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Propagating Edge States for a Magnetic Hamiltonian

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Abstract

We study the quantum motion of a charged particle in a half plane, subject to a perpendicular constant magnetic field B and to an arbitrary weak impurity potential W_B (i.e. $||W_B||_\infty < \delta B$, for some δ small enough). We show that there exist states propagating with a speed of size $B^{1/2}$ along the edge, no matter how fast W_B fluctuates. As a consequence, the spectrum of the Hamiltonian is purely absolutely continuous in a spectral interval of size γB ($0 < \gamma < 1$) between the Landau levels of the system without edge or potential, so that the corresponding eigenstates are extended. This then provides a rigorous proof of a phenomenon pointed out by Halperin in his work on the quantum Hall effect.

1 Introduction

It is well known that a classical charged particle, constrained to a plane and subjected to a perpendicular magnetic field will move along physical boundaries when those are present. In the case of a particle moving in a half plane ($x > 0, y \in \mathbb{R}$), it is easy to see that the circular trajectories that are at a distance less than $\sqrt{E}/B$ from the edge will bounce of it in such a way that the particle speeds alongside the edge with a velocity of the order of $\sqrt{E}$, where E denotes the energy of the particle. If, on the other hand, the centre of the trajectory is too far from the edge, it will not affect the motion of the particle.

If, as one would expect, this picture is to carry over to the quantum mechanical situation, then an initial state localized close to the edge in a region of size $B^{-1/2}$ – an *edge state* – should move ballistically along the edge with a speed of order $\sqrt{B}$: here we used that the lowest Landau level, in absence of the edge, is of order B. On the other hand, although states further away from the edge – *bulk states* – should, due to the uncertainty principle, not remain completely localized in the y-direction, as in the classical case, they should nevertheless move much more slowly than the edge states. This picture has long been known to be correct, but as a preparation for the case when an impurity potential is present, we give a precise statement of the above properties in Corollary 2.1. We consider the Hamiltonian

$$H_0 = \frac{1}{2}p_x^2 + \frac{1}{2}(p_y - Bx)^2, \tag{1.1}$$

with a Dirichlet boundary condition at $x = 0$. Corresponding to each Landau band, we introduce the notion of H_0-invariant *edge* and *bulk* spaces, with the following properties. The y-component of the velocity, given by $i[H_0, Y]$, is of order $\sqrt{B}$ on an edge space, whereas it is exponentially small in B on a bulk space (Corollary 2.1). Furthermore, states belonging to the edge spaces are negligibly small at distances much larger than the magnetic length scale $1/\sqrt{B}$, reflecting the intuitively clear fact that the presence of the edge makes itself felt only in a region of size $1/\sqrt{B}$ from the edge. In this sense the edge states are quasi one-dimensional. The eigenfunctions of the restriction of H_0 to the edge spaces are extended along the entire edge.

The existence of non-localized current-carrying quasi one-dimensional edge states plays a role in certain theories of the quantum Hall effect [1] (see [2], [3] and [4] for further details). It is therefore of importance to understand if such states exist in systems exhibiting a quantized Hall resistance. This is argued to be the case in [1], in the case when the full Hamiltonian is obtained by adding a weak impurity potential to H_0. In other words, such potentials are not supposed to destroy the edge states existing in the free case. A very simple and rigorous proof of this statement is given in the present paper. A weak potential is a potential $W_B \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ satisfying $\delta_B \equiv ||W_B||_\infty < \frac{1}{2}B$. Since the distance between successive Landau levels equals B, such a potential can not close the gaps between the Landau levels of the infinite system without a edge, even though its size can be of order B: in this sense it is weak. It can however fluctuate arbitrarily fast, and in particular on the magnetic length scale, which is of order $1/\sqrt{B}$ (about $50-100$ Angstrom in realistic situations): this is important since, as explained in [2], the weak impurity potential is created by impurities at a distance of order $1/\sqrt{B}$ or less of the layer and can vary rapidly on this length scale. As a typical form for W_B we can keep in mind a potential of the type

$$W_B = \delta B \sum_{i \in \mathbb{Z}^+ \times \mathbb{Z}} u_i(B^\alpha(\vec{x} - \frac{i}{B^\beta}))$$

for some compactly supported site-potentials u_i and exponents $\alpha \geq 0$ and $\beta \geq 1/2$.

For weak potentials, we show that, in a spectral interval of size B between the Landau levels, there are no bound states and that the speed in the y-direction is still of order $\sqrt{B}$. As a consequence, we obtain that in the same spectral interval, the spectrum is absolutely continuous, implying the corresponding eigenstates are extended.

The results described above in the case when no impurity potential is present have been known for a long time and can be obtained by studying explicitly the spectrum and eigenfunctions of H_0, since it is an explicitly solvable Hamiltonian. Such an approach would however not easily extend to the case when an impurity potential is added. Instead, we show in Proposition 2.1 that the magnitude of the speed in the y-direction is strictly positive on the spectral subspaces corresponding to suitable spectral intervals between Landau levels. Such a positive commutator estimate is then shown to be stable under perturbations in section 3, yielding the main results via the virial theorem and the Mourre theory of positive commutators. (Theorem 3.1).

The idea that positive commutator methods and the virial theorem can be used to obtain information about magnetic Hamiltonians in the presence of boundaries was first proposed in [5]. They consider a model with a *soft* edge, modeled by a positive potential V, supported on the negative axis and steeply rising from 0, and prove the absence of eigenvalues in certain regions between the Landau levels in this case. The conjugate operator used in this approach is the quantum observable C_y corresponding to the y-coordinate of the centre of the classical circular orbit: $C_y = y - (p_x/B)$. Classically this is indeed a monotonic function of time for orbits close to the edge, since the Poisson bracket $\{C_y, H\} = \frac{1}{B}(\partial_x V + \partial_x W) < 0$ in that region, provided the impurity potential W has a small enough derivative.

