\left(w_1 + \varepsilon\,\left(i\kappa' w_0/2 + w_0' + w_0\,\left(\frac{1}{2}\frac{d}{d\varphi}\,\ln\,k_E + \omega_+\right)\right)\right) + O\,(\varepsilon^2)\\ &= e^{i\kappa}\,w_1(\varphi+\varepsilon)\,\left(1 + \varepsilon\left(i\kappa'/2 + \frac{1}{2}\frac{d}{d\varphi}\,\ln\,k_E + \omega_+\right)\right) + O\,(\varepsilon^2)\\ &= w_1(\varphi+\varepsilon)\,\exp\left(i\kappa\,(\varphi+\varepsilon/2) + \frac{\varepsilon}{2}\frac{d}{d\varphi}\,\ln\,k_E + \varepsilon\omega_+\right) + O\,(\varepsilon^2). \end{aligned}$$

We have indicated explicitly the cases where the argument of the functions differs from φ. This, together with the definition for T_{11} given by (4.18), implies the desired result. □

4.5. **Transformation of the difference equation into an integral one.** To construct analytic solutions of equation (4.17), we transform it into an integral equation. This transformation consists in three steps.

4.5.1. Note that, in view of Proposition 4.1, the matrix T is close to a diagonal one. First, we transform the functions A and B so that the diagonal elements of the new matrix be close to 1. Therefore, we define

$$A\,(\varphi) = e^{i\theta_A(\varphi)}\,A_1(\varphi), \quad B\,(\varphi) = e^{i\theta_B(\varphi)}\,B_1(\varphi). \tag{4.22}$$

Hence, equation (4.17) takes the form

$$\begin{pmatrix} A_1(\varphi+\varepsilon) \\ B_1(\varphi+\varepsilon) \end{pmatrix} = \hat{T}\,(\varphi)\,\begin{pmatrix} A_1(\varphi) \\ B_1(\varphi) \end{pmatrix}, \quad \varphi,\ \varphi+\varepsilon\in D, \tag{4.23}$$

with

$$\hat{T} = \begin{pmatrix} e^{-\theta_A(\varphi+\varepsilon)\,+\,\theta_A(\varphi)}\,T_{11} & e^{-\theta_A(\varphi+\varepsilon)\,+\,\theta_B(\varphi)}\,T_{12} \\ e^{-\theta_B(\varphi+\varepsilon)\,+\,\theta_A(\varphi)}\,T_{21} & e^{-\theta_B(\varphi+\varepsilon)\,+\,\theta_B(\varphi)}\,T_{22} \end{pmatrix}.$$

Fix a point φ_0 in D and set

$$\theta_A(\varphi) = \frac{i}{\varepsilon}\int_{\varphi_0}^{\varphi}\kappa\,d\varphi + \frac{1}{2}\,\ln\,k_E + \int_{\varphi_0}^{\varphi}\omega_+\,d\varphi \tag{4.24}$$

$$\theta_B(\varphi) = -\frac{i}{\varepsilon}\int_{\varphi_0}^{\varphi}\kappa\,d\varphi + \frac{1}{2}\,\ln\,k_E + \int_{\varphi_0}^{\varphi}\omega_-\,d\varphi. \tag{4.25}$$

In this case, in view of Proposition 4.1,

$$\hat{T} = I + \hat{T}_1, \quad \hat{T}_1 = \begin{pmatrix} O(\varepsilon^2) & e^{-2\theta_0} O(\varepsilon) \\ e^{2\theta_0} O(\varepsilon) & O(\varepsilon^2) \end{pmatrix}, \tag{4.26}$$

where

$$\theta_0 = \frac{i}{\varepsilon} \int_{\varphi_0}^{\varphi} \kappa \, d\varphi, \tag{4.27}$$

and the symbol $O(\varepsilon^l)$ denotes functions satisfying the estimate

$$|O(\varepsilon^l)| \le C \varepsilon^l, \quad \varphi \in D,$$

uniformly in φ.

4.5.2. A simple example of how to pass from a difference equation to an integral equation is given by the

Lemma 4.2. *Let $f(z)$ be a function of $z \in \mathbb{C}$ analytic in the strip $Y_1 \le \operatorname{Im} z \le Y_2$, $Y_1, Y_2 \in \mathbb{R}$. Fix z_1 and z_2 so that $\operatorname{Im} z_1 = Y_1$, $\operatorname{Im} z_2 = Y_2$. Let γ_z be a smooth vertical curve going from z_1 to z_2 via z. The function*

$$g(z) = \frac{1}{2i} \int_{\gamma_z} \cot \pi(\zeta - z - 0) \, f(\zeta) \, d\zeta$$

is a solution of the equation

$$g(z+1) - g(z) = f(z)$$

analytic in the strip $Y_1 < \operatorname{Im} z < Y_2$.

The proof of this lemma immediately follows from the residue theorem. For analogous statements see, for example, [3].

We will follow the idea of this lemma to transform equation (4.23) into an integral equation.

4.5.3. We are ready to make the last two steps of the desired transformation. However, to construct the solutions $f_\pm$ described in Theorem 1.1, one has to make these two steps in two different ways depending on whether one is dealing with f_+ or f_-. We shall concentrate on f_+. The solution f_- can be constructed similarly and we shall discuss the details of this construction in the end of section 3.

Fix two points ζ_1 and ζ_2 in D so that $\operatorname{Im} \zeta_1 < \operatorname{Im} \zeta_2$ and so that they can be connected by a vertical curve $\gamma \subset D$. For a function $f(\varphi)$ sufficiently regular on γ, define

$$L_+ f(\varphi) = \frac{1}{2i\varepsilon} \int_{\gamma} \left(\cot \left[\frac{\pi(\varphi' - \varphi - 0)}{\varepsilon} \right] - i \right) f(\varphi') \, d\varphi'. \tag{4.28}$$

To construct a solution of (4.23) analytic in the domain

$$\{\varphi \in D; \ \operatorname{Im} \zeta_1 < \operatorname{Im} \varphi < \operatorname{Im} \zeta_2\},$$

we consider the equation

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + L_+ \hat{T}_1 \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}, \quad \varphi \in \gamma \subset D. \tag{4.29}$$

4.5.4. We need to make one more transformation. We fix α such that

$$0 < \alpha < 1,$$

and let

$$B_1 = \varepsilon^\alpha e^{2\theta_0} B_2, \tag{4.30}$$

so that equation (4.29) be transformed into

$$\begin{pmatrix} A_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} L_+ O(\varepsilon^2) & L_+ O(\varepsilon^{1+\alpha}) \\ K_+ O(\varepsilon^{1-\alpha}) & K_+ O(\varepsilon^2) \end{pmatrix} \begin{pmatrix} A_1 \\ B_2 \end{pmatrix}, \tag{4.31}$$

where $\varphi \in \gamma \subset D$ and

$$K_+ f(\varphi) = e^{-2\theta_0(\varphi)} L_+ \left(e^{2\theta_0(.)} f(.) \right)(\varphi). \tag{4.32}$$

In the next subsection, we shall show that, if γ is a canonical curve, then, after a proper choice of the functional space, the operator in the right hand side of (4.31) is small, and thus, that the equation (4.31) has a unique solution. Then we show that this solution can be analytically continued outside the contour γ. This will allow us to construct an analytic solution of the system (4.17).

We note that an operator similar to K_+ was studied in [3]. It was the integral operator with the kernel

$$\mathcal{L}(z,z') = \frac{1}{2ih}\left(\tan\frac{\pi(z'-z)}{h} - i\right) e^{\frac{2i}{h}\int_z^{z'} p(\zeta)\,d\zeta},$$

acting in the space of functions analytic in a vicinity of a vertical curve going from $-i\infty$ to $+i\infty$. One can immediately see that, if, along this curve, $\mathrm{Im}\int^z p\,d\zeta$ is strictly monotonically increasing and $\mathrm{Im}\int^z (p-\pi)\,d\zeta$ is strictly monotonically decreasing, then $|\mathcal{L}(z,z')| \le \mathrm{Const}\, h^{-1}\, e^{-\delta\,|\mathrm{Im}\, z - \mathrm{Im}\, z'|/h}$, $\delta > 0$, and, thus, for small h, the corresponding operator is bounded. In our case, the estimates are more complicated since K_+ is a singular integral operator.

4.6. **Estimates of the norms of the integral operators.** In this subsection we estimate the norms of L_+ and K_+ as the operators acting in a proper functional space. We assume that the curve γ introduced above is sufficiently smooth. In the sequel, the letter C will denote positive constants independent of ε.

4.6.1. *Functional space.* Let γ' be the curve γ without the end points. Fix two positive numbers a and b so that

$$0 < b < a < 1. \tag{4.33}$$

For a function f defined on γ', we let

$$\|f\|_{a,b} = \sup_{\varphi\in\gamma'} |\, f_a(\varphi)| + \sup_{\varphi,\,\varphi'\in\gamma'} \frac{|f_a(\varphi) - f_a(\varphi')|}{|\varphi-\varphi'|^b}, \tag{4.34}$$

where

$$f_a(\varphi) = |\varphi-\zeta_1|^a\, |\varphi-\zeta_2|^a\, f(\varphi).$$

We say that f belongs to $C_{a,b}(\gamma)$ if $\|f\|_{a,\,b} < \infty$. One can easily see that

1. The set $C_{a,b}(\gamma)$ equipped with the norm (4.34) is a Banach space.
2. For any $f \in C^1(\gamma)$,

$$\|f\|_{a,b} \le C\,\|f\|_{C^1} \tag{4.35}$$

and

$$\|f\|_{a,b} \le C\,\|f\|_C + C\,\|f\|_C^{1-b}\,\|f'\|_C^b \tag{4.36}$$

with some $C > 0$ independent of f.

