ODE Tutorial RTG summer program 7/21/08

Start w/ example

\[ y' = 3y \quad (1) \]
\[ y(0) = c \quad (2) \]

Solution?

\[ y(t) = ce^{3t} \]

family of integral curves (picture*)

In general

\[ y' = f(t, y) \]
\[ y(t_0) = y_0 \]

\[ f : [a, b] \times E \rightarrow \mathbb{R}, t, y \in [a, b], y_0 \in E, E \subseteq \mathbb{R}^n \text{ open?} \]

Looking for \( y : [t_0, T] \rightarrow E \).

Equivalent to (by fundamental calc)

\[ y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) \, ds \]
ordinary $\iff$ derivative/dependence on variable
first order $\iff$ first derivative

3 important questions

(1) local existence: Does our equation (or system) have a solution $y(t)$ defined near $t_0$?

(2) existence in the large (e.g. global existence) on what $t$-ranges do we have a solution? (this may depend on $y_0$)

(3) uniqueness of solutions.

Examples

\begin{align*}
y' &= y^2 \quad y(0) = c \ (> 0) \\
y(t) &= \frac{c}{1 - ct} \quad \text{solution?} \quad \checkmark \\
\end{align*}

only exists for $-\infty < t < \frac{1}{c}$. Domain of sol. can depend on i.e.
2) \[ y' = \sqrt{y} \quad y(0) = 0 \]

\[ y(t) = \begin{cases} 0 & \text{if } t < c \\ \frac{(t-a)^2}{4} & \text{if } t \geq c \end{cases} \]

And for any \( c > 0 \)

Check it: nonuniqueness, graph.

**Picard's Thm**

Let \( f: [t_0, t_0 + a] \times \mathbb{R} \rightarrow \mathbb{R} \) cont. \( E = \{ y' \mid y'(t_0) = f(t_0, y_0) \} \) and uniformly Lipschitz cont wrt \( y \). Let \( M \) be a bound for \( |f(t, y)| \) on \( \mathbb{R} \), \( a = \min(a, \frac{b}{M}) \).

Then

\[ y' = f(t, y) \quad y(t_0) = y_0 \]

has a unique sol \( y = y(t) \) on \([t_0, t_0 + a] \).

**Def** We say \( f \) is Lipschitz if \( \exists K \) s.t.

\[ |f(x) - f(y)| \leq K|x-y| \quad \text{for all } x, y \in \text{dom } f. \]
(Sketch of proof)

\[ y_0(t) := y_0 \]
\[ y_1(t) := y_0 + \int_{t_0}^{t} f(s, y_0) \, ds \]
\[ y_n(t) := y_0 + \int_{t_0}^{t} f(s, y_n(s)) \, ds. \]

Assume \( \lim_{n \to \infty} y_n(t) \) exists. \( y(t) := \lim_{n \to \infty} y_n(t) \)

\[ \lim_{n \to \infty} y_n(t) = y_0 + \lim_{n \to \infty} \int_{t_0}^{t} f(s, y_n(s)) \, ds. \]

\[ y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) \, ds. \] bounded.

How do we know \( \lim_{n \to \infty} y_n(t) \) exists? Analysis requires Lipschitz assumption.

Note what do we mean by

\[ \lim_{n \to \infty} y_n(t) = y(t) \]

Fix \( t^* \), \( y_n(t^*) = y(t^*) \) "pointwise convergence"

In some function space norm.
\[ \|y\|_{L^\infty} = \sup_{t \in [a, b]} |y(t)| \]
\[ \|y\|_{L^1} = \int_a^b |y(t)| \, dt \]
\[ \|y\|_{L^2} = \left( \int_a^b |y(t)|^2 \, dt \right)^{1/2} \]

\[ \lim_{n \to \infty} y_n(t) = y(t) \text{ in } \| \cdot \| \text{ "strong convergence"} \]

\[ \lim_{n \to \infty} \|y_n - y\|_* = 0 \text{ in real \#s sense.} \]

**Examples**

\[ y_n(t) = t^n \text{ on } [0, 1] \]

Pointwise limit \[ f(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } t = 1 \end{cases} \]

Strong limit \[ \| \cdot \|_1 \text{ is } g(t) = 0 \]

\[ \|y_n - g\|_{L^1} = \int_0^1 |y_n(t) - g(t)| \, dt \]
\[ = \int_0^1 t^n - 0 \, dt \]
\[ = \left( \frac{1}{n+1} t^{n+1} - 0 \right) \bigg|_0^1 = \frac{1}{n+1} - 0 \to 0 \]
Strong limit \( \| \cdot \|_\infty \) DNE

\[ \| y_n - f \|_\infty = \sup_{t \in [0,1]} |y_n(t) - f(t)| = \]
\[ = \sup_{t \in [0,1]} |t^n| = 1 \]

OR

\[ \| y_n - g \|_\infty = \sup_{t \in [0,1]} |t^n - 1| = 1. \]

Little o and different norms work differently \( \Rightarrow \) Big O.

In Picard we are talking strong limit in \( L_\infty \) norm.