In the present paper, we deal with the problem with a hard edge, as described before. We use the y-coordinate itself as a conjugate operator, proving that the speed in the y-direction, $i[H, Y]$, is strictly negative on edge states. This is marginally surprising since it is not true classically, but it turns out to be extremely simple to understand in terms of the band structure of the free Hamiltonian H_0. Using y also has the important advantage of not introducing derivatives of the potential in the commutator, as is the case when using C_y, and therefore eliminating the need to control their size. In addition, it renders the interpretation of the results in terms of propagation along the edge more transparent. On the down side, it is not obvious the present method will adapt itself easily to cases where the edge is not straight.

Let us point out that we could treat the soft edge in the same way. It seems however that this model does not lend itself to an analysis of the high field regime, which is important for the quantum Hall effect. In that case, the particles will, even in the lowest Landau level, penetrate deeply into the region $x < 0$, so that there is an effective edge around those values of x where $V(x) \sim B$, where V is the edge potential. The high field behaviour of the speed, for example, will then depend crucially on the precise shape of the edge, and this is not satisfactory. We will therefore not deal any further with the soft edge in the following.

The results of [5] on the soft edge have recently been extended [6] to a proof of absence of singular continuous spectrum in suitable intervals between the Landau levels, using the same conjugate operator as in [5] to prove a positive commutator estimate. While finishing the present work, we learned that those results were further extended, still using C_y as a conjugate operator, to the case of the *hard* edge in [7]. A result comparable to our Theorem 3.1 is proven there, but under the additional assumptions that both the first and second derivatives of the impurity potential are small, so that rapid fluctuations in the impurity potential are no longer allowed. In addition, our proof is technically considerably less complicated partially because, in the model with a hard edge, the operator C_y is symmetric but not self-adjoint, leading to complications in applying the Mourre theory of positive commutators.

2 The free Hamiltonian: edge and bulk spaces

To study H_0 in (1.1), we first use the translational invariance in the y-direction to write

$$H_0 = \int_{\mathbb{R}}^{\oplus} dk\ H(k), \quad \text{with} \quad H(k) = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}(k - Bx)^2, \quad x > 0, \tag{2.1}$$

acting on $L^2(\mathbb{R}_+ \times \mathbb{R}, dxdk)$, k being the Fourier transform variable conjugate to y. We recall that H_0 is essentially self-adjoint on the space of functions $\varphi \in C_0^\infty(\overline{\mathbb{R}_+} \times \mathbb{R})$

vanishing on the boundary [8].

The spectrum of $H(k)$ consists of isolated non-degenerate eigenvalues $E_n(k)$, $n \in \mathbb{N}_0$, with *normalized* eigenfunctions $\varphi_n(x,k)$. We will write $\mathcal{H}_n$ for the n^{th} *band space*, namely the space consisting of vectors of the form $f(k)\varphi_n(x,k)$, $f \in L^2(\mathbb{R}, dk)$. This is an H_0-invariant subspace of $L^2(\mathbb{R}_+ \times \mathbb{R}, dxdk)$; we shall on occasion view it as a subspace of $L^2(\mathbb{R}_+ \times \mathbb{R}, dxdy)$ as well, with the same notation. To understand the behaviour of the $E_n(k)$ and the $\varphi_n(x,k)$, and in particular their dependence on B, we introduce the following scaling:

$$\tilde{x} = \sqrt{B}x,\ \tilde{y} = \sqrt{B}y,\ H_0 = B\tilde{H}_0,\ \tilde{H}_0 = -\frac{1}{2}\frac{\partial^2}{\partial \tilde{x}^2} + \frac{1}{2}(\frac{1}{i}\frac{\partial}{\partial \tilde{y}} - \tilde{x})^2. \tag{2.2}$$

Note that, strictly speaking, H_0 is unitarily equivalent to $B\tilde{H}_0$, not equal to it, but since the unitary transformation is just the rescaling of the variables, we allow ourselves this slight abuse of notation. Again

$$\tilde{H}_0 = \int_{\mathbb{R}}^{\oplus} d\kappa\ \tilde{H}(\kappa),$$

with

$$\tilde{H}(\kappa) = -\frac{1}{2}\frac{d^2}{d\tilde{x}^2} + \frac{1}{2}(\kappa - \tilde{x})^2, \qquad \tilde{x} > 0. \tag{2.3}$$

Here κ is the Fourier transform variable conjugate to $\tilde{y}$, so that $ky = \kappa\tilde{y}$ and hence $k = \sqrt{B}\kappa$. The spectrum of $\tilde{H}(\kappa)$ consists of isolated eigenvalues $\alpha_n(\kappa)$. The normalized eigenfunctions of $\tilde{H}(\kappa)$, at each fixed κ, $\tilde{\varphi}_n(\cdot,\kappa)$ are given by

$$\tilde{\varphi}_n(\tilde{x},\kappa) = C_n D_{\alpha_n(\kappa)-1/2}(\sqrt{2}(\tilde{x} - \kappa)), \tag{2.4}$$

where $D_{\alpha-1/2}$ is the Whittaker function ([9] p686) with parameter α and $\alpha_n(\kappa)$ is determined by the boundary condition

$$D_{\alpha_n(\kappa)-1/2}(-\sqrt{2}\kappa) = 0. \tag{2.5}$$

One can check that the eigenvalues $\alpha_n(\kappa)$ are smooth functions of κ. The other properties of the $\alpha_n(\kappa)$ that we shall be needing are collected in the following Lemma:

Lemma 2.1 *(i)* $\alpha_n(0) = 2n + 3/2, \qquad \alpha_n'(0) = -\dfrac{(2n+2)!}{n!(n+1)!\pi^{1/2}2^{2n}},$

(ii) $\alpha_{n+1}(\kappa) - \alpha_n(\kappa) > 1,$

(iii) $\alpha_n'(\kappa) = -\frac{1}{2}|\tilde{\varphi}_n'(0,\kappa)|^2 < 0,$

(iv) $\alpha_n(\kappa) > (n + \frac{1}{2})$ *and there exist* $C_n > 0$ *so that, for all* $\kappa \geq 0$,

$$\alpha_n(\kappa) - (n + \frac{1}{2}) \leq C_n \exp -\frac{1}{4}(\kappa - \sqrt{n})^2.$$

(v) For all $n \in \mathbb{N}_0$ *and all* $\epsilon > 0$, *there exist positive constants* X_n, $K_{n,\epsilon}, C_{n,\epsilon}$ *so that, for all* $\kappa \geq K_{n,\epsilon}$ *and all* $\tilde{x} \in [0, X_n]$,

$$|\tilde{\varphi}_n(\tilde{x}, \kappa)|^2 \leq C_{n,\epsilon} \exp\left\{-\frac{1}{2}(1-\epsilon)(\tilde{x} - \kappa)^2\right\},$$

and

$$|\alpha_n'(\kappa)| \leq C_{n,\epsilon} \exp\left\{-\frac{1}{2}(1-\epsilon)\kappa^2\right\}.$$

(vi) For all $\kappa < 0$, $|\alpha_n'(\kappa)| > |\kappa|$ *and* $\lim_{\kappa \to -\infty} \alpha_n(\kappa) = \infty$.