We have chosen the space $C_{a,b}(\gamma)$ as, in particular, it is well known (see, for example, [10]) that the singular integral operator defined by the formula

$$Sf(\varphi) = \frac{1}{2\pi i}\int_\gamma \frac{f(\varphi')}{\varphi'-\varphi-0}\,d\varphi', \quad \varphi\in\gamma', \tag{4.37}$$

is a bounded operator in this space.

4.6.2. We shall need the following two lemmas

Lemma 4.3. *Let γ be a bounded vertical curve. Then, there exists $\delta > 0$ such that the angle θ between γ and the horizontal axis satisfies the inequality $\delta < \theta < \pi - \delta$.*

and

Lemma 4.4. *Let γ be a bounded canonical curve. Then, there exists $\delta > 0$ such that*

$$Im\int_\varphi^{\varphi'} \kappa\, d\varphi \ge \delta\,\mathrm{Im}\,(\varphi'-\varphi), \tag{4.38}$$

$$Im\int_\varphi^{\varphi'} (\kappa-\pi)\, d\varphi \le -\delta\,\mathrm{Im}\,(\varphi'-\varphi). \tag{4.39}$$

The lemmas immediately follow from the definitions of a vertical and a canonical curve and from the compactness of a bounded curve.

4.6.3. *Estimates of the norms .* The operators L_+ and K_+ are bounded operators acting in $C_{a,b}(\gamma)$. The estimate of the norm of L_+ does not require any additional condition on γ. The estimate of the norm of K_+ is the principal point in the proof of Theorem 1.1. It is in this estimate where one has to use the definition of the canonical curves. We shall first describe the estimates of the norms of these two operators, and then, prove them. The estimates given below are rather rough, but we do not need any sharper ones.

Begin with the operator L_+.

Lemma 4.5. *Let γ be sufficiently smooth vertical curve. The norm of the operator L_+ in $C_{a,b}(\gamma)$ satisfies the estimate*

$$\|L_+\| \leq C\,\varepsilon^{-1-a}. \tag{4.40}$$

One can see that the norm of L_+ can be estimated from below be $C\,/\,\varepsilon$. At the same time, if γ is a canonical curve, then the norm of K_+ satisfies a much better estimate. We recall that a and b are two arbitrary positive numbers fixed so that $0 < b < a < 1$. One has

Proposition 4.2. *Let γ be sufficiently smooth canonical curve. The norm of the operator K_+ in $C_{a,b}(\gamma)$ satisfies the estimate*

$$\|K_+\| \leq C\,\varepsilon^{-a-b}. \tag{4.41}$$

Proof of Lemma 4.5. For $f \in C_{a,b}(\gamma)$, represent L_+f in the form

$$L_+f = R_L\,f + S\,f,$$

where S is the singular integral operator (4.37) and

$$R_Lf\,(\varphi) = \frac{1}{2i\varepsilon}\int_{\gamma'} r_L\,\left(\frac{\varphi'-\varphi}{\varepsilon}\right)\,f\,(\varphi')\,d\varphi',$$

where

$$r_L\,(\varphi) = \frac{1}{\varphi}\,\left(\,\varphi\,(\,\cot\,(\pi\varphi) - i\,) - \frac{1}{\pi}\right).$$

The norm of S is bounded by a constant independent of ε. Let us estimate the norm of R_L.

Note that, in view of Lemma 4.3, along any given vertical curve passing via the point $\varphi_0 = 0$,

$$|r_L\,(\varphi)| \leq C, \tag{4.42}$$

and

$$|r_L'\,(\varphi)| \leq C\left\{\begin{array}{ll} 1 & |\eta| \leq 1, \\ e^{-2\pi\,|\eta|} + \frac{1}{|\eta|^2}\,, & |\eta| \geq 1, \end{array}\right. \qquad \eta = \operatorname{Im}\varphi. \tag{4.43}$$

The constants here depend only on the vertical curve. Below, we shall denote $\operatorname{Im}\varphi$ by η.

First, we estimate $\|R_L\,f\|_C$. Fix $\varphi' \in \gamma$. Since the transformation $\varphi \mapsto \frac{\varphi-\varphi'}{\varepsilon}$ does not change the angles on the complex plane $\mathbb{C}$, the inequality (4.42) takes the form

$$\left|r_L\,\left(\frac{\varphi-\varphi'}{\varepsilon}\right)\right| \leq C, \quad \varphi, \varphi' \in \gamma,$$

with C is independent of ε, φ and φ'. This immediately implies that

$$\|R_L\,f\|_C \leq C\,\varepsilon^{-1}\int_{\eta_1}^{\eta_2}\frac{d\eta'}{(\eta'-\eta_1)^a\,(\eta_2-\eta')^a}\,\|f\|_{a,\,b} = C\,\varepsilon^{-1}\,\|f\|_{a,\,b}. \tag{4.44}$$

Here $\eta_{1,2} = \operatorname{Im}\zeta_{1,2}$.

Similarly, using (4.42), we get

$$|(R_Lf)'(\varphi)| \leq C\,\varepsilon^{-2}\,\left(\,I_1\,(\eta) + I_2\,(\eta) + I_3\,(\eta)\,\right)\,\|f\|_{a,\,b}, \quad \eta = \operatorname{Im}\varphi,$$

where

$$I_1 = \int_{\eta_1,\ |\eta-\eta'|\le\varepsilon}^{\eta_2} \frac{d\eta'}{(\eta'-\eta_1)^a\,(\eta_2-\eta')^a},$$

$$I_2 = \varepsilon^2 \int_{\eta_1,\ |\eta-\eta'|\ge\varepsilon}^{\eta_2} \frac{d\eta'}{|\eta'-\eta|^2\,(\eta'-\eta_1)^a\,(\eta_2-\eta')^a},$$

$$I_3 = \int_{\eta_1,\ |\eta-\eta'|\ge\varepsilon}^{\eta_2} \frac{e^{-2\pi\,|\eta-\eta'|/\varepsilon}\,d\eta'}{(\eta'-\eta_1)^a\,(\eta_2-\eta')^a}.$$

Now, elementary estimates show that

$$\sup_{\eta_1<\eta<\eta_2} |I_j| \le C\,\varepsilon^{1-a}, \quad j=1,2,3.$$

Thus,

$$\|(R_Lf)'\|_C \le C\,\varepsilon^{-1-a}\|f\|_{a,b}. \tag{4.45}$$

In view of (4.35), (4.44) and (4.45) imply the desired result. □

Proof of Proposition 4.2. As in the proof of Lemma 4.5, we represent K_+f in the form

$$K_+f = R_K\,f + S\,f,$$

where

$$R_Kf\,(\varphi) = \frac{1}{2i\varepsilon}\int_{\gamma'} r_K\,(\varphi,\,\varphi')\,f\,(\varphi')\,d\varphi',$$

$$r_K\,(\varphi,\varphi') = \frac{\varepsilon}{\varphi'-\varphi}\left(\frac{(\varphi'-\varphi)}{\varepsilon}\,e\,(\varphi,\varphi') - \frac{1}{\pi}\right),$$

$$e\,(\varphi,\varphi') = \left(\cot\left(\frac{\pi(\varphi'-\varphi)}{\varepsilon}\right) - i\right)\,e^{\frac{2i}{\varepsilon}\int_\varphi^{\varphi'}\kappa\,(s)\,ds}.$$

We begin by estimating $\|R_K\,f\|_C$. As before, let $\eta = \operatorname{Im}\varphi$. Note that

$$\left|\int_\varphi^{\varphi'}\kappa\,(s)\,ds\right| \le C\,|\varphi-\varphi'|, \quad \varphi,\varphi'\in\gamma.$$

Therefore,

$$e\,(\varphi,\varphi') = \frac{\varepsilon}{\varphi-\varphi'}\left(\frac{1}{\pi} + \frac{1}{\varepsilon}\,O\,(\varphi-\varphi')\right), \quad \varphi,\varphi'\in\gamma, \quad |\eta-\eta'|\le\varepsilon,$$

where the error estimate is uniform in ε, φ and φ'. So,

$$|r_K\,(\varphi,\,\varphi')| \le C, \qquad |\eta-\eta'|\le\varepsilon, \qquad \varphi,\varphi'\in\gamma. \tag{4.46}$$

Now, we assume that γ is canonical. Due to the definition of a canonical curve, we show that

$$|e\,(\varphi,\varphi')| \le C\,e^{-\frac{\delta}{\varepsilon}\,|\eta-\eta'|}, \qquad |\eta-\eta'|\ge\varepsilon, \qquad \varphi,\varphi'\in\gamma, \tag{4.47}$$

where δ is a positive constant independent of ε.

First, we consider the case where $\eta-\eta'\ge\varepsilon$. In this case, since γ is vertical, due to Lemma 4.3, we have

$$\left|\cot\left(\frac{\pi(\varphi'-\varphi)}{\varepsilon}\right) - i\right| \le C\,e^{-\frac{2\pi}{\varepsilon}\,(\eta-\eta')}, \tag{4.48}$$

and, therefore,

$$|e\,(\varphi,\varphi')| \le C\,e^{\frac{2}{\varepsilon}\operatorname{Im}\int_{\varphi'}^{\varphi}(\kappa\,(s)-\pi)\,ds}, \qquad \varphi,\varphi'\in\gamma.$$

In view of Lemma 4.4, we know that

$$\operatorname{Im}\int_{\varphi'}^{\varphi}(\kappa\,(s)-\pi)\,ds \le -\delta_2\,(\eta-\eta'),$$

with some positive constant δ_2 (see (4.38)). This implies (4.47).

