Problem 2.1. Prove the fundamental theorem of asset pricing.

(Hint. Use the fact that \( \langle W \rangle \) is a convex set which does not intersect the (convex and compact) unit simplex \( S^n = \{(c_1, \ldots, c_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n c_i = 1\} \).

Problem 2.2. Show that

\[
\dim(\langle W \rangle) = \sum_{\xi \in \mathcal{N}^-} \text{rank}(D(\xi^+) + q(\xi^+)),
\]

where \( \mathcal{N}^- \) denotes the set of all non-terminal nodes and \( D(\xi^+) + q(\xi^+) \) is a \( b(\xi) \times J \)-matrix, whose rows correspond to all children of \( \xi \), and columns to all contracts.

(Hint: Construct the matrix \( \tilde{W} \) be multiplying each row of \( W \) by \( \pi(\xi) \), where \( \xi \) is the node corresponding to that row and \( \pi \) is a present-value price process. \( \tilde{W} \) has the same rank as \( W \) and you can use the “martingale” property \( \pi(\xi)q_j(\xi) = \sum_{\xi' > \xi} \pi(\xi')(Dj(\xi') + q_j(\xi')) \) to perform appropriate operations on its rows to make it block-diagonal.)

Problem 2.3. Consider a market where \( \Omega = \{\omega_1, \omega_2, \omega_3\} \), \( T = 1 \), \( A_0 = \{\Omega, \emptyset\} \) and \( A_1 = \mathcal{P}(\Omega) \) (where \( \mathcal{P}(X) \) denotes the power set of \( X \), i.e., the set of all subsets of \( X \).) Moreover, there are two contracts, both issued at \( \xi_0 \), with dividend processes

\[
D^1(\xi_1) = 10, \quad D^1(\xi_2) = -20, \quad D^1(\xi_3) = 60,
\]

and

\[
D^2(\xi_1) = 20, \quad D^2(\xi_2) = 30, \quad D^2(\xi_3) = 10.
\]

(1) Characterize the set \( Q \) of all price-processes \( q = (q^1, q^2) \), such that there is no arbitrage.

(2) For each price process \( q \in P \), characterize the set of all vectors of present-value prices \( \Pi(q) \).

(3) Pick a vector \( q \notin P \), and construct an arbitrage portfolio, i.e., a portfolio process (a pair of numbers, really) \( z \) such that \( Wz \geq 0, Wz \neq 0 \).