The proof of this lemma, which uses only standard techniques of Schrödinger operator theory, is postponed to Section 4. Numerical computation of the $\alpha_n(\kappa)$ indicates that they are convex functions of κ (see Figure 1). We have not been able to prove this result and shall not use it. If it is true, the statements of our results below can be simplified somewhat.

It is clear from the above that the spectrum of $\tilde{H}_0$ and hence of H_0 is absolutely continuous and fills the entire half-axis from 1/2 to infinity. The bands $E_n(k)$ can now be written

$$E_n(k) = B\alpha_n(\frac{k}{\sqrt{B}}). \tag{2.6}$$

Writing $\varphi_n(x, k)$ for the normalized eigenfunctions of $H(k)$ (at each fixed k), we have

$$\varphi_n(x, k) = B^{1/4}\tilde{\varphi}_n(\sqrt{B}x, \frac{k}{\sqrt{B}}). \tag{2.7}$$

These simple observations will now allow us to define within $\mathcal{H}_n$ *edge spaces* and *bulk spaces* as follows. We define, for each $\sigma > 0, \gamma > 0$:

$$\mathcal{H}_{n,e}(\sigma, \gamma) \cong L^2(]-\infty, \sigma B^\gamma], dk) \subset \mathcal{H}_n, \tag{2.8}$$
$$\mathcal{H}_{n,b}(\sigma, \gamma) \cong L^2([\sigma B^\gamma, \infty), dk) \subset \mathcal{H}_n, \tag{2.9}$$
$$\mathcal{H}_n = \mathcal{H}_{n,e} \oplus \mathcal{H}_{n,b}. \tag{2.10}$$

Note that these spaces are H_0 invariant. We will call $\mathcal{H}_{n,e}(\sigma, \gamma)$ an edge space for all $\gamma \leq 1/2$ and $\mathcal{H}_{n,b}(\sigma, \gamma)$ a bulk space for all $\gamma > 1/2$. For a different approach to the definition of bulk and edge spaces, in the case of a bounded geometry, we refer to [10].

To understand those definitions, recall first that a standard stationary phase argument shows that $-\partial_k E_n(k_0) = -\sqrt{B}\alpha_n'(k_0/\sqrt{B})$ is the group speed in the y-direction of a wave packet $f(k)\varphi_n(x,k)$ with the support of f close to k_0. If k_0 is inside an interval $(-\infty, k_B]$ where k_B is of order $\sqrt{B}$ or smaller, the wave packet belongs to the edge space $\mathcal{H}_{n,e}(\sigma, 1/2)$ and it follows from Lemma 2.1 that such a wave packet speeds along the edge in the y direction with a velocity of order $\sqrt{B}$. In addition, it follows from standard exponential estimates on the eigenfunctions $\tilde{\varphi}_n$ (as in the proof of Lemma 2.1) that in this case the wave packet is exponentially small for x much bigger than $1/\sqrt{B}$. If, on the other hand, k_0 belongs to an interval of the form $[k_B, \infty[$ with k_B of order $B^\gamma, \gamma > \frac{1}{2}$, then the group velocity is exponentially small in B (see Lemma 2.1 (iv)). In addition, if $f(k)\varphi_n(x,k) \in \mathcal{H}_{n,b}(\sigma,\gamma)$, with $\gamma > \frac{1}{2}$, then Lemma 2.1(iv) immediately implies that

$$\int_0^{\frac{1}{\sqrt{B}}} \int_{-\infty}^{\infty} |f(k)\varphi_n(x,k)|^2 dk dx \leq C_{n,\epsilon} \exp -(1-\epsilon)(\sigma^2 B^{2\gamma-1} - 1),$$

so that the wave packet is exponentially small in the region $0 \leq x \leq \frac{1}{\sqrt{B}}$ close to the edge. We note also that the spectrum of H_0 restricted to a bulk space $\mathcal{H}_{n,b}(\sigma,\gamma)$ is an exponentially small interval (in B) just above the nth Landau level (Lemma 2.1 (iii)), that we will refer to as the bulk spectrum. The spectrum of H_0 restricted to an edge space $\mathcal{H}_{n,e}(\sigma, 1/2)$ – *the edge spectrum* – is on the other hand of the form $[B(n + \frac{1}{2} + c_\sigma), \infty)$. In particular, it fills up an interval of size B below the $(n+1)$th Landau level, including the latter.

To give a formulation of the above statements that is at once more precise and does not use the band structure of the Hamiltonian H_0, so that it has a chance to pass to the perturbed Hamiltonian, we now turn to the statement and proof of a *positive commutator estimate.* We will show that the speed $V_y = i[H_0, Y]$, where the operator Y is multiplication by y, is strictly negative away from the Landau levels. This is the content of the following proposition, which will be generalized to the perturbed Hamiltonian in the next section.

Let $L_n = (n + 1/2, n + 3/2]$ be the nth Landau band when $B = 1$.

