Introduction to Wavelet Based Numerical Homogenization

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Suppose

\[ L_{j+1} u = f, \quad L_{j+1} \in \mathcal{L}(V_{j+1}, V_{j+1}) \quad u, f \in V_{j+1}, \]

is a discretization (e.g. FD, FEM) of a differential equation on scale-level \( j + 1 \) where \( L_{j+1} \) contains small scales.

Want to find an effective discrete operator \( \bar{L}_{j'} \), with \( j' \ll j \) that computes the coarse part of \( u \).

C.f. classical homogenization.
Wavelet based numerical homogenization
[Beylkin, Brewster, Engquist, Dorobantu, Levy, Gilbert, O.R., ...]

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**Example (Elliptic eq, Haar)**

\[
\partial_x r(x/\varepsilon) \partial_x u_\varepsilon = f, \quad \Rightarrow \quad L_{j+1} = \frac{1}{h^2} \Delta + R^\varepsilon \Delta_-. 
\]

where \( R^\varepsilon \) is diagonal matrix sampling \( r(x/\varepsilon) \), and \( 2^j \sim 1/\varepsilon \). Here one could use

\[
\bar{L}_{j'} = \frac{1}{h^2} \Delta + \bar{R} \Delta_.
\]
Wavelet transforms

Simple to extract the coarse and fine part of $u = \{u_k\}$:

$$\mathcal{W} u = \begin{pmatrix} U_f \\ U_c \end{pmatrix}, \quad u \in V_{j+1}, \quad U_f \in W_j, \quad U_c \in V_j.$$

For compactly supported wavelets, $\mathcal{W}$ is sparse. It is also orthonormal, $\mathcal{W}^T \mathcal{W} = I$.

In Haar basis on $[0, 1]$,

$$\mathcal{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 1 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & 0 & 1 & -1 \\ 1 & 1 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix} \in \mathbb{R}^{2^{j+1} \times 2^{j+1}}.$$
Wavelet based numerical homogenization

Wavelet decomposition of operator

Start from equation

\[ L_{j+1} u = f, \quad L_{j+1} \in \mathcal{L}(V_{j+1}, V_{j+1}) \quad u, f \in V_{j+1}. \]

Decompose equation in coarse and fine part (use \( \mathcal{W}^T \mathcal{W} = I \))

\[ \mathcal{W} L_{j+1} \mathcal{W}^T \mathcal{W} u = \mathcal{W} f \quad \Rightarrow \]

\[
\begin{pmatrix}
A_j & B_j \\
C_j & L_j
\end{pmatrix}
\begin{pmatrix}
U^f \\
U^c
\end{pmatrix}
= \begin{pmatrix}
F^f \\
F^c
\end{pmatrix}.
\]
Wavelet based numerical homogenization
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Decompose equation in coarse and fine part (use \( \mathcal{W}^T \mathcal{W} = I \))

\[ \mathcal{W} L_{j+1} \mathcal{W}^T \mathcal{W} u = \mathcal{W} f \quad \Rightarrow \]

\[ \begin{pmatrix} A_j & B_j \\ C_j & L_j \end{pmatrix} \begin{pmatrix} U^f \\ U^c \end{pmatrix} = \begin{pmatrix} F^f \\ F^c \end{pmatrix}. \]

Eliminate \( U^f \),

\[ (L_j - C_j A_j^{-1} B_j) U^c = F^c - C_j A_j^{-1} F^f. \]
Start from equation

$$L_{j+1} u = f, \quad L_{j+1} \in \mathcal{L}(V_{j+1}, V_{j+1}) \quad u, f \in V_{j+1}. $$

Decompose equation in coarse and fine part (use $\mathcal{W}^T \mathcal{W} = I$)

$$\mathcal{W}L_{j+1}\mathcal{W}^T \mathcal{W} u = \mathcal{W} f \quad \Rightarrow$$

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\begin{pmatrix}
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\end{pmatrix}
\begin{pmatrix}
U^f \\
U^c
\end{pmatrix}
= 
\begin{pmatrix}
F^f \\
F^c
\end{pmatrix}.
$$

Eliminate $U^f$,

$$(L_j - C_j A_j^{-1} B_j) U^c = F^c - C_j A_j^{-1} F^f. $$

Supposing $f$ smooth so $F^f = 0$ and $F^c = f$.

$$(L_j - C_j A_j^{-1} B_j) U^c = f.$$
We call the matrix

\[ \bar{L}_j = L_j - C_j A_j^{-1} B_j, \quad \bar{L}_j \in \mathcal{L}(V_j, V_j), \]

the (numerically) homogenized operator. Since
- Half the size of original \( L_{j+1} \).
- Given \( \bar{L}_j, f \) we can solve for coarse part of solution, \( U^c \).
- Takes influence of fine scales into account.