Now, we consider the case where $\eta - \eta' \le -\varepsilon$. In this case, instead of (4.48), we get

$$\left| \cot\left(\frac{\pi(\varphi' - \varphi)}{\varepsilon}\right) - i \right| \le C,$$

and, therefore,

$$|e(\varphi, \varphi')| \le C\, e^{-\frac{2}{\varepsilon} \operatorname{Im} \int_\varphi^{\varphi'} \kappa(s)\, ds}, \qquad \varphi, \varphi' \in \gamma.$$

Again, by Lemma 4.4, we obtain

$$\operatorname{Im} \int_\varphi^{\varphi'} \kappa(s)\, ds \ge \delta\,(\eta' - \eta),$$

with some positive δ, which again leads to (4.47).

The estimate (4.47) and the definition of r_K imply the inequality

$$|r_K(\varphi, \varphi')| \le C\, e^{-\frac{\delta}{\varepsilon}|\eta - \eta'|} + \frac{C\varepsilon}{|\eta - \eta'|}, \qquad |\eta - \eta'| \ge \varepsilon, \qquad \varphi, \varphi' \in \gamma. \tag{4.49}$$

The inequalities (4.46), (4.49) and the definition of R_K imply that

$$|R_K\, f(\varphi)| \le C\,(\,I_4(\eta) + I_5(\eta) + I_6(\eta)\,)\ \|f\|_{a,b}, \quad \varphi \in \gamma,$$

where

$$I_4 = \frac{1}{\varepsilon} \int_{\eta_1,\ |\eta - \eta'| \le \varepsilon}^{\eta_2} \frac{d\eta'}{(\eta' - \eta_1)^a\,(\eta_2 - \eta')^a},$$

$$I_5 = \int_{\eta_1,\ |\eta - \eta'| \ge \varepsilon}^{\eta_2} \frac{d\eta'}{|\eta' - \eta|\,(\eta' - \eta_1)^a\,(\eta_2 - \eta')^a},$$

$$I_6 = \frac{1}{\varepsilon} \int_{\eta_1,\ |\eta - \eta'| \ge \varepsilon}^{\eta_2} \frac{e^{-\delta\,|\eta - \eta'|/\varepsilon}\, d\eta'}{(\eta' - \eta_1)^a\,(\eta_2 - \eta')^a}.$$

Elementary estimates show that

$$\sup_{\eta_1 < \eta < \eta_2} |I_j| \le C\,\varepsilon^{-a}, \quad j = 4, 5, 6.$$

And, as a result, we get

$$\|R_K f\|_C \le C\,\varepsilon^{-a} \|f\|_{a,b}. \tag{4.50}$$

Having estimated $\|R_K f\|_C$, we estimate $\|(R_K f)'\|_C$. We first show that, for $\varphi, \varphi' \in \gamma$,

$$\left|\frac{\partial r_K}{\partial \varphi}(\varphi, \varphi')\right| \le C\,\varepsilon^{-1} \begin{cases} 1 & \text{when } |\eta - \eta'| \le \varepsilon, \\ e^{-2\pi\,|\eta - \eta'|/\varepsilon} + \frac{\varepsilon^2}{|\eta - \eta'|^2}, & \text{when } |\eta - \eta'| \ge \varepsilon, \end{cases} \tag{4.51}$$

Note that

$$\frac{\partial r_K}{\partial \varphi}(\varphi, \varphi') = \left(\frac{\pi}{\varepsilon}\,\left(\sin^2\zeta\,(\cot\,\zeta - i)\,\right)^{-1} - \frac{2i}{\varepsilon}\,\kappa(\varphi)\right)\, e(\varphi, \varphi') - \frac{\varepsilon}{\pi(\varphi - \varphi')^2}, \tag{4.52}$$

where

$$\zeta = \frac{\pi(\varphi' - \varphi)}{\varepsilon}.$$

Now, first, we assume that $|\eta - \eta'| \le \varepsilon$. We recall that $\gamma \subset D$, and, thus, the curve γ is situated at some distance from the branching points of the quasi-momentum κ. Therefore,

$$\int_\varphi^{\varphi'} \kappa(s)\, ds = (\varphi' - \varphi)\,\kappa(\varphi) + O\left((\varphi - \varphi')^2\right), \quad \varphi, \varphi' \in \gamma,$$

with the error estimate uniform in φ' and ε. This allows us to get

$$e(\varphi, \varphi') = \frac{\varepsilon}{\pi(\varphi' - \varphi)}\,\left(1 - i\frac{\pi(\varphi' - \varphi)}{\varepsilon} + 2i\kappa(\varphi)\,\frac{(\varphi' - \varphi)}{\varepsilon} + O\left(\frac{(\varphi - \varphi')^2}{\varepsilon^2}\right)\right),$$
$$\varphi, \varphi' \in \gamma, \quad |\eta - \eta'| \le \varepsilon.$$

And, substituting this representation in (4.52), one easily comes to the first inequality of (4.51).

To treat the case where $|\eta-\eta'|\geq\varepsilon$, one notes that, for $|\eta-\eta'|\geq\varepsilon$ and $\varphi,\varphi'\in\gamma$, by Lemma 4.3 one has

$$\left|\sin^2\zeta\,(\cot\,\zeta-i)\right|^{-1}\leq C.$$

This estimate and the estimate (4.47) lead to the second inequality from (4.51).

Using the estimates (4.51), one obtains

$$|(R_K f)'(\varphi)|\leq C\,\varepsilon^{-2}\,\left(I_1\,(\eta)+I_2\,(\eta)+I_3\,(\eta)\,\right)\,\|f\|_{a,\,b},\quad \varphi\in\gamma,$$

with the same I_1, I_2 and I_3 as in the proof of Lemma 4.5. This gives the inequality

$$\|(R_K f)'\|_C\leq C\,\varepsilon^{-1-a}. \tag{4.53}$$

similar to (4.45). Now, combining (4.50) and (4.53), by means of (4.36), we get

$$\|R_K f\|_{a,b}\leq C\left(\varepsilon^{-a}+\varepsilon^{-a\,(1-b)}\,\varepsilon^{-(1+a)\,b}\right)\,\|f\|_{a,\,b}\leq C\,\varepsilon^{-a-b}\,\|f\|_{a,\,b}.$$

This implies the announced estimate of $\|K_+\|$. $\square$

4.7. **Analytic solution of equation (4.17).** We are now ready to solve equation (4.17). In this subsection, we prove

Proposition 4.3. *Let K be a bounded canonical domain related to some points $\zeta_1,\,\zeta_2\in\mathbb{C}$, $\operatorname{Im}\zeta_1<\operatorname{Im}\zeta_2$. Assume that K does not contain any pole of $\psi^0_\pm$. For sufficiently small $\varepsilon>0$, there exists a solution of (4.17) analytic in the strip*

$$\operatorname{Im}\zeta_1<\operatorname{Im}\varphi<\operatorname{Im}\zeta_2,\ \ \varphi\in K. \tag{4.54}$$

This solution has the asymptotics:

$$\begin{pmatrix} A\,(\varphi)\\ B\,(\varphi)\end{pmatrix}=e^{\theta_A\,(\varphi)}\begin{pmatrix}1+O\,(\varepsilon^\beta)\\ O\,(\varepsilon^\beta)\end{pmatrix},\quad \varphi\in K,\quad \varepsilon\to 0. \tag{4.55}$$

Here, β is any fixed number,

$$0<\beta<1,$$

and θ_A is the function defined by (4.24) The error estimate is locally uniform in $\varphi\in K$.

The rest of this subsection will be devoted to the proof of Proposition 4.3.

4.7.1. Let $\gamma\in K$ be one of the canonical curves connecting ζ_1 and ζ_2. First, construct a solution of (4.31) with the components A_1 and B_2 from the class $C_{a,\,b}(\gamma)$. In view of Lemma 4.5 and Proposition 4.2, for sufficiently small ε, the components of the matrix integral operator from the right hand side of equation (4.31) satisfy the estimates

$$\begin{aligned}&\|L_+\,(O\,(\varepsilon^2)\ .\ )\|\leq C\,\varepsilon^{1-a}, && \|L_+\,(O\,(\varepsilon^{1+\alpha})\ .\ )\|\leq C\,\varepsilon^{\alpha-a},\\ &\|K_+\,(O\,(\varepsilon^{1-\alpha})\ .\ )\|\leq C\,\varepsilon^{1-\alpha-a-b}, && \|K_+\,(O\,(\varepsilon^2)\ .\ )\|\leq C\,\varepsilon^{2-a-b},\end{aligned} \tag{4.56}$$

where $\|\ .\ \|$ denotes the operator norm induced by $\|\ .\ \|_{a,b}$. Fix the numbers α, a and b so that

$$0<b<a<\alpha<1,\quad \alpha+a+b<1. \tag{4.57}$$

Clearly, in this case, for sufficiently small ε, equation (4.31) has a unique solution in $C_{a,\,b}$. This solution satisfies

$$A_1=1+O\,(\varepsilon^{\alpha-a}),\quad B_2=O\,(\varepsilon^{1-\alpha-a-b}).$$

Here the error terms are estimated in the $\|\ .\ \|_{a,b}$-norm. Letting

$$\beta_1=\min\{\alpha-a,\,1-a-b\} \tag{4.58}$$

we get

$$A_1=1+O\,(\varepsilon^{\beta_1}),\quad B_2=O\,(\varepsilon^{\beta_1-\alpha}). \tag{4.59}$$

Note that $0<\beta_1<1$.