Proposition 2.1 *Let $\Delta \subset L_n$ be a closed interval with $|\Delta| < 1$. Let*

$$\nu_-(\Delta) = \inf_{\{n',\kappa \ | \ \alpha_{n'}(\kappa) \in \Delta\}} |\alpha_{n'}'(\kappa)| > 0 \tag{2.11}$$

and

$$\nu_+(\Delta) = \sup_{\{n',\kappa \ | \ \alpha_{n'}(\kappa) \in \Delta\}} |\alpha_{n'}'(\kappa)| > 0. \tag{2.12}$$

If $\tilde{Y}$ is multiplication by $\tilde{y}$ and $\tilde{P}_0(\Delta)$ is the spectral projection of $\tilde{H}_0$ onto Δ, then

$$\nu_-(\Delta)\tilde{P}_0(\Delta) \leq \tilde{P}_0(\Delta)i[\tilde{Y},\tilde{H}_0]\tilde{P}_0(\Delta) \leq \nu_+(\Delta)\tilde{P}_0(\Delta), \tag{2.13}$$

and consequently, for ψ with $||\psi|| = 1$ in the range of $\tilde{P}_0(\Delta)$

$$-\nu_+(\Delta)t \leq \langle \psi_t, \tilde{Y}\psi_t \rangle - \langle \psi_0, \tilde{Y}\psi_0 \rangle \leq -\nu_-(\Delta)t. \tag{2.14}$$

Proof: It follows from Lemma 2.1 that $\alpha_{n'}^{-1}(\Delta) \cap \alpha_{n''}^{-1}(\Delta) = \emptyset$ if $n' \neq n''$, since $|\Delta| < 1$. Thus for any ψ we can write

$$\tilde{P}_0(\Delta)\psi(\tilde{x},\kappa) = \sum_{n'=0}^{n} \psi_{n'}(\tilde{x},\kappa), \tag{2.15}$$

where

$$\psi_{n'}(\tilde{x},\kappa) = \beta_{n'}(\kappa)\mathbf{1}_{\alpha_{n'}^{-1}(\Delta)}(\kappa)\tilde{\varphi}_{n'}(\tilde{x},\kappa) \tag{2.16}$$

and

$$\beta_{n'}(\kappa) = \int_{\mathbb{R}_+} \overline{\psi(\tilde{x},\kappa)}\ \tilde{\varphi}_{n'}(\tilde{x},\kappa)dx. \tag{2.17}$$

Since $i[\tilde{Y},\tilde{H}_0] = \tilde{x} - p_{\tilde{y}}$, it is clear that $\langle \psi_{n'}, i[\tilde{Y},\tilde{H}_0]\psi_{n''} \rangle = 0$ if $n' \neq n''$ since the supports of $\psi_{n'}$ and of $\psi_{n''}$ are disjoint in the κ variable. On the other hand,

$$\langle \psi_{n'}, i[\tilde{Y},\tilde{H}_0]\psi_{n'} \rangle = \int_{\alpha_{n'}^{-1}(\Delta)} d\kappa\ |\beta_{n'}(\kappa)|^2 \int_{\mathbb{R}_+} d\tilde{x}(\tilde{x}-\kappa)|\varphi_{n'}(\tilde{x},\kappa)|^2, \tag{2.18}$$

and, by the Feynman-Hellman theorem

$$\int_{\mathbb{R}_+} d\tilde{x}(\tilde{x}-\kappa)|\tilde{\varphi}_{n'}(\tilde{x},\kappa)|^2 = -\alpha'_{n'}(\kappa) = |\alpha'_{n'}(\kappa)|.$$

The proposition is now immediate.

□

Using the scaling behaviour in B we now have the following Corollary:

Corollary 2.1
(i) Let $n \in \mathbb{N}$ be fixed and let $\Delta \subset ((n+\frac{1}{2})B,(n+\frac{3}{2})B]$ be a closed interval with $|\Delta| < B$. Then

$$\sqrt{B}\nu_-(B^{-1}\Delta)P_0(\Delta) \leq P_0(\Delta)i[Y,H_0]P_0(\Delta) \leq \sqrt{B}\nu_+(B^{-1}\Delta)P_0(\Delta). \tag{2.19}$$

where $B^{-1}\Delta = \{E/B \mid E \in \Delta\}$.

(ii) For all $n \in \mathbb{N}$, for all $\sigma > 0$ there exists a constant $C_{n,\sigma} > 0$ so that for all $\psi \in \mathcal{H}_{n,e}(\sigma, 1/2)$ and for all B

$$\langle \psi, i[Y, H_0]\psi \rangle \geq \sqrt{B} \inf_{\kappa \leq \sigma} |\alpha_n'(\kappa)| \; ||\psi||^2 \; > C_{n,\sigma}\sqrt{B}||\psi||^2. \tag{2.20}$$

(iii) Let $\epsilon > 0$. Then for all $n \in \mathbb{N}_0$, for all $\sigma > 0$ there exists a constant $C_{n,\sigma,\epsilon} > 0$ so that for all B and for all $\psi \in \mathcal{H}_{n,b}(\sigma, 1/2 + \epsilon)$

$$\langle \psi, i[Y, H_0]\psi \rangle \leq \sqrt{B} \sup_{\kappa \geq \sigma B^\epsilon} |\alpha_n'(\kappa)| \; ||\psi||^2 \; < C_{n,\sigma,\epsilon}\sqrt{B} \exp\left\{ -\frac{1}{2}(1-\epsilon)\sigma^2 B^{2\epsilon} \right\} ||\psi||^2. \tag{2.21}$$

Proof: This is now an immediate consequence of Lemma 2.1 and of the proof of Proposition 2.1.

□

Remark 2.1 *Parts (ii) and (iii) of the corollary state that the speed in the y direction is at least of order $\sqrt{B}$ for any edge state and at most of order $\exp -B^{2\epsilon}$ for any bulk state.*

3 Adding a weak impurity potential

We now consider the Hamiltonian

$$H = H_0 + W_B$$

where $W_B \in L^\infty(\mathbb{R}_+ \times \mathbb{R}, dxdy)$ is a real potential satisfying $||W_B||_\infty \leq AB$ where $A < \infty$ is independent of B. Let

$$\tilde{H} = \tilde{H}_0 + \tilde{W}_B; \tilde{W}_B(\tilde{x}, \tilde{y}) = B^{-1} W_B(\frac{\tilde{x}}{\sqrt{B}}, \frac{\tilde{y}}{\sqrt{B}})$$

and let $\tilde{P}(\cdot)$ denote the spectral family of $\tilde{H}$.