Compare with classical homogenization:

\[ L = \nabla R(x / \varepsilon) \nabla \ \Rightarrow \ \bar{L} = \nabla \int R(x) dx \nabla - \nabla \int R(x) \frac{\partial \chi}{\partial x} dx \nabla \]

where \( \chi \) solves the (elliptic) cell problem.
Reduction can be repeated,

\[ L_j \rightarrow L_{j-1} \rightarrow L_{j-2} \rightarrow \ldots, \quad L_j \in \mathcal{L}(V_j, V_j), \]

to discard suitably many small scales / to get a suitably coarse grid. Also, condition number improves

\[ \kappa(L_j) < \kappa(L_{j+1}). \]
Problem: $L$ sparse (banded) $\not\rightarrow \bar{L}$ sparse (banded). (Must invert $A_j$.)

However: Approximation properties of wavelets imply elements of $A_j^{-1}$ decay rapidly away from diagonal.

Therefore: $\bar{L}$ diagonally dominant in many important cases and can be well approximated by a banded matrix. (Cf. a (local) differential operator.)
Different Approximation Strategies

1. “Crude” truncation to $\nu$ diagonals,

2. Band projection to $\nu$ diagonals, defined by

   $$Mx = \text{band}(M, \nu)x, \quad \forall x \in \text{span}\{v_1, v_2, \ldots, v_\nu\}.$$  

   $$v_j = \{1^{j-1}, 2^{j-1}, \ldots, N^{j-1}\}^T, \quad j = 1, \ldots, \nu.$$  

   C.f. “probing”, [Chan, Mathew], [Axelsson, Pohlman, Wittum].

3. The above methods used on the matrix $H$ instead, where e.g.

   $$L_{j+1} = \frac{1}{h^2} \Delta \big|_+ R \Delta \big|_- \quad \Rightarrow \quad \bar{L}_j = \frac{1}{(2h)^2} \Delta \big|_+ H \Delta \big|_-.$$  

   $H$ can be seen as the effective material coefficient.

4. The above methods used on the matrix $A_j^{-1}$ instead, where

   $$\bar{L}_j = L_j - C_j A_j^{-1} B_j.$$

   [Levy, Chertock]
Elliptic 1D case

Consider the elliptic one-dimensional problem

$$\partial_x a^\varepsilon(x) \partial_x u = 1, \quad u(0) = u'(1) = 0,$$

with standard second order discretization.

Try two cases:

$$a^\varepsilon(x) = "\text{noise}"$$

$$a^\varepsilon(x) = "\text{narrow slit}"$$
Elliptic 1D case – noise
Different approximation strategies

- Exact
  - $\nu = 13$
  - $\nu = 15$
  - $\nu = 17$

- $\text{trunc}(L, \nu)$

- Exact
  - $\nu = 3$
  - $\nu = 5$
  - $\nu = 7$

- $\text{trunc}(H, \nu)$

- $\text{band}(L, \nu)$

- Exact
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Elliptic 1D case – narrow slit
Different approximation strategies

- $\text{trunc}(L, \nu)$
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- $\text{trunc}(H, \nu)$
  - Exact
  - $\nu=3$
  - $\nu=5$
  - $\nu=7$

- $\text{band}(L, \nu)$
  - Exact
  - $\nu=7$
  - $\nu=9$
  - $\nu=11$

- $\text{band}(H, \nu)$
  - Exact
  - $\nu=1$
  - $\nu=3$
  - $\nu=5$
Elliptic 1D case – narrow slit
Matrix element size
Simulate a wave hitting a wall with a small opening modeled by Helmholtz

\[ \nabla a(x, y) \nabla u + \omega^2 u = 0, \]
Examples

Helmholtz 2D case
Examples

Helmholtz 2D case

Untruncated operator

v=5

v=7

v=9
Helmholtz 2D case
Matrix element size

\[ nz = 19026 \]