4.7.2. Now, let us check that A_1 and B_2 can be analytically continued in the strip defined by (4.54). Clearly, the poles in φ' of the kernel of the operator L_+ are situated at the points $\varphi + 0 + \varepsilon l,\ l \in \mathbb{Z}$. Therefore, the right hand side in (4.31) can be analytically continued in the strip

$$S_\varepsilon = \left\{\varphi \in K:\ \exists \zeta \in \gamma',\, 0 < \operatorname{Re}\varphi - \operatorname{Re}\zeta < \varepsilon\right\},$$

where, as before, γ' is the curve γ without the endpoints. Deforming the integration contour in (4.28) inside S_ε, we see that, in fact, A_1 and B_2 can be analytically continued in the strip $S_{2\varepsilon}$. Repeating these arguments, we prove that these functions can be continued in the part of the strip defined by (4.54) situated at the right of γ.

We check that A_1 and B_2 are analytic at the left of γ. Denote the second term in (4.31) by $\mathcal{L}\begin{pmatrix} A_1 \\ B_2 \end{pmatrix}(\varphi)$. We assume first that φ is located at the right of γ and deform the integration contour so that it passes between the points φ and $\varphi + h$. Since the kernels of L_+ and K_+ have simple poles at $\varphi' = \varphi$, instead of (4.31), one obtains

$$\left[I + O\left(\varepsilon^{1-\alpha}\right)\right]\begin{pmatrix} A_1(\varphi) \\ B_2(\varphi) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tilde{\mathcal{L}}\begin{pmatrix} A_1 \\ B_2 \end{pmatrix}(\varphi). \tag{4.60}$$

Here, the second term in the brackets in the left hand side is a 2×2-matrix with analytic coefficients which can be estimated by $C\,\varepsilon^{1-\alpha}$ for $\varphi \in K$. This term is the residue that appeared in the course of the contour deformation. We have continued A_1 and B_2 to the right of γ by using (4.31). Applying almost the same arguments to (4.60), one continues these functions to the left of γ.

4.7.3. Now, we discuss analytic properties of the functions A_1 and B_2 in some small vicinities of the end points of the curve γ. Clearly, for $\varphi \in \gamma$ being in a sufficiently small vicinity V_1 of ζ_1, equation (4.31) can be rewritten in the form

$$\begin{pmatrix} A_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + S\begin{pmatrix} A_1 \\ B_2 \end{pmatrix} + \mathcal{L}_1\begin{pmatrix} A_1 \\ B_2 \end{pmatrix},$$

where S is the singular integral operator (4.37), and $\mathcal{L}_1$ is a matrix integral operator with some kernel analytic in $\varphi \in V_1$ and in $\varphi' \in \gamma$. Therefore,

$$\begin{pmatrix} A_1 \\ B_2 \end{pmatrix}(\varphi) = S\begin{pmatrix} A_1 \\ B_2 \end{pmatrix}(\varphi) + f(\varphi),$$

where f is analytic in V_1. We recall that A_1 and B_2 belong to $C_{a,b}(\gamma)$. The analytic behavior of the singular integral in a vicinity of the ends of the integration contour is well known (see, e.g, [10]). Using classical results, we see that ζ_1 is a branching point of A_1 and B_2, and that these functions admit the representations

$$(\varphi - \zeta_1)^a\, A_1(\varphi) = c_1 + O\left(|\varphi - \zeta_1|^{b_1}\right), \quad (\varphi - \zeta_1)^a\, B_2(\varphi) = c_2 + O\left(|\varphi - \zeta_1|^{b_1}\right), \tag{4.61}$$

where $c_{1,2}$ are some complex constants, and b_1 is a positive number. Similarly, in a sufficiently small vicinity V_2 of ζ_2,

$$(\varphi - \zeta_2)^a\, A_1(\varphi) = c_3 + O\left(|\varphi - \zeta_2|^{b_2}\right), \quad (\varphi - \zeta_2)^a\, B_2(\varphi) = c_4 + O\left(|\varphi - \zeta_2|^{b_2}\right). \tag{4.62}$$

We can assume that $0 < b_1 = b_2 \le b$. We set

$$\beta = \min\{\alpha - a,\, 1 - a - b_1\}. \tag{4.63}$$

Clearly, $0 < \beta < 1$.

4.7.4. Consider the functions A_1 and B_2 along some other canonical curve $\gamma_1 \in K$. We show that, for sufficiently small ε, they have the asymptotic representation

$$A_1 = 1 + O\left(\varepsilon^{\beta}\right), \quad B_2 = O\left(\varepsilon^{\beta-\alpha}\right) \quad \text{in} \quad C_{a,b_1}(\gamma_1) \tag{4.64}$$

similar to (4.59)

Therefore, instead of (4.31), we consider the analogous equation along the curve γ_1. For sufficiently small ε, this new equation has a unique solution in $C_{a,b_1}(\gamma_1)$, and the new functions A_1 and B_2, e.g. the components of this solution, do have the desired asymptotics. But, the representations (4.61) – (4.62) for the "old" functions A_1 and B_2 imply that they belong to this functional space. And just deforming the curve γ to γ_1, one can see these two "old" functions satisfy the "new" equation. This implies the desired result.

4.7.5. Let $\mathcal{C}$ be a compact subset of K. Now, we can easily show that

$$A_1(\varphi) = 1 + O(\varepsilon^{\beta}), \quad B_2(\varphi) = O(\varepsilon^{\beta-\alpha}), \quad \varphi \in \mathcal{C}, \tag{4.65}$$

with the uniform error estimates.

Let $\varphi_0 \in \mathcal{C}$. One can consider φ_0 as an internal point of a domain $K_0 \subset K$ with the boundary Γ_0 consisting of two segments of two canonical curves γ_1 and γ_2 situated in K. We represent the functions A_1 and B_2 by the Cauchy integrals along Γ_0. Along $\Gamma_0 \cap \gamma_1$ (resp. $\Gamma_0 \cap \gamma_2$), the functions A_1 and B_2 admit the representation (4.65) in $C_{a,b_1}(\gamma_1)$ (resp. $C_{a,b_1}(\gamma_2)$). This leads to the desired asymptotic formulae for any fixed $\varphi \in K_0$. Moreover, the error estimates are uniform in φ in a sufficiently small fixed vicinity of φ_0.

Thus, we see that any point φ_0 of the compact $\overline{\mathcal{C}}$ has an open vicinity V_0 such that the desired asymptotic formulae are uniform in $\varphi \in V_0$. This implies the needed result.

4.7.6. To finish the proof, we reconstruct in terms of B_2 the function B_1 by formula (4.30). The vector $\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}(\varphi)$ satisfies equation (4.29). By means of the residue theorem, one checks that

$$L_+\hat{T}_1 \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}(\varphi+\varepsilon) - L_+\hat{T}_1 \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}(\varphi) = \hat{T}_1 \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}(\varphi),$$
$$\operatorname{Im}\zeta_1 < \operatorname{Im}\varphi < \operatorname{Im}\zeta_2.$$

This implies that this vector satisfies (4.23), and, by means of (4.22) we reconstruct a vector solution $\begin{pmatrix} A \\ B \end{pmatrix}$ of the difference equation (4.17). Clearly, this solution is analytic in K; moreover the asymptotic formulae (4.65) and the relations (4.22), (4.30) imply (4.55). □

4.8. **Consistent basis solutions.** In this subsection, we complete the proof of Theorem 1.1.

4.8.1. First, we construct the solution f_+ described in Theorem 1.1. Denote the components of the vector solution of (4.17) described in Proposition 4.3 by A^+ and B^+. Let $\psi^1_\pm$ be the solutions of (1.2) constructed in subsection 4.2. We consider the function

$$f_+(x,\varphi) = A^+(\varphi)\,\psi^1_+(x,\varphi) + B^+(\varphi)\,\psi^1_-(x,\varphi), \tag{4.66}$$

where $\operatorname{Im}\zeta_1 < \operatorname{Im}\varphi < \operatorname{Im}\zeta_2$, $u \in \mathbb{R}$. This function is a solution of (1.2) analytic in $\varphi \in K$. By construction, it satisfies the condition (1.7).
Finally, we note that the asymptotics (4.55) and (4.12) imply

$$f_+(x,\varphi) = e^{\theta_A(\varphi)}\left(\psi^0_+(x,\varphi) + O(\varepsilon^{\beta})\right), \quad \varphi \in K, \quad -X \le x \le X.$$

Remembering the definitions of θ_A, $\psi^0_\pm$ and $\Psi^0_\pm$, we see that this formula coincides with (1.10).

4.8.2. Up to this moment, we were assuming that K does not contain any pole of $\psi^0_\pm$. Now, we get formula (1.10), and see that the leading term in this formula is the canonical Bloch solution. By Lemma 3.1, it has no poles. Having found the canonical Bloch solutions, we could have used them everywhere instead of $\psi^0_\pm$. This remark enables us to remove the assumption about the absence of the poles of $\psi^0_\pm$ in K.

4.8.3. Let us discuss the construction of the second solution of (1.2) described in Theorem 1.1. It is similar to the one leading to f_+. So, we shall only single out its new elements.