Our main result is then the following theorem, which should be compared to Proposition 2.1. For $\lambda < 1$, let $L_n^\lambda = (n + 1/2 + \lambda, n + 3/2]$. Let

$$\nu(n, \lambda) = \nu_-(L_n^\lambda) = \inf\{|\alpha_{n'}'(\kappa)| \mid n' \leq n, \; n{+}1/2{+}\lambda < \alpha_{n'}(\kappa) \leq n{+}3/2\} > 0. \tag{3.1}$$

Theorem 3.1 *Let $n \in \mathbb{N}$ be fixed. Let λ, $\lambda' > 0$ with $\lambda + \lambda' < 1$ and let $L_n^{\lambda,\lambda'} = (n + 1/2 + \lambda, n + 3/2 - \lambda')$. There exists $\delta(n, \lambda, \lambda') > 0$ such that if $||W_B||_\infty < \delta(n, \lambda, \lambda')B$ and $\epsilon < \delta(n, \lambda, \lambda')$, then, for all $\alpha \in L_n^{\lambda,\lambda'}$, for the interval $\Delta \equiv (\alpha - \epsilon, \alpha + \epsilon)$,*

$$\tilde{P}(\Delta) i[\tilde{Y}, \tilde{H}] \tilde{P}(\Delta) \geq \frac{1}{2} \nu(n, \lambda/2) \tilde{P}(\Delta). \tag{3.2}$$

Consequently if $||W_B||_\infty < \delta(n, \lambda, \lambda')B$, then

$$\sigma_{\text{sing}}(\tilde{H}) \cap L_n^{\lambda,\lambda'} = \emptyset. \tag{3.3}$$

Clearly we can give a scaled up version of this theorem:

Corollary 3.1 *Let $n \in \mathbb{N}$ be fixed and let λ, $\lambda' > 0$ with $\lambda + \lambda' < 1$. There exists $\delta(n, \lambda, \lambda') > 0$ such that if $||W_B||_\infty < \delta(n, \lambda, \lambda')B$, then*

$$\sigma_{\text{sing}}(H) \cap (B(n + 1/2 + \lambda), B(n + 3/2 - \lambda')) = \emptyset. \tag{3.4}$$

It is useful to have the following variant of Theorem 3.1. Here we fix a bound on $||W_B||_\infty / B$ and give the dependence on this bound of the endpoints a,b of the interval $(a, b) \subset L_n$, such that (a, b) contains only absolutely continuous spectrum.

Theorem 3.2 *Let $n \in \mathbb{N}$ be fixed. Suppose that $||W_B||_\infty < \delta B$ where $\delta < 1/2$. Let λ_n, $\lambda'_n \in (0, 1/2)$ be such that $\lambda_n \nu(n, \lambda_n/2)^2 > 2^9(n + 2)\delta$ and $\lambda'_n \nu(n, 1/4)^2 > 2^9(n + 2)\delta$ then for all $\alpha \in L_n^{\lambda_n,\lambda'_n}$, there exists an interval Δ containing α such that*

$$\tilde{P}(\Delta) i[\tilde{Y}, \tilde{H}] \tilde{P}(\Delta) \geq \frac{1}{2} \nu(n, \lambda_n/2) \tilde{P}(\Delta). \tag{3.5}$$

Therefore

$$\sigma_{\text{sing}}(\tilde{H}) \cap L_n^{\lambda_n,\lambda'_n} = \emptyset. \tag{3.6}$$

Note that, given δ, no λ_n and λ'_n satisfying the conditions of the theorem might exist. Nevertheless, it is clear that for sufficiently small δ, the above results guarantee the existence of an interval of absolutely continuous spectrum between the Landau levels. The scaled up version of this theorem is then:

Corollary 3.2 *Under the conditions of Theorem 3.2,*

$$\sigma_{\text{sing}}(H) \cap (B(n + 1/2 + \lambda_n), B(n + 3/2 - \lambda'_n)) = \emptyset. \tag{3.7}$$

Proof of Theorem 3.1: Note first that $i[\tilde{Y},\tilde{H}_0]=i[\tilde{Y},\tilde{H}]$, so that the result would follow from Proposition 2.1 if we could replace $\tilde{P}(\Delta)$ by $\tilde{P}_0(\Delta)$. This can indeed be achieved with a few tricks and at not too high a cost, provided one replaces the interval Δ by an auxiliary one Δ', that is larger but for which $\nu(\Delta')$ is not too small. Let $\sigma\equiv\min(\lambda,\lambda')/4$ and let Δ' be the interval $[\alpha-\sigma,\alpha+\sigma]\subset L_n^{\lambda/2,\lambda'/2}$. Let Δ be the interval $[\alpha-\epsilon,\alpha+\epsilon]$, where $\epsilon\le\sigma$. Let $\psi\in\tilde{P}(\Delta)\mathcal{H}$. Then, recalling that $||\tilde{W}_B||_\infty\le A$,

$$||(\tilde{H}_0-\alpha)\psi||\le||(\tilde{H}-\alpha)\tilde{P}(\Delta)\psi||+A||\psi||\le(\epsilon+A)||\psi||.$$

Hence

$$||\tilde{P}_0({\Delta'}^c)\psi||\le||\frac{1}{\tilde{H}_0-\alpha}\tilde{P}_0({\Delta'}^c)||\ ||(\tilde{H}_0-\alpha)\psi||\le\sigma^{-1}(\epsilon+A)||\psi||, \tag{3.8}$$

since $\min\{|\lambda-\alpha|\mid\lambda\in\Delta'^c\}\ge\sigma$. Clearly

$$\begin{aligned} i\langle\psi,[\tilde{Y},\tilde{H}]\psi\rangle &\ge i\langle\tilde{P}_0(\Delta')\psi,[\tilde{Y},\tilde{H}_0]\tilde{P}_0(\Delta')\psi\rangle \\ &\quad -2||[\tilde{Y},\tilde{H}_0]\tilde{P}_0({\Delta'}^c)\psi||\ ||\psi||. \end{aligned}\tag{3.9}$$

