Consider the wave equation in 2 spatial dimensions, \( x = (x_1, x_2) \)
\[
    u_{tt} - u_{x_1x_1} - u_{x_2x_2} = 0 ,
\]
with initial data
\[
    u|_{t=0} = e^{-100|x|^2} e^{ik(-x_1 + x_2^2)} ,
    \quad u_t|_{t=0} = \left[ ik \sqrt{1 + 4x_2^2} \right] e^{-100|x|^2} e^{ik(-x_1 + x_2^2)} ,
\]
so that the waves propagate in the positive \( x_1 \) direction.

1. Derive the Eikonal and transport equations for solutions of the form,
\[
    u = A(t, x)e^{ik\phi(t, x)} .
\]

2. Write a function in some programming language (Matlab would be the easiest using ode45) to solve the ODEs that define the bicharacteristics for the Eikonal equation. In the notation from class, the bicharacteristics are
\[
    (T(s), X(s), \tau(s), \xi(s)) ,
\]
where \( X = (X_1, X_2) \) and \( \xi = (\xi_1, \xi_2) \).
Your code should take as inputs, \( t, X(0) = (y_1, y_2) \) and \( \xi(0) = (\eta_1, \eta_2) \) and return \( (T(s_0), X(s_0), \tau(s_0), \xi(s_0)) \) for \( s_0 \) such that \( T(s_0) = t \). You can use the analytic solution to find this \( s_0 \).
Note: You do not need to numerically integrate the \( \tau \) equation, since \( \tau^2 = |\xi|^2 \) for all \( s \). Choose the root for \( \tau \) which gives you propagation in the positive \( x_1 \) direction.

3. Enlarge your ODE system to also compute \( \phi \) and its second derivatives on the bicharacteristic originating from \( (0, y_1, y_2, \eta_1, \eta_2) \). You will need initial conditions for \( \phi \) and all second derivatives involving \( x_1 \) and \( x_2 \). Remember that you can get derivatives involving \( t \) using derivatives of the Eikonal equation directly. Also, compute the amplitude on this bicharacteristic (it will need an initial value as well).

4. Consider a representation of the initial data as follows,
\[
    A(x)e^{ik\phi(x)} \approx \frac{k}{2\pi} \int_{\Omega} A(y)e^{ik(T(y)|\phi(x)+i|x-y|^2/2)} dy ,
\]
where,
\[
    A(x) = e^{-100|x|^2} \quad \phi(x) = -x_1 + x_2^2 .
\]
and $T^y_2[\phi](x)$ is the second order Taylor polynomial of $\phi$ about the point $y$ as a function of $x$. The domain $\Omega$ is the square $[-0.2, 0.2]^2$.

Now, looking at $T^y_2[\phi](x) + i|x-y|^2/2$ as the entire initial phase and $A(y)$ as the amplitude, decide what you need to send as input to your code from the previous part, so that you can calculate the phase, its derivatives, and the amplitude at a given time $t$ for the characteristics originating from $(y_1, y_2)$ at $t = 0$.

5. For a fixed $y$, fixed $t$, and the appropriate initial conditions from the previous part, calculate $\phi$, its first and second derivatives and the amplitude at $s_0$ (this is the same $s_0$ as before). Then form

$$
\psi(t, x; y) = \phi(s_0) + \nabla_x \phi(s_0) \cdot (x - y) + \frac{1}{2}(x - y) \cdot H_x \phi(s_0)(x - y)
$$

$$
A(t, x; y) = A(s_0),
$$

where $\nabla_x \phi = (\phi_{x_1}, \phi_{x_2})$ and $H_x \phi$ is the $2 \times 2$ Hessian matrix of $\phi$ containing its $x$ derivatives.

Finally, compute the wave field for one Gaussian beam for $k = 10^4$,

$$
v(t, x; y) = \frac{k}{2\pi} A(t, x; y) e^{ik\psi(t, x; y)}.
$$

Computationally, you will need to evaluate $v(t, x; y)$ on some grid: fix a value for $t$ and $y$ (say something like $t = .25$, $y = (0, 0)$, but you should be able to vary these values later) and create a mesh for $(x_1, x_2)$ (say on the rectangle $[-0.2, 1.2] \times [-0.2, 0.2]$), then evaluate $v(t, x; y)$ on this grid.

6. Finally, loop over your code in the last part to compute

$$
u(t, x) = \int_{\Omega} v(t, x; y) dy.
$$

As with $x$, you will need a grid on $\Omega$ to calculate this integral. Test your code by computing $u(t, x)$ at $t = 0$ and comparing the result to the initial condition for $u$. Make several plots showing $u$ (its real, and absolute values) at $t = 0, 0.25, 0.50, 0.75, 1.00$. You may find it useful to look at the article “Superpositions and higher order Gaussian beams” available at http://www.intlpress.com/CMS/2008/issue6-2/ and more specifically, sections 3.1 and 3.4.