Now, the central objects of the analysis are the operators

$$L_- f(\varphi) = \frac{1}{2i\varepsilon}\int_\gamma \left(\cot\left(\frac{\pi(\varphi'-\varphi-0)}{\varepsilon}\right) + i\right) f(\varphi')\,d\varphi',$$

and

$$K_- f(\varphi) = e^{2\theta_0(\varphi)}\, L_-\left(e^{-2\theta_0(.)}\, f(.)\right)(\varphi).$$

acting on the "old" space $C_{a,b}(\gamma)$. As in subsection 4.6.3, one checks that, for γ a canonical curve,

$$\|L_-\| \le C\,\varepsilon^{-1-a}, \quad \|K_-\| \le C\,\varepsilon^{-a-b}.$$

To reduce the analysis of (4.23) to the analysis of these two operators, we consider instead of (4.29) the equation

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + L_- \hat{T}_1 \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}, \quad \varphi \in \gamma \subset K.$$

Note that it differs from (4.29) not only by replacing L_+ by L_-, but also by replacing the term $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The latter allows to obtain, by means of the transformation

$$A_1 = \varepsilon^{\alpha}\, e^{-2\theta_0}\, A_2,$$

the integral equation

$$\begin{pmatrix} A_2 \\ B_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} K_- O(\varepsilon^2) & K_-\, O(\varepsilon^{1-\alpha}) \\ L_-\, O(\varepsilon^{1+\alpha}) & L_- O(\varepsilon^2) \end{pmatrix} \begin{pmatrix} A_2 \\ B_1 \end{pmatrix},$$

for $\varphi \in \gamma \subset K$.
The analysis of this equation is similar to the analysis of (4.31), and, as a result, one constructs the solution f_- of (1.2) described in Theorem 1.1. Omitting the details, we note only that

$$f_-(x, \varphi) = A^-(\varphi)\, \psi_+^1(x,\, \varphi) + B^-(\varphi)\, \psi_-^1(x,\, \varphi), \tag{4.67}$$

for $\operatorname{Im}\zeta_1 < \operatorname{Im}\varphi < \operatorname{Im}\zeta_2$, $x \in \mathbb{R}$.
The functions A^- and B^- are analytic in the strip defined by (4.54) and admit the asymptotic representations

$$A^-(\varphi) = e^{\theta_B(\varphi)}\, O(\varepsilon^{\beta}), \quad B^-(\varphi) = e^{\theta_B(\varphi)}\, (1 + O(\varepsilon^{\beta})), \tag{4.68}$$

for $\varphi \in K$ when $\varepsilon \to 0$, with the error estimate being locally uniform. Here, β is any fixed number, $0 < \beta < 1$, and θ_B is the function defined by (4.25). ☐

4.8.4. Finally, we check that the Wronskian of $f_\pm(x,\, \varphi)$ is independent of φ, and, thus, that they do form a consistent basis of solutions for the equation family (1.2). By (4.66) and (4.67),

$$w\{f_+, f_-\} = \left(A^+(\varphi)\, B^-(\varphi) - A^-(\varphi)\, B^+(\varphi)\right)\, w_1(\varphi).$$

Here $w_1(\varphi)$ is the Wronskian of the solutions $\psi_\pm^1$. Now, using the asymptotics of A^+ and B^+ described in Proposition 4.3, the asymptotics of A^- and B^- given by (4.68), and the asymptotics of w_1 (see Lemma 4.1), we get

$$w\{f_+, f_-\} = w_0(\varphi)\, \exp\left(\ln k_E' + \int_{\varphi_0}^{\varphi} (\omega_+ + \omega_-)\, d\varphi\right)\, (1 + o(1)). \tag{4.69}$$

By (3.3)

$$\omega_+ + \omega_- = -\frac{\partial}{\partial\varphi}\, \ln \int_0^1 \psi_+^0(x,\, \varphi)\, \psi_-^0(x,\, \varphi) dx.$$

Using Lemma 4.1, we transform this expression to the form

$$\omega_+ + \omega_- = -\frac{\partial}{\partial\varphi}\, \ln\left(k_E'\, w_0(\varphi) \right).$$

This allows to get finally

$$w\{f_+, f_-\} = k_E'\, w_0\big|_{\varphi=\varphi_0}\, (1 + g(\varphi)), \quad g(\varphi) = o(1).$$

This formula shows only that the leading terms of the asymptotics of the Wronskian is independent of φ.

Let us discuss the term $g(\varphi)$ in the last formula. Since $f_\pm$ satisfy (1.7), it is ε-periodic in φ. Moreover, its estimate is locally uniform in $\varphi \in K$.

Let us change the notations, denoting by f_+ the "old" function f_+ divided by $1 + g$. The "new" functions $f_\pm$, clearly, have all the properties announced in Theorem 1.1. This completes the proof of the theorem. ☐

5. An example

We now restrict ourselves to the case $W(x) = \cos x$ i.e. we consider the equation

$$-\frac{d^2}{dx^2}\psi(x) + V(x) + \cos(\varepsilon x + \varphi)\,\psi(x) = \psi(x), \quad x \in \mathbb{R}. \tag{5.1}$$

As said, here, we will not give the complete proofs of the results we will now describe; these will be published somewhere else (nevertheless see [6]).
One of the objects important for the spectral study of (5.1) is the monodromy matrix (see [7]). Let us recall its definition. Let $(f_\pm)$ be a consistent basis of solutions of (5.1). The functions $f_\pm(x, \varphi + 2\pi)$ are also two solutions of equation (5.1). Therefore, one can write

$$f(x, \varphi + 2\pi) = M(\varphi)\, f(x, \varphi), \tag{5.2}$$

where f is the vector

$$f^T(x, \varphi) = (f_-(x, \varphi), f_+(x, \varphi)) \tag{5.3}$$

and ^{T} is the symbol of transposition; $M(\varphi)$ is a matrix with coefficients independent of x. We call this matrix *the monodromy matrix* corresponding to the consistent basis $(f_{1,2})$. M is ε-periodic and its determinant is equal to 1. For further properties and remarks on M, we refer to [7, 6].

We will now explain how, using our WKB constructions, one can compute the asymptotics of M when $\varepsilon \to 0$.

5.1. **The complex momentum.** Let us first describe the complex momentum associated to (5.1). We recall that the conformal properties of κ depend on the spectral parameter E and on the lengths of the spectral zones and the spectral gaps of the operator $H_0 = -\frac{d^2}{dx^2} + V$. For the sake of definiteness, we shall assume that

$$E - 1 \leq E_1 - \delta_E, \quad E_1 + \delta_E < E + 1 < E_2 - \delta_E, \quad E_2 < E_3, \tag{5.4}$$

here $\delta_E > 0$ is fixed independent of ε.
The branching points of the complex momentum κ are described in formula (3.1). Each of the E_l gives rise to an infinite 2π-periodic lattice of branching points of the complex momentum. There are two branching points corresponding to each of E_l and situated in the strip $0 \leq \mathrm{Re}\varphi < 2\pi$. Let us describe them. The

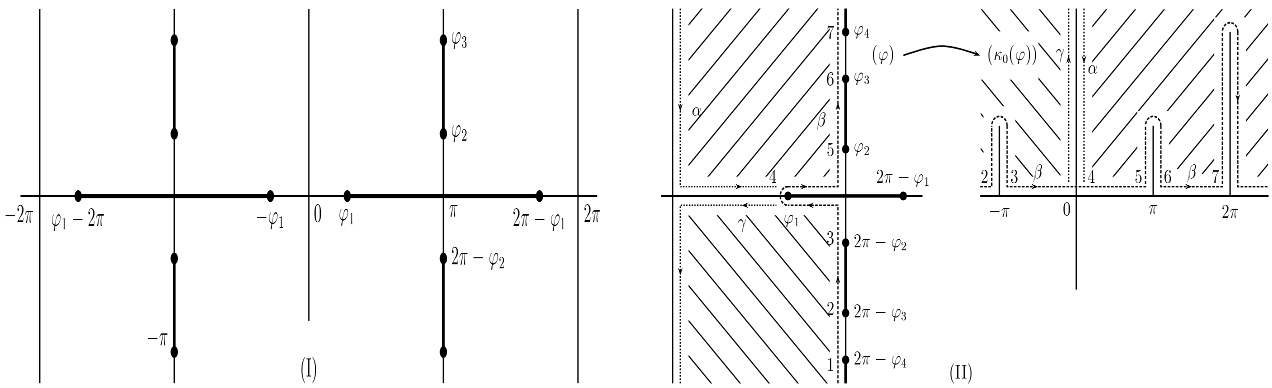

Figure 2: The mapping κ_0.

branching points corresponding to E_1 are real. They are symmetric with respect to the point π. We denote them by φ_1 and $2\pi - \varphi_1$ where $0 < \varphi_1 < \pi$.

The branching points corresponding to E_l, $l > 1$, are situated on the line $\mathrm{Re}\,\varphi = \pi$. They are also symmetric with respect to the point π. The points with positive imaginary part will be denoted by φ_l. The branching points are indexed in such a way that $\mathrm{Im}(\varphi_{l+1}) \geq \mathrm{Im}(\varphi_l)$ for $l = 1, 2, \ldots$.

The whole picture of the branching points is 2π-periodic. It is partly shown on Fig. 2 (I).

The conformal properties of the complex momentum near the branching points are determined by the behavior of the Bloch quasimomentum k near the points E_l (see subsection 2.1). In particular, the branching points φ_1 and $2\pi - \varphi_1$ related to E_1 are of square root type.
One can fix a single valued branch of the complex momentum in a vicinity of $\varphi = 0$ by the conditions

$$\operatorname{Im} \kappa_0(0) > 0, \quad \operatorname{Re} \kappa_0(0) = 0. \tag{5.5}$$

The branch κ_0 can be analytically continued along any curve not passing through any branching point. In Fig. 2 (II), we draw several curves on the complex plane $\varphi \in \mathbb{C}$, and then, we show the images of these curves by $\kappa_0(\varphi)$. The results shown in these figures easily follow from the discussions of subsection 2.1.