The required positivity will come from the first term, so we only have to control the last one. We find

$$\begin{aligned} ||[\tilde{Y},\tilde{H}_0]\tilde{P}_0({\Delta'}^c)\psi|| &\le 2\langle\tilde{P}_0({\Delta'}^c)\psi,\tilde{H}_0\tilde{P}_0({\Delta'}^c)\psi\rangle^{1/2} \\ &\le 2||\tilde{H}_0\tilde{P}_0({\Delta'}^c)\psi||^{1/2}||\tilde{P}_0({\Delta'}^c)\psi||^{1/2}. \end{aligned}$$

But

$$\begin{aligned} ||\tilde{H}_0\tilde{P}_0({\Delta'}^c)\psi||^{1/2} &\le (||\tilde{H}\psi||+A||\psi||)^{1/2} \\ &\le (n+3/2+A)^{1/2}||\psi||^{1/2} \\ &\le (n+2)^{1/2}||\psi||^{1/2}, \end{aligned}$$

if $A\le1/2$. Therefore

$$||[\tilde{Y},\tilde{H}_0]\tilde{P}_0({\Delta'}^c)\psi||\le2(n+2)^{1/2}\sigma^{-1/2}(\epsilon+A)^{1/2}||\psi||.$$

Inserting this into (3.9) yields

$$\begin{aligned} i\langle\psi,[\tilde{Y},\tilde{H}]\psi\rangle &\ge i\langle\tilde{P}_0(\Delta')\psi,[\tilde{Y},\tilde{H}_0]\tilde{P}_0(\Delta')\psi\rangle \\ &\quad -4(n+2)^{1/2}\sigma^{-1/2}(\epsilon+A)^{1/2}||\psi||^2. \end{aligned}\tag{3.10}$$

On the other hand, since $|\Delta'|=2\sigma<1$, Proposition 2.1 states that

$$i\langle\tilde{P}_0(\Delta')\psi,[\tilde{Y},\tilde{H}_0]\tilde{P}_0(\Delta')\psi\rangle\ge\nu(\Delta')||\tilde{P}_0(\Delta')\psi||^2\ge\nu(n,\lambda/2)||\tilde{P}_0(\Delta')\psi||^2,$$

where $\nu(n, \lambda/2)$ is defined in (3.1). Inserting this into (3.10) and using (3.8) together with the observation that $||\psi||^2 = ||\tilde{P}_0(\Delta')\psi||^2 + ||\tilde{P}_0(\Delta'^c)\psi||^2$, yields

$$i\langle\psi, [\tilde{Y}, \tilde{H}]\psi\rangle \geq \nu(n, \frac{\lambda}{2})[1 - \left(\frac{(\epsilon + A)^2}{\sigma^2} + \frac{4(n+2)^{1/2}(\epsilon + A)^{1/2}}{\sigma^{1/2}\nu(n, \frac{\lambda}{2})}\right)]||\psi||^2. \quad (3.11)$$

Let $\delta(n, \lambda, \lambda') = \min\left(\frac{\sigma\nu(n,\lambda/2)^2}{2^9(n+2)}, \frac{\sigma}{4}, \frac{1}{2}\right)$. Then if $A < \delta$ and $\epsilon < \delta$, one has that $4(n+2)^{1/2}\sigma^{-1/2}(\epsilon + A)^{1/2} < \frac{1}{4}\nu(n, \lambda/2)$ and $((\epsilon + A)/\sigma)^2 \leq 1/4$, so that the first statement in the theorem follows. To prove (3.3) it is now sufficient to use (3.2) and to apply the Mourre theory of positive commutators. For a textbook treatment, we refer to [11]; see also [12] for a concise review of the domain questions involved. The latter are trivial in the present case. Indeed, the commutator $[H_0, Y] = [H, Y]$ is obviously relatively H_0 bounded, the domain of the Hamiltonian is invariant under the unitary group $\exp isY$ and the second commutator $[[H_0, Y], Y]$ is bounded.

□

4 Proof of Lemma 2.1

(i) This follows from a computation using standard properties of the Hermite polynomials.

(ii) The Whittaker functions satisfy the recurrence relations ([9] p688):

$$-D'_{\alpha+1/2}(x) - \frac{1}{2}xD_{\alpha+1/2}(x) = -(\alpha + 1/2)D_{\alpha-1/2}(x) \quad (4.1)$$

$$-D'_{\alpha-3/2}(x) + \frac{1}{2}xD_{\alpha-3/2}(x) = D_{\alpha-1/2}(x). \quad (4.2)$$

Consider $H_e(\kappa)$ given by the same expression as for $\tilde{H}(\kappa)$ in (2.3) but with elastic boundary condition

$$\psi'(0) = \lambda\psi(0), \quad 0 \leq \lambda < \infty,$$

instead of the Dirichlet boundary condition ($\lambda = \infty$). Let $\beta_n(\kappa)$, $n = 0, 1, 2, \ldots$ be the eigenvalues of $H_e(\kappa)$. Then standard arguments (see [14], Chapter 1, section 3) show that

$$\beta_0(\kappa) < \alpha_0(\kappa) < \beta_1(\kappa) < \alpha_1(\kappa) < \beta_2(\kappa) < \ldots \quad (4.3)$$

If $\kappa \geq 0$, put $\lambda = \kappa$. Then from (4.1) it follows that the eigenfunctions of $H_e(\kappa)$ are $D_0(\sqrt{2}(\tilde{x} - \kappa))$, $D_{\alpha_{n-1}(\kappa)+1/2}(\sqrt{2}(\tilde{x} - \kappa))$, $n = 1, 2, \ldots$ with eigenvalues $\beta_0(\kappa) = 1/2$, $\beta_n(\kappa) = \alpha_{n-1}(\kappa) + 1$, $n = 1, 2, \ldots$ Thus

$$\frac{1}{2} < \alpha_0 < \alpha_0 + 1 < \alpha_1 < \alpha_1 + 1 < \ldots$$

If $\kappa \le 0$ put $\lambda = -\kappa$. In this case it follows from (4.2) that the eigenfunctions of $H_e(\kappa)$ are $D_{\alpha_n(\kappa)-3/2}(\sqrt{2}(\tilde{x}-\kappa))$, $n = 0, 1, \ldots$ with eigenvalues $\beta_n(\kappa) = \alpha_n(\kappa) - 1$. Therefore

