Market environments, stability and equilibria

Gordan Žitković

Department of Mathematics
University of Texas at Austin

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A Toy Model

The Information Flow

- two states of the world: $\Omega = \{\omega_1, \omega_2\}$
- one period $t \in \{0, 1\}$
- nothing is known at $t = 0$, everything is known at $t = 1$: $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = 2^\Omega$.

Agents

two economic agents characterized by

- random endowments (stochastic income)

$$\mathcal{E}^1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathcal{E}^2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

- utility functions

$$U^1(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \frac{1}{2} \log(x_1) + \frac{1}{2} \log(x_2)$$

$$U^2(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \frac{1}{7} x_1^{1/3} + \frac{6}{7} x_2^{1/3}$$
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THE FINANCIAL INSTRUMENT

\[ S_0 = p, \ S_1 = \begin{cases} 1 \\ 0 \end{cases}, \ B_0 = B_1 = 1: \]

\[ \Delta (p) = \arg\max_q U^i (E^i + q(S_1 - p)) \]

Market clearing

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  \[ \Delta^1(p) + \Delta^2(p) = 0 \]
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![Diagram with nodes labeled 0, 1