Note that on the shadowed domain in Fig. 2 (II), we have

$$\kappa_0(\varphi) = k_0(E - \cos\varphi),$$

5.2. **The Stokes lines.** The definition of the Stokes lines is fairly standard. The integral $\int \kappa\, d\varphi$ has the same branching points as the complex momentum. Let φ_0 be one of them.
Consider the curves beginning at φ_0 and described by the equation

$$\operatorname{Im} \int_{\varphi_0}^{\varphi} (\kappa(\tilde{\varphi}) - \kappa(\varphi_0))\, d\tilde{\varphi} = 0. \tag{5.6}$$

These curves are the *Stokes lines* beginning at φ_0. Note that the Stokes line definition is independent of the choice of the branch of κ in (5.6).
Assume, as before, that E satisfies (5.4). In this case, at any branching point $\pm\varphi_j + 2\pi l$, $j = 1, 2, 3$, begin exactly three Stokes lines. The angle between any two of them at this point are equal to $\frac{2\pi}{3}$. Some of the Stokes lines are finite: they connect pairs of finite branching points; some of the Stokes lines are infinite: they go from finite branching points to infinity. Some of the Stokes lines in the domain $0 < \operatorname{Re} z < 2\pi$ are shown in Fig. 3.

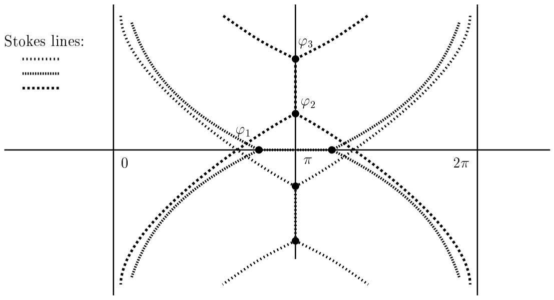

Figure 3: Stokes lines

Consider the branching point φ_1. Two of the Stokes lines beginning at φ_1 go to $\pm i\infty$ and have the asymptote $\operatorname{Re}\varphi = 0$. The last Stokes line starting at φ_1 is finite; it goes along the real axis to $2\pi - \varphi_1$. In the same way, for each $l > 1$, two of the Stokes lines beginning at the point φ_l are infinite. And the last one is finite and runs along the line $\operatorname{Re}\varphi = \pi$ from one branching point to another.
The whole picture of the Stokes lines in the domain $0 \leq \operatorname{Re}\varphi \leq 2\pi$ is symmetric with respect to the real line as well as with respect to the line $\operatorname{Re}\varphi = \pi$.
At last, we note that, if φ is a branching point for κ, then so is the point $\varphi + 2\pi$. The Stokes lines passing through $\varphi + 2\pi$ are just the 2π-translates of the Stokes lines beginning φ.

5.3. **Canonical domains.** In Fig. 4, we show two canonical domains K_0 and K_1. K_0 is represented on the left side (I) and K_1 on the right one (II). For both domains, part A corresponds to the case where $\zeta_1 = +i\infty$ and $\zeta_2 = -i\infty$ (see section 3.1). Part B corresponds to ζ_1 and ζ_2 finite. The domain K_0 is canonical with respect to the branch κ_0. The boundary of K_0 (part A) consists of segments of Stokes lines; as for part B, its boundary also consists of segments of the four lines passing through the points ζ_1 and ζ_2 where, either $\operatorname{Im} \int^{\varphi} \kappa d\varphi$, or $\operatorname{Im} \int^{\varphi} (\kappa - \pi) d\varphi$ is constant. These domains exist if $E - 1 < E_1$.
The domain K_1 exists if $E_1 < E + 1 < E_2$. It is canonical with respect to the branch fixed by the condition $0 < \kappa_1(\pi) < \pi$. The boundary of the canonical domain K_1 (see Fig. 4) consists of segments of Stokes lines and of segments of the four lines passing through the points ζ_1 and ζ_2 where, either $\operatorname{Im} \int^{\varphi} \kappa\, d\varphi$, or $\operatorname{Im} \int^{\varphi} (\kappa - \pi) d\varphi$ is constant.

5.4. **Admissible sub-domains.** Fix $\delta > 0$. Consider a canonical domain, say K_0, and remove the δ-neighborhood of its boundary. Denote by $\mathcal{C}$ the remaining domain and call it an admissible sub-domain. It is bounded. And, our asymptotic formulas (1.10) are valid uniformly in $\mathcal{C}$. So, in the case, $W(x) = \cos x$, one has to consider bounded canonical domains. In the sequel, by a slight abuse of language, a "canonical domain" will mean an "admissible sub-domain" of the considered canonical domain.

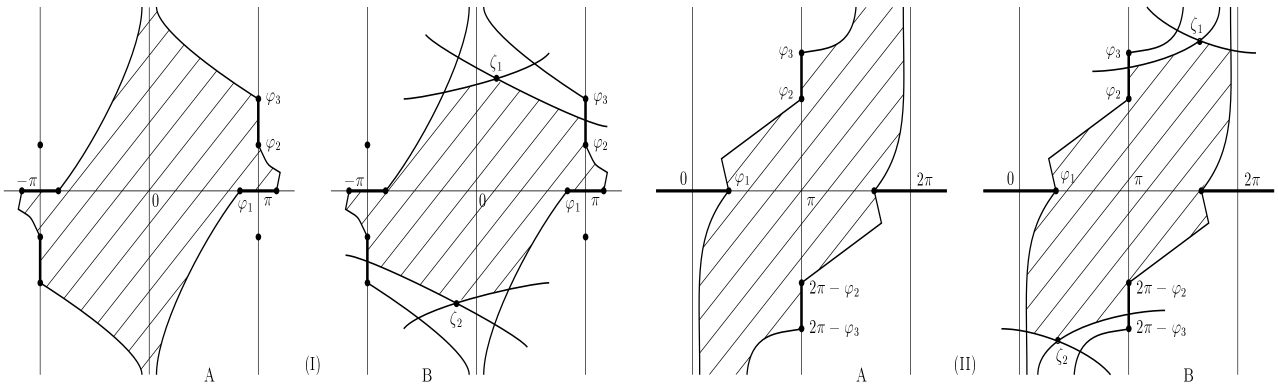

Figure 4: The canonical domains K_0 and K_1.

5.5. **Asymptotics of the monodromy matrices.** Now we describe the strategy used to compute the asymptotics of a monodromy matrix. The monodromy matrix we consider is the one associated to the consistent basis of solutions of (5.1) constructed in Theorem 1.1 for the canonical domain K_0 (see Fig. 4). We denote these solutions by $(f^0_\pm)$. The monodromy matrix associated to this basis is then defined by

$$F^0(x,\varphi+2\pi) = M(\varphi)F^0(x,\varphi) \tag{5.7}$$

where $F^0 = (f^0_+, f^0_-)^T$.

In a subset of K_0 where we know both the asymptotic behavior of $F^0(u,\varphi+2\pi)$ and $F^0(u,\varphi)$, it will be easy to compute the asymptotics of M. That is, we need to know the asymptotics of $F^0(u,\varphi)$ in K_0 and $K_2 = K_0 + 2\pi$. Unfortunately, a quick look at section 5.3 and Fig. 5 shows us that $K_0 \cap (K_0 + 2\pi) = \emptyset$. Therefore we need to introduce an auxiliary canonical domain.
The auxiliary domain we need, we have already defined: it is K_1 (see Fig. 4 and subsection 5.3). Let us denote by $(f^1_\pm)$ the consistent basis constructed by Theorem 1.1 in K_1. As $(f^0_\pm)$ and $(f^1_\pm)$ are both basis of solutions of (1.2), we define

$$F^0(u,\varphi) = T_1(\varphi)F^1(u,\varphi). \tag{5.8}$$

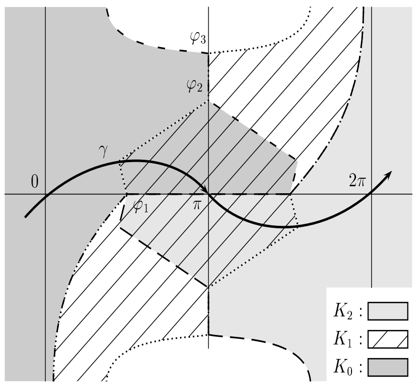

Figure 5: K_0, K_1 and K_2

We will be able to compute the asymptotic behavior of T_1.
The domain K_2 is canonical for the branch of κ obtained by continuing κ_0 along the curve γ shown on Fig. 5. Moreover it is immediate that if we define $f^2_\pm(\varphi) = f^0_\pm(\varphi - 2\pi)$, then for K_2, this defines a consistent basis. As $(f^2_\pm)$ and $(f^1_\pm)$ are both basis of solutions of (1.2), we define

$$F^1(u,\varphi) = T_2(\varphi)F^2(u,\varphi). \tag{5.9}$$

Putting (5.8) and (5.9) together, we get that

$$F^0(u,\varphi) = T_1(\varphi)T_2(\varphi)F^2(u,\varphi) = T_1(\varphi)T_2(\varphi)F^0(u,\varphi-2\pi).$$

Hence the monodromy matrix reads

$$M(\varphi) = T_1(\varphi+2\pi)T_2(\varphi+2\pi). \tag{5.10}$$

5.6. **How to compute a coefficient of a transfer matrix: an example.** Let us define the coefficients of T_1 by

$$T_1(\varphi) = \begin{pmatrix} a_1(\varphi) & b_1(\varphi) \\ c_1(\varphi) & d_1(\varphi) \end{pmatrix}.$$

We will compute $c_1(\varphi)$; this computation is rather long but is the most instructive of the four. One computes

$$c_1(\varphi) = \frac{1}{w(f_+^1, f_-^1)} w(f_-^0, f_-^1) \tag{5.11}$$

where $w(f, g)$ denotes the Wronskian of f and g. We shall now compute the asymptotics of the Wronskians used in (5.11). We are interested in a uniform control in a strip of the form

$$-Y \leq \operatorname{Im} \varphi \leq Y, \quad \operatorname{Im}\varphi_2 < Y < \operatorname{Im}\varphi_3. \tag{5.12}$$

Let us now take ζ_1 and ζ_2 sufficiently far from the real line, so that the points of intersection of the segments of the Stokes lines bordering K_1 and of the four lines passing through the points ζ_1 and ζ_2 where, either $\operatorname{Im} \int^\varphi \kappa \, d\varphi$ or $\operatorname{Im} \int^\varphi (\kappa - \pi) d\varphi$ are constant, are outside the strip (5.12).

