1 The Fourier integral transform

Let \( f \) denote a real-or complex-valued function of a real variable \( x \) such that \( f(x) \) is defined over \( \mathbb{R} \). The Fourier transform on \( f \) arises when considering linear integral transformations of the form

\[
T_\xi f = \int_{-\infty}^{+\infty} f(x)K(x,\xi)dx
\]

for a "nice" function \( K \). Let’s call \( K \) the kernel of the transformation. In general there are many choices for the kernel and each is useful in its own right. In our case, the goal is to choose a kernel and the domain of the parameter \( \xi \) so that \( T \) transforms derivatives \( f' \) (let’s assume derivatives exist) into products

\[
T_\xi f' = \xi T_\xi f
\]

To this end, we assume henceforth that the functions \( K \) have derivatives in \( x \) and are bounded on \( \mathbb{R} \) whereas the functions \( f \) live in the set \( C^m_0 \) of continuously differentiable functions with \( m \) derivatives and the functions together with the derivatives decay to zero sufficiently fast so that (1) converges. Ok, these are a lot of assumptions and the natural question is: can we weaken some of these conditions? The answer is "Yes!" but one would need to consider \( L_p \) spaces and the Lebesgue integral to achieve this task.

**Definition 1.** The Fourier transform of a function \( f \in C^m_0 \) is given by \( \Phi_\xi(f(x)) = \int_{-\infty}^{+\infty} f(x)e^{-i\xi x}dx \).

One might wonder how we picked the kernel. Well, a quick integration by parts of (1) yields \( K'(x,\xi) = -\xi K(x,\xi) \) and the exponential function satisfies this and all the other above mentioned requirements. Moreover, repeated integration by parts yields the following theorem.

**Theorem 1.** If \( f \in C^m_0 \) then \( \Phi_\xi(f^{(m)}(x)) = (i\xi)^m \Phi_\xi(f(x)) \).

Now suppose we are given the Fourier transform of a function. In general, it is not obvious what the function is so the question rises of how we can retrieve our function. The following theorem answers this question.

**Theorem 2.** If \( f \in C^m_0 \), then for all real \( x \) we have \( f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi_\xi(f(x))e^{i\xi x}d\xi = \Phi^{-1}_\xi(\Phi_\xi(f(x))) \).

**Theorem 3.** (Plancherel)

Let \( f \in C^m_0 \). Then \( \int_{-\infty}^{+\infty} |f(x)|^2dx = \int_{-\infty}^{+\infty} |\Phi_\xi(f(x))|^2d\xi \).

Now we list some basic properties of \( \Phi_\xi(f) \). Assume that \( c \) is a real constant.

\[
\Phi_\xi(f(x-c)) = e^{-ic\xi}\Phi_\xi(f(x))
\]
\(\Phi_\xi(f(-x)) = f(-\xi)\) \hspace{1cm} (4)

\(\Phi_\xi(f(cx)) = \frac{1}{|c|} \Phi_\xi(\frac{\xi}{c}), \ c \neq 0\) \hspace{1cm} (5)

\(\Phi_\xi(e^{icx}f(x)) = \Phi_\xi(\xi - c)\) \hspace{1cm} (6)

2 Fourier series for functions on periodic intervals

Certain functions have the property of periodicity. A function \(f\) is said to be periodic with period \(L\) if \(f(x) = f(x + L)\). Perhaps the most familiar functions with this property are the sine and cosine functions. The next theorem gives us a nice way of thinking about periodic functions:

**Theorem 4. (Dirichlet’s Condition)**

i) \(f\) is periodic with period \(2L\) over \(\mathbb{R}\)

ii) \(f\) has at most, a finite number of local max and local min for \(-L \leq x \leq L\)

iii) \(f\) has at most a finite number of jump discontinuities over \([-L, L]\)

then \(f(x) \frac{f(x^+) + f(x^-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)]\), where \(a_n = \frac{1}{L} \int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx\) and \(b_n = \frac{1}{L} \int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\).

By squaring the function \(f(x)\) in the above theorem and its Fourier series representation and integrating both sides one can derive Parseval’s theorem. Note that this just the Plancherel version for Fourier series.

**Theorem 5.** With \(f\) as above we have that \(\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\).

In the more general case, suppose \(f\) is a function periodic in \([-\frac{L}{2}, \frac{L}{2}]\). If one uses the complex form for the sine and cosine function then the series looks like

\[f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \left(\frac{2\pi nx}{L}\right)}\] \hspace{1cm} (7)

where the coefficients are given by the expression

\[c_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-i \left(\frac{2\pi nx}{L}\right)} dx\] \hspace{1cm} (8)