$$\alpha_0 - 1 < \alpha_0 < \alpha_1 - 1 < \alpha_1 < \alpha_2 - 1 < \ldots.$$

(iii) We put $V_\kappa(\tilde{x}) = \frac{1}{2}(\tilde{x}-\kappa)^2$ and use the Feynman-Hellman formula to write (see [13])

$$\begin{aligned}\alpha_n'(\kappa) &= -\int_0^\infty V_\kappa'(\tilde{x})\tilde{\varphi}_n^2(\tilde{x},\kappa)\, d\tilde{x} \\ &= 2\int_0^\infty V_\kappa(\tilde{x})\tilde{\varphi}_n(\tilde{x},\kappa)\tilde{\varphi}_n'(\tilde{x},\kappa)\, d\tilde{x} \\ &= \int_0^\infty \varphi_n''(\tilde{x},\kappa)\varphi_n'(\tilde{x},\kappa)\, d\tilde{x} + 2\alpha_n(\kappa)\int_0^\infty \varphi_n(\tilde{x},\kappa)\varphi_n'(\tilde{x},\kappa)\, d\tilde{x},\end{aligned}$$

from which the result follows. Note that by uniqueness $\varphi_n'(0)$ cannot be zero.

(iv) Here we will use a perturbative argument, treating the Dirichlet boundary condition at 0 as a perturbation. We note first that, by the min-max principle, $\alpha_n(\kappa) > n + \frac{1}{2}$. Now, let h_n denote the nth Hermite function and let $h_{n,\kappa}(x) = h_n(x-\kappa)$. Let θ be a smooth function such that $\theta(x) = 0$ for $x \le 0$ and $\theta(x) = 1$ for $x \ge 1$ We compute

$$(\tilde{H}(\kappa) - (n+\frac{1}{2}))\theta h_{n,\kappa} = [\tilde{H}(\kappa), \theta]h_{n,\kappa} = \frac{1}{2}(-\theta'' - 2i\theta' p)h_{n,\kappa}.$$

Now, since the supports of θ' and θ'' are contained in $[0,1]$, and since

$$||\theta' p h_n||^2 = \langle h_{n,\kappa}, [p, \theta'^2] p h_{n,\kappa}\rangle + \langle \theta'^2 h_{n,\kappa}, p^2 h_{n,\kappa}\rangle,$$

one easily concludes there exists a constant C_n so that

$$||(\tilde{H}(\kappa) - (n+\frac{1}{2}))\theta h_{n,\kappa}|| \le C_n ||\mathbf{1}_{[0,1]} h_{n,\kappa}||^{\frac{1}{2}},$$

where $\mathbf{1}_{[0,1]}$ denotes the characteristic function of $[0,1]$. Standard properties of the Hermite functions then imply that, for κ large enough

$$||(\tilde{H}(\kappa) - (n+\frac{1}{2}))\theta h_{n,\kappa}|| \le C_n \exp -\frac{1}{4}(\kappa - \sqrt{n})^2.$$

This shows that, for n fixed,

$$\mathrm{dist}(\sigma(\tilde{H}(\kappa)), (n+\frac{1}{2})) \le 2C_n \exp -\frac{1}{4}(\kappa - \sqrt{n})^2.$$

For $n = 0$, $|\alpha_0(\kappa) - \frac{1}{2}| = \mathrm{dist}(\sigma(\tilde{H}(\kappa)), \frac{1}{2}) \le 2C_0 \exp -\frac{1}{4}(\kappa)^2$, since $\alpha_1(\kappa) > 3/2$, and (iv) then follows by induction on n.

(v) This only involves a rather straightforward application of the standard method for proving exponential decay estimates on eigenfunctions in a classically forbidden region (see, for example [15, 8]). With $V_\kappa(\tilde{x}) \equiv \frac{1}{2}(\tilde{x}-\kappa)^2$ as before, we first define, for all $\kappa > 0$ large enough, $0 < x_n(\kappa) < \kappa$ by $V_\kappa(x_n(\kappa)) = \alpha_n(\kappa) + 1$. Clearly, for all $0 \le \tilde{x} \le x_n(\kappa)$, $\tilde{\varphi}_n(\tilde{x},\kappa)$ and $\tilde{\varphi}''_n(\tilde{x},\kappa)$ have the same sign, which we can assume to be strictly positive. Also, on the same region $\varphi'_n(\tilde{x},\kappa) > 0$. As a result, for any $a \in [0, x_n(\kappa) - 2]$, one has

$$|\tilde{\varphi}_n(a,\kappa)|^2 \le \int_a^{a+1} |\tilde{\varphi}_n(y,\kappa)|^2 dy.$$

Let, for $0 \le \tilde{x} \le x_n(\kappa)$,

$$f_n(\tilde{x},\kappa) = \int_{\tilde{x}}^{x_n(\kappa)} \sqrt{2(V_\kappa(y) - \alpha_n(\kappa) - 1)}dy.$$

Note that

$$\frac{1}{2} f'_n(\tilde{x},\kappa)^2 - (V_\kappa(\tilde{x}) - \alpha_n(\kappa) - 1) = 0.$$

We introduce $\eta_n(\tilde{x},\kappa)$, a smooth characteristic function of the interval $[0, x_n(\kappa)-1]$, with supp $\eta'_n \subset [x_n(\kappa)-1, x_n(\kappa)-1/2]$. Then

$$\begin{aligned}\int_a^{a+1} |\tilde{\varphi}_n(y,\kappa)|^2 dy &\le \exp -2f_n(a+1,\kappa) \int_a^{a+1} \exp 2f_n(y,\kappa) |\tilde{\varphi}_n(y,\kappa)|^2 dy \\ &\le \exp -2f_n(a+1,\kappa) \langle \psi_n, (V_\kappa - \alpha_n(\kappa) - \frac{1}{2} f'^2_n)\psi_n \rangle, \quad (4.4)\end{aligned}$$

where $\psi_n = \eta_n(\exp f_n)\tilde{\varphi}_n$. A simple computation shows

$$(\exp f_n)(\tilde{H} - \alpha_n)(\exp -f_n)\psi_n = \frac{\tilde{p}^2}{2}\psi_n + (V_\kappa - \alpha_n - \frac{1}{2} f'^2_n)\psi_n + \frac{1}{2}(f'_n \frac{d}{d\tilde{x}} + \frac{d}{d\tilde{x}} f'_n)\psi_n,$$

so that

$$\mathrm{Re}\langle \psi_n, (\exp f_n)(\tilde{H} - \alpha_n)(\exp -f_n)\psi_n \rangle \ge \langle \psi_n, (V_\kappa - \alpha_n - \frac{1}{2} f'^2_n)\psi_n \rangle.$$