We divide this strip into three different smaller strips called (I), (II) and (III) (see Fig. 6). Each of these strip will require a different type of computation.

Region (I). In region (I), there is no intersection between K_0 and K_1. But there is a simple way to overcome this difficulty. Let φ_0, φ_+ and φ_- be complex points such that $\operatorname{Im} \varphi_- = \operatorname{Im} \varphi_+ = \operatorname{Im} \varphi_0$ and $\operatorname{Re} \varphi_- < \operatorname{Re} \varphi_0 < \operatorname{Re} \varphi_+$ (see Fig. 4 A). One can rigorously justify a well known WKB heuristic that says that, if a function f admits on $[\varphi_-, \varphi_0]$ an asymptotic representation that is defined and exponentially increasing from φ_- to φ_+, then the f admits the same representation on the interval $[\varphi_0, \varphi_+]$.

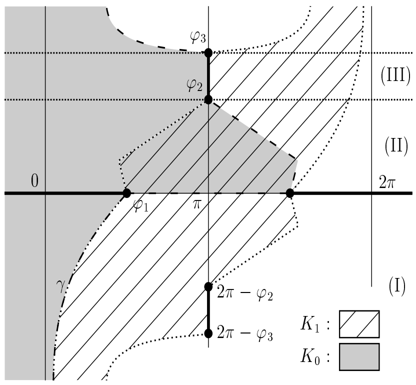

Figure 6: K_0 and K_1

As $\operatorname{Im}(\kappa_0) > 0$, we can continue the asymptotic expansion for f_-^0 given by Theorem 1.1 in K_0 across ∂K_0 along a horizontal line (see Fig. 7 A). So the asymptotic expansion for f_-^0 on $[\varphi_b, \varphi_+]$ is

$$f_-^0(u, \varphi) = e^{-\frac{i}{\varepsilon}\int_0^\varphi \kappa_0 d\varphi} \cdot \left(\sqrt{k_E'} e^{\int_0^\varphi \omega_-^0 d\varphi} (\psi_-^0 + o(1)) \right).$$

Here we integrate along paths going from 0 to φ staying below the point φ_1. Now, for φ on the Stokes line separating K_0 from K_1, in the region (I), we know that

$$\kappa_1(\varphi + 0) = -\kappa_0(\varphi - 0), \tag{5.13}$$

$$\psi_\pm^1(u, \varphi + 0) = \psi_\mp^0(u, \varphi - 0), \tag{5.14}$$

$$\omega_\pm^1(\varphi + 0) = \omega_\mp^0(\varphi - 0) \tag{5.15}$$

Here κ_1 is the branch obtained from κ_0 by continuation along γ from K_0 to K_1 (see Fig. 5). This is the branch of κ corresponding to K_1. Formulae (5.13)– (5.15) are obtained using the analysis done in section 2.2.

We use (5.13) and compute

$$\begin{aligned} \int_0^\varphi \kappa_0 d\varphi &= \int_0^{\varphi_1} \kappa_0 d\varphi - \int_{\varphi_1}^{\varphi} \kappa_1 d\varphi \\ &= \int_0^{\varphi_1} \kappa_0 d\varphi - \int_{\varphi_1}^{\pi} \kappa_1 d\varphi - \int_\pi^\varphi \kappa_1 d\varphi. \end{aligned} \tag{5.16}$$

In the first two integrals, we integrate along the real axis; in the last integral, the contour is located in K_1. Similarly

$$\int_0^\varphi \omega_-^0 d\varphi = \int_0^{\varphi_1} \omega_-^0 d\varphi + \int_{\varphi_1}^\varphi \omega_+^1 d\varphi,$$

where the first integral is taken along the real line; the integration contour in the second integral lies in K_1. Finally, we get

$$\int_0^{\varphi} \omega_-^0 d\varphi = \int_0^{\varphi_1} \omega_-^0 d\varphi - \int_\pi^{\varphi_1} \omega_+^1 d\varphi + \int_\pi^{\varphi} \omega_+^1 d\varphi, \tag{5.17}$$

We also note that, on the different sides of γ in region (I) (see Fig. 7 A and section 2.2), we have

$$\sqrt{k'_E}_{|\varphi+0} = i\sqrt{k'_E}_{|\varphi-0} \tag{5.18}$$

Combining (5.16) – (5.18), we get

$$f_-^0(u,\varphi) = -ie^{-\frac{i}{\varepsilon}\int_0^{\varphi_1}\kappa_0 d\varphi+\frac{i}{\varepsilon}\int_{\varphi_1}^{\pi}\kappa_0 d\varphi+\int_0^{\varphi_1}\omega_-^0-\int_\pi^{\varphi_1}\omega_+^0} e^{\frac{i}{\varepsilon}\int_\pi^{\varphi}\kappa_1 d\varphi}(\Psi_+^1(u,\varphi)+o(1)).$$

On the other hand, we know that

$$f_-^1(u,\varphi) = e^{-\frac{i}{\varepsilon}\int_\pi^{\varphi}\kappa_1 d\varphi}(\Psi_-^1(u,\varphi)+o(1)).$$

Hence, we easily obtain

$$w(f_-^0, f_-^1) = -i\cdot e^{-\frac{i}{\varepsilon}\int_0^{\pi}\kappa_0 d\varphi} e^{2\frac{i}{\varepsilon}\int_{\varphi_1}^{\pi}\kappa_0 d\varphi} e^{\int_0^{\varphi_1}\omega_-^0 d\varphi-\int_\pi^{\varphi_1}\omega_+^0 d\varphi}(w(f_+^1, f_-^1)+o(1)),$$

that is

$$c_1(\varphi) = -i\cdot e^{-\frac{i}{\varepsilon}\int_0^{\pi}\kappa_0 d\varphi} e^{2\frac{i}{\varepsilon}\int_{\varphi_1}^{\pi}\kappa_0 d\varphi} e^{\int_0^{\varphi_1}\omega_-^0 d\varphi-\int_\pi^{\varphi_1}\omega_+^0 d\varphi}(1+o(1)) \tag{5.19}$$

where the integrals are taken along the real line. This holds uniformly in any strip $\{-Y \le \operatorname{Im}\varphi \le -\delta\}$ (where Y is defined by (5.12) and $\delta > 0$ fixed (see subsection 5.4)). Note that the principal term is independent of φ.

Region (III). In the strip (III), the common boundary of K_0 and K_1 consists in the interval $[\varphi_2, \varphi_3]$ (see Fig. 5, 6 and 7 B). This interval is the Stokes line joining φ_2 to φ_3. Along $[\varphi_2, \varphi_3]$, we have

$$\kappa_1(\varphi+0) = 2\pi - \kappa_0(\varphi-0), \tag{5.20}$$

$$\psi_\pm^1(u,\varphi+0) = \psi_\mp^0(u,\varphi-0), \tag{5.21}$$

$$\omega_\pm^1(\varphi+0) = \omega_\mp^0(\varphi-0). \tag{5.22}$$

As $\operatorname{Im}(\kappa_0) > 0$, as above, we again continue the asymptotics given for f_-^0 by Theorem 1.1 to the interval $[\varphi_-, \varphi_+]$ (see Fig. 7 B). We get

$$f_-^0 = e^{-\frac{i}{\varepsilon}\int_0^{\varphi}\kappa_0 d\varphi}\left(\sqrt{k'_E}e^{\int_0^{\varphi}\omega_-^0 d\varphi}\psi_-^0 + o(1)\right) \tag{5.23}$$

In (5.23), the integration contour first goes from 0 to φ_- in K_0, then to φ along $[\varphi_-, \varphi_+]$. Using (5.20), we rewrite the integrals in (5.23) to get

$$\int_0^{\varphi}\kappa_0 d\varphi = \int_0^{\pi}\kappa_0 d\varphi + 2\int_\pi^{\varphi_2}\kappa_0 d\varphi + 2\pi(\varphi-\varphi_2) - \int_\pi^{\varphi}\kappa_1 d\varphi,$$

where, in the first integral, we integrate along a path in $K_0 \cap K_1$, and in the next two, along a path in K_1.
The same way we get

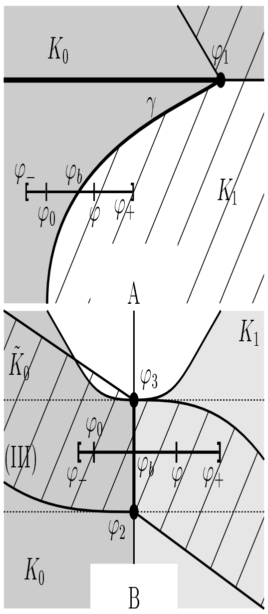

Figure 7: From K_0 toK_1

$$\int_0^{\varphi}\omega_-^0 d\varphi = \int_0^{\pi}\omega_-^0 d\varphi + \int_\pi^{\varphi_2}(\omega_-^0 - \omega_+^0)d\varphi + \int_\pi^{\varphi}\omega_+^0\varphi.$$