Uniqueness? Suppose \( z(t) \) exists s.t.

\[ z(t) = y_0 + \int_{t_0}^{t} f(s, z(s)) \, ds. \]

we show \( y_n(t) \to z(t) \) uniformly
Systems of Equations

\[ \begin{align*}
  y'_1 &= F_1(t, y_1, \ldots, y_n), \quad y_1(0) = y_1^0 \\
  y'_n &= F_n(t, y_1, \ldots, y_n), \quad y_n(0) = y_n^0
\end{align*} \]

If \( F_1, \ldots, F_n \) and partials \( \frac{\partial F_i}{\partial y_j} \) cont. in a region \( R \) of \( \mathbb{R}^N \) and \((t_0, y_1^0, \ldots, y_n^0) \in R\),

then there is an interval \((t_0-h, t_0+h)\) in which there exists a unique sol. to the system (\(*\)).

**Note on superposition**

\[ \begin{align*}
  y' &= f(t, y) \quad \text{where } f \text{ linear in } y \\
  z' &= g(t, z)
\end{align*} \]

\[ \begin{align*}
  (\alpha y + \beta z)' &= \alpha y' + \beta z' = \alpha f(t, y) + \beta g(t, z) \\
                             &= f(t, \alpha y + \beta z)
\end{align*} \]

Linear combos of sols. to linear eqns. are also sols. to the eqn.

**Example**

\[ y' - p(t) y = 0 \]

Sols. of form \[ y(t) = C \exp \left\{ \int_0^t p(s) \, ds \right\} \]
Gronwall's (if there is time)

Suppose we know that for some c (t) function \( \beta(t) \)

\[ u'(t) \leq \beta(t)u(t) \quad t > 0 \]

then

\[ u(t) \leq u(0) \exp \int_0^t \beta(s) \, ds \quad \text{for all } t > 0. \]

\[ \frac{\text{d}}{\text{d}t} \left( \frac{u(t)}{v(t)} \right) = \frac{u' v - v' u}{v^2} \leq \frac{\beta u v - \beta v u}{v^2} = 0 \]

\[ \frac{u(t)}{v(t)} \leq \frac{u(0)}{v(0)} = u(0) \]

\[ u(t) \leq u(0) v(t) = u(0) \exp \int_0^t \beta(s) \, ds \]

Add a bit on Euler scheme + graph done!

(+ pictures)
1st and 2nd order linear ODEs constant coefficients

\[ ay'' + by' + cy = f(x) \]

characteristic eqn. (solve homogeneous eqn 1st)

\[ ar^2 + br + c = 0 \]

\[ r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Solutions \( y_1(t) = e^{r_1t} \), \( y_2(t) = e^{r_2t} \)

\[ ay''(t) + by'(t) + cy = 0 \]

\[ = ar^2 e^{r_1t} + br e^{r_1t} + ce^{r_1t} = (ar^2 + br_1 + c)e^{r_1t} = 0 \]

complex roots

Double roots

complex roots \( \leftrightarrow \) oscillations!

\[ y_1 = e^{(\lambda - \mu)t} \quad y_2 = e^{(\lambda + \mu)t} \]

\[ e^{(\lambda - \mu)t} + e^{(\lambda + \mu)t} = e^{\lambda t}(\cos\mu t + i \sin\mu t) + e^{\lambda t}(\cos\mu t - i \sin\mu t) = 2e^{\lambda t} \cos\mu t \]

\[ e^{(\lambda + \mu)t} - e^{(\lambda - \mu)t} = 2i e^{\lambda t} \sin\mu t \quad \text{similarly} \]
Summary of 2\textsuperscript{nd} order linear ODE

\[ y'' + p(t)y' + q(t)y = g(t) \] \text{ inhomogeneous}
\[ y'' + p(t)y' + q(t)y = 0 \] \text{ homogeneous}

- homogeneous, \( p, q \) constants \( \Rightarrow \) we can solve it.
- solutions to 2\textsuperscript{nd} order ODE 2 \text{ linearly independent} \text{ fundamental solutions}
sols to homogeneous eqn + particular sol to inhomogeneous problem describes the sol space.

homogeneous problem,
\[ \text{if we have one fundamental sol, we can find another (linearly independent) one. Called Reduction of Order} \]

\[ \rightarrow \text{ Obtaining particular solutions to inhomogeneous problem.} \]
\[ \text{Method of Undetermined Coefficients} \]
\[ \text{clever guessing for constant coefficient case.} \]

- \text{Variation of Parameters}
\[ \text{must know find sols., to find a particular solution.} \]
First order linear ODE

\[ y' + p(t) y = g(t) \]

Choose integrating factor \( \mu(t) \) s.t.

\[ \mu(t) p(t) y = \mu'(t) y \quad \text{WHY?} \]

\[ \mu(t) y' + \mu(t) p(t) y = \mu(t) g(t) \]

\[ \mu(t) y' + \mu'(t) y = \mu(t) g(t) \]

\[ (\mu(t) y)' = \mu(t) g(t) \]

\[ \mu(t) y = \int^{t}_{0} \mu(s) g(s) \, ds + c \]

\[ y = \frac{1}{\mu(t)} \int^{t}_{0} \mu(s) g(s) \, ds + c \]

What's \( \mu(t) \)

\[ \mu(t) p(t) = \mu'(t) \]

\[ \frac{\mu'(t)}{\mu(t)} = p(t) \]

\[ \ln(\mu(t)) = \int^{t}_{0} p(s) \, ds \]

\[ \mu(t) = \exp \left\{ \int^{t}_{0} p(s) \, ds \right\} \]