On the other hand, using the definition of ψ_n and $(\tilde{H} - \alpha_n)\tilde{\varphi}_n = 0$, one has

$$\mathrm{Re}\langle \psi_n, (\exp f_n)(\tilde{H} - \alpha_n)(\exp -f_n)\psi_n \rangle = \mathrm{Re}\langle \tilde{\varphi}_n, (\exp 2f_n)\eta_n \frac{1}{2}[\tilde{p}^2, \eta_n]\tilde{\varphi}_n \rangle.$$

Consequently,

$$|\tilde{\varphi}_n(a,\kappa)|^2 \le (\exp -2f_n(a+1,\kappa)) \mathrm{Re}\langle \tilde{\varphi}_n, W_n \tilde{\varphi}_n \rangle,$$

where

$$W_n = (\exp 2f_n)\ \eta_n \frac{1}{2}[\tilde{p}^2, \eta_n] = -\frac{1}{2}(\exp 2f_n)(\eta_n\eta_n'' + 2i\eta_n\eta_n'\tilde{p}),$$

so that

$$\mathrm{Re}W_n = -\frac{1}{2}\exp 2f_n(\eta_n\eta_n'' - 2f_n'\eta_n\eta_n' - (\eta_n\eta_n')').$$

It follows from the support properties of η_n' and the definition of f_n that there exists a constant C_n so that for all κ, one has $|\mathrm{Re}W_n| \leq C_n$. As a result, a simple computation shows that, for all ϵ, there exists constants $C_{n,\epsilon}, K_{n,\epsilon}$ so that for all $\kappa > K_{n,\epsilon}$, and for all $0 \leq \tilde{x} < x_n(\kappa) - 2$

$$|\tilde{\varphi}_n(\tilde{x}, \kappa)|^2 \leq C_{n,\epsilon} \exp\left\{-\frac{1}{2}(1-\epsilon)(\kappa - \tilde{x})^2\right\}.$$

This proves the first statement of (v). To prove the second estimate, it is clear from part (iii) that we need to prove the above estimate holds for $|\tilde{\varphi}_n'(\tilde{x}, \kappa)|^2$ as well. If χ is a smooth characteristic function of the interval $[0,1]$ with support in $[0,2]$, one has, using the eigenvalue equation and two partial integrations that

$$\begin{aligned} |\tilde{\varphi}_n'(0,\kappa)|^2 &\leq \int_0^1 |\tilde{\varphi}_n'(\tilde{x},\kappa)|^2 d\tilde{x} \\ &\leq \int_0^\infty \chi(\tilde{x})\tilde{\varphi}_n'(\tilde{x},\kappa)^2 d\tilde{x} \leq C\int_0^1 |\tilde{\varphi}_n(\tilde{x},\kappa)|^2 d\tilde{x}, \end{aligned}$$

from which the result follows. The last part of the Lemma is an immediate consequence of the Feynman-Hellman formula.

□

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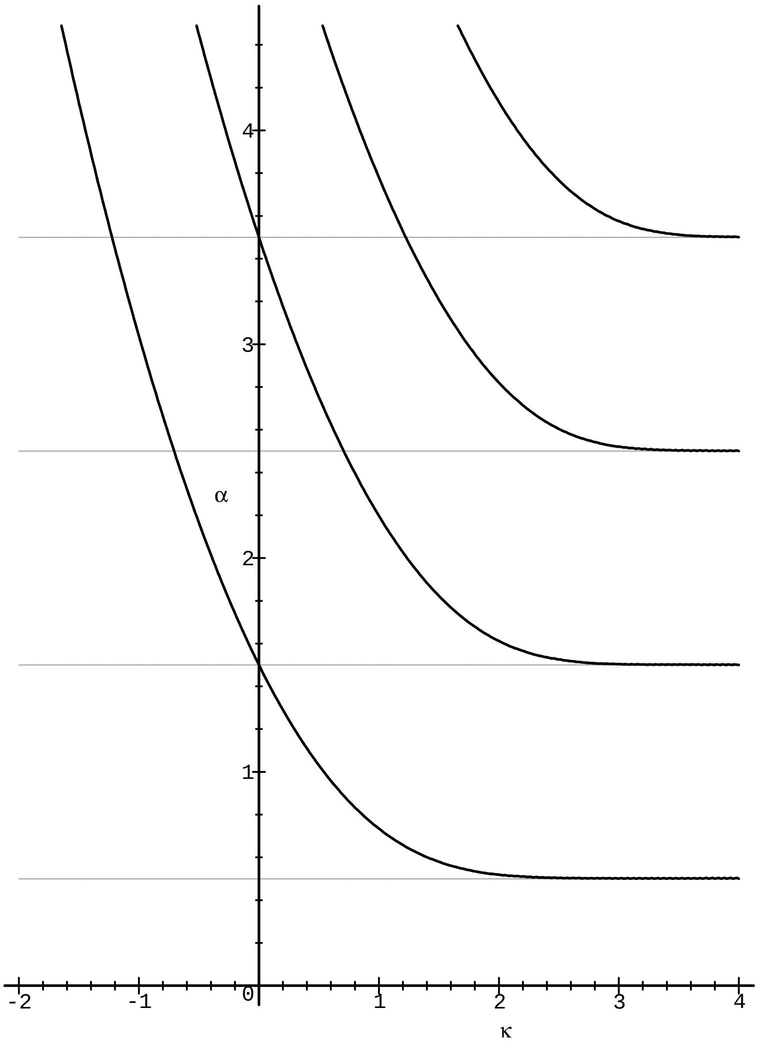

Figure 1: $\alpha_n(\kappa)$ for $n = 0, 1, 2, 3$