Problem 1. Let $W$ be a Brownian motion under $\mathbb{P}$, set $M_t = 1 + W_t$, and let $\tau = \inf\{t : M_t \leq 0\}$ denote the first time that $M$ hits 0. Now fix some $T > 0$, and set $Z = M_T \wedge \tau$.

1. Show that $\mathbb{E}^\mathbb{P}[Z] = 1$.
2. Now define $Q(A) = \mathbb{E}^\mathbb{P}(\mathbb{1}_A Z)$. Compute $Q(\tau \leq T)$?
3. Are $\mathbb{P}$ and $Q$ equivalent?

Solution:

1. Set $N_t = M_t \wedge \tau$. Then $N_t$ is a martingale by the optional stopping theorem, so
$$\mathbb{E}^\mathbb{P}[Z] = \mathbb{E}^\mathbb{P}[N_T] = \mathbb{E}^\mathbb{P}[\mathbb{E}^\mathbb{P}[N_T | F_0]] = \mathbb{E}^\mathbb{P}[N_0] = 1$$

2. We have
$$Q(\tau \leq T) = \mathbb{E}^\mathbb{P}[\mathbb{1}_{\{\tau \leq T\}} M_T \wedge \tau]$$
$$= \mathbb{E}^\mathbb{P}[\mathbb{1}_{\{\tau \leq T\}} M_\tau] = 0$$

3. No. $Q(\tau \leq T) = 0$, but $\mathbb{P}(\tau \leq T) > 0$

Problem 2. Let $W$ be a Brownian motion under $\mathbb{P}$, let $Z$ solve $dZ_t = -\lambda_t Z_t dW_t$, and set $\hat{W}_t = W_t + \int_0^t \lambda_s ds$.

1. Compute $d(1/Z)_t$ to show that process $1/Z$ can be written as a stochastic integral against the process $\hat{W}_t$.
2. Suppose that $dM_t = H_t dW_t$. Compute $d(M/Z)_t$ to show that $M/Z$ can be written as a stochastic integral against the process $\hat{W}_t$.
3. Let $(F_t)$ denote the filtration generated by $W$, and let $Q$ denote the measure with $dQ/d\mathbb{P} = Z_T$. If all processes are adapted to $F$, argue that every $Q$-martingale can be written as a stochastic integral against $\hat{W}_t$ over the time interval $[0, T]$.

Solutions:

1. We have $d(1/Z)_t = (\lambda_t/Z_t) d\hat{W}_t$.
2. We have $d(M/Z)_t = \frac{H_t + \lambda_t M_t}{Z_t} d\hat{W}_t$.
3. Let $\hat{M}$ be a $Q$ martingale. Then $MZ$ is a $\mathbb{P}$ martingale and we can find $H$ such that $M_t Z_t = C + \int_0^t H_s ds$. The previous part then show that we can write $M = MZ/Z$ as a stochastic integral against $\hat{W}$.
Problem 3 (Brownian Representation). Let \((W_t)\) be a Brownian motion (with \(W_0 = 0\)) and let \((\mathcal{F}_t)\) denote the filtration generated by \(W\). Now fix some \(T > 0\), and set \(X_t = \mathbb{E}[W_T^3 \mid \mathcal{F}_t]\), so \(X_t\) is a martingale. Since \(W\) has the predictable representation property, there exists a constant \(C\) and an integrand \(H\) such that \(X_t = C + \int_0^t H_u dW_u\). We would like to find \(C\) and \(H\). One approach is outlined below:

1. Use Itô’s lemma to write \(W_T^3\) in the form
   \[
   W_T^3 = W_t^3 + \int_t^T A_u du + \int_t^T B_u dW_u
   \]
   for appropriate \(A\) and \(B\).

2. Use the previous expression to compute \(\mathbb{E}[W_T^3 \mid \mathcal{F}_t]\) explicitly. You may pull integrals over time out of conditional expectations (which are essentially integrals over \(\Omega\)). This should be enough to determine \(C\).

3. Set \(f(t, W_t) = \mathbb{E}_t[W_T^3 | \mathcal{F}_t]\) and use the explicit formula above and compute \(df(t, W_t)\) to find \(H\).

Solution:

1. We have \(A_t = 3W_t\) and \(B_t = 3W_t^2\).

2. We notice that
   \[
   \mathbb{E}[W_T^3 \mid \mathcal{F}_t] = W_t^3 + \mathbb{E}\left[\int_t^T 3W_u du + \int_t^T 3W_u^3 du \mid \mathcal{F}_t\right]
   = W_t^3 + \int_t^T 3 \mathbb{E}[W_u \mid \mathcal{F}_t] du
   = W_t^3 + \int_t^T 3W_t du = W_t^3 + 3(T-t)W_t.
   \]
   This means that \(C = \mathbb{E}[W_T^3] = 0\).

3. With \(f(t, x) = x^3 + 3(T-t)x\), we have \(df(t, W_t) = 3(W_t^2 + (T-t))dW_t\), so \(H_t = 3(W_t^2 + (T-t))\).