We also note that, for $\varphi \in [\varphi_2, \varphi_3]$, we have

$$\sqrt{k'_E}_{|\varphi+0} = -i\sqrt{k'_E}_{|\varphi-0}. \tag{5.24}$$

Combining these formulae, we obtain

$$f_-^0 = ie^{-\frac{i}{\varepsilon}\int_0^{\pi}\kappa_0 d\varphi-2\frac{i}{\varepsilon}\int_\pi^{\varphi_2}(\kappa_0-\pi)d\varphi} e^{-2\pi\frac{i}{\varepsilon}(\varphi-\pi)} e^{\int_0^{\varphi_2}\omega_-^0 d\varphi-\int_\pi^{\varphi_2}\omega_+^0 d\varphi} e^{\frac{i}{\varepsilon}\int_\pi^{\varphi}\kappa_1 d\varphi}(\Psi_+^1 + o(1)).$$

Using this in conjunction with

$$f_-^1 = e^{-\frac{i}{\varepsilon}\int_\pi^\varphi \kappa_1 d\varphi}(\Psi_-^1 + o(1)),$$

we get

$$c_1(\varphi) = ie^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi - 2\frac{i}{\varepsilon}\int_\pi^{\varphi_2}(\kappa_0-\pi)d\varphi} e^{-2\pi\frac{i}{\varepsilon}(\varphi-\pi)} e^{\int_0^{\varphi_2}\omega_-^0 d\varphi - \int_\pi^{\varphi_2}\omega_+^0 d\varphi}(1+o(1)). \tag{5.25}$$

This asymptotic is valid for $\operatorname{Im}\varphi_2 + \delta \leq \varphi \leq$, $\delta > 0$ fixed.

Let us now derive global information on $c_1(\varphi)$. As $c_1(\varphi)$ is ε-periodic, we expand it in a Fourier series

$$c_1(\varphi) = \sum_{n\in\mathbb{Z}} \gamma_n e^{2i\pi n\frac{\varphi-\pi}{\varepsilon}}.$$

Then, by the asymptotic (5.19), for $n > 0$, we have

$$|\gamma_n| \leq C|e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}|e^{-\frac{2\pi}{\varepsilon}|n|Y}.$$

The same formula gives

$$\gamma_0 = -ie^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi + 2\frac{i}{\varepsilon}\int_{\varphi_1}^\pi \kappa_0 d\varphi} e^{\int_0^{\varphi_1}\omega_-^0 d\varphi - \int_\pi^{\varphi_1}\omega_+^0 d\varphi}(1+o(1)).$$

We use the asymptotic given by region (III) to estimate (γ_n) for $n < 0$. For some φ_0 in (III), we compute

$$\begin{aligned}
\gamma_{-1} &= \frac{1}{\varepsilon}\int_{\varphi_0}^{\varphi_0+\varepsilon} c_1(\varphi)e^{2i\pi\frac{\pi-\varphi}{\varepsilon}}d\varphi \\
&= ie^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi - 2\frac{i}{\varepsilon}\int_\pi^{\varphi_2}(\kappa_0-\pi)d\varphi} e^{\int_0^{\varphi_2}\omega_-^0 d\varphi - \int_\pi^{\varphi_2}\omega_+^0 d\varphi}(1+o(1))
\end{aligned}$$

Now we recall that the segment $[\varphi_2, \varphi_3]$ is a Stokes line for κ_0 and $\kappa_0(\varphi_2) = \pi$; hence $\operatorname{Im}\int_\pi^{\varphi_2}(\kappa_0 - \pi)d\varphi = 0$. For $n < -1$, we use (5.25) to estimate

$$\begin{aligned}
|\gamma_n| &= \left|\frac{1}{\varepsilon}\int_{\varphi_0}^{\varphi_0+\varepsilon} c_1(\varphi)e^{2in\pi\frac{\pi-\varphi}{\varepsilon}}d\varphi\right| \\
&\leq C\left|e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi} e^{-2\frac{i}{\varepsilon}\int_\pi^{\varphi_2}(\kappa_0-\pi)d\varphi} e^{\int_0^{\varphi_2}\omega_-^0 d\varphi - \int_\pi^{\varphi_2}\omega_+^0 d\varphi}\right| \frac{1}{\varepsilon}\left|\int_{\varphi_0}^{\varphi_0+\varepsilon} e^{2i(n+1)\pi\frac{\varphi}{\varepsilon}}d\varphi\right| \\
&\leq C|e^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}|e^{-\frac{2\pi}{\varepsilon}|n+1|Y}.
\end{aligned}$$

Putting all this together, we get that, for φ in the strip $\{|\operatorname{Im} z| \leq Y\}$, we have

$$\begin{aligned}
c_1(\varphi) = -ie^{-\frac{i}{\varepsilon}\int_0^\pi \kappa_0 d\varphi}\Big(& e^{2\frac{i}{\varepsilon}\int_{\varphi_1}^\pi \kappa_0 d\varphi} e^{\int_0^{\varphi_1}\omega_-^0 d\varphi - \int_\pi^{\varphi_1}\omega_+^0 d\varphi}(1+o(1)) \\
& - e^{-2\frac{i}{\varepsilon}\int_\pi^{\varphi_2}(\kappa_0-\pi)d\varphi} e^{\int_0^{\varphi_2}\omega_-^0 d\varphi - \int_\pi^{\varphi_2}\omega_+^0 d\varphi} e^{-2\pi\frac{i}{\varepsilon}(\varphi-\pi)}(1+o(1))\Big)
\end{aligned}$$

where $o(1)$ is uniform in the strip $\{|\operatorname{Im} z| \leq Y\}$.

We can rewrite this in a better understandable form; define

$$\begin{aligned}
& t = exp\left(2\frac{i}{\varepsilon}\int_0^{\varphi_1}\kappa d\varphi\right), \quad t_1 = exp\left(2\frac{i}{\varepsilon}\int_{\varphi_2}^\pi(\kappa-\pi)d\varphi\right), \\
& \phi_1 = \frac{2}{\varepsilon}\int_{\varphi_1}^\pi \kappa d\varphi \quad \text{and} \quad \phi_2 = \frac{i\varepsilon}{2\pi}\int_{\varphi_1}^{\varphi_2}(\omega_- - \omega_+)d\varphi.
\end{aligned} \tag{5.26}$$

Here κ is the branch of κ obtained from κ_0 by continuation along the path γ shown on Fig. 5. Note that, all the four coefficients defined here are real. The coefficients t_1 and t are exponentially small in ε. We write

$$c_1(\varphi) = -i\frac{u}{\sqrt{t}}e^{-i\phi_1/2}\left(e^{i\phi_1+o(1)} - t_1 e^{-\frac{2\pi i}{\varepsilon}(\phi-\pi+\phi_2)+o(1)}\right)$$

where

$$u = exp\left(\int_0^{\varphi_1}\omega_-^0 d\varphi - \int_\pi^{\varphi_1}\omega_+^0 d\varphi\right)$$

Along the same lines as above, one can compute the asymptotics of all the coefficients of T_1 and T_2. For the monodromy matrix, one then obtains

$$M \sim \begin{pmatrix} ie^{i\phi_1} & ie^{i\phi_1} + -it_1 e^{iz} \\ -ie^{i\phi_1} + it_1 e^{-iz} & \frac{2}{t}\cos(\phi_1) - 2\frac{t_1}{t}\cos(z) \end{pmatrix} \tag{5.27}$$

where $z = \frac{2\pi}{\varepsilon}\varphi - \pi + \phi_2$. Here $\sim$ means three things

- first, M is similar to the matrix in the right hand side, the similarity being independent of φ,
- second, we only kept the first Fourier coefficients, the others being exponentially small with respect to any of those we have written,
- third, the Fourier coefficients are described up to a factor $(1 + o(1))$ when $\varepsilon \to 0$.

We see that, up to the exponentially small terms (the term $t_1/t\cos(z)$ may be exponentially large but it has to be compared to $2/t\cos(\phi_1)$), M is a constant matrix. It is the exponentially small terms that are important for the analysis of the spectrum of (5.1) (see [7, 5]).

References

[1] V. Buslaev. Adiabatic perturbation of a periodic potential. *Teor. Mat. Fiz.*, 58:223–243, 1984. in russian.

[2] V. Buslaev and A. Fedotov. The complex WKB method for Harper's equation. Preprint, Mittag-Leffler Institute, Stocholm, 1993.

[3] V. Buslaev and A. Fedotov. The complex WKB method for Harper's equation I. *St Petersburg Math. Jour.*, 6:495–517, 1995.

[4] M. Fedoryuk. *Asymptotic analysis.* Springer Verlag, Berlin, 1 edition, 1993.

[5] A. Fedotov and F. Klopp. Coexistence of different spectral types for quasi-periodic Schrödinger equations in dimension 1. In *Proceedings of the conference QMath 7, Prague 22-26/06/1998.*

[6] A. Fedotov and F. Klopp. The monodromy matrix for a family of almost periodic equations in the adiabatic case. Preprint, Fields Institute, Toronto, 1997.

[7] A. Fedotov and F. Klopp. Anderson transitions for quasi-periodic Schrödinger operators in dimension 1. Technical Report 98-29, Université Paris-Nord, 1998.

[8] V. Marchenko and I. Ostrovskii. A characterization of the spectrum of Hill's equation. *Math. USSR Sbornik*, 26:493–554, 1975.

[9] H. McKean and E. Trubowitz. The spectrum of Hill's equation. *Inventiones Mathematicae*, 30:217–274, 1975.

[10] N.I. Muskhelishvilli. *Singular Integral Equations.* Noordhoff, Groningen, 1953.

[11] Y. Sibuya. *Global theory of second order linear ordinary differential equations with a polynomial coefficient.* North-Holland, Amsterdam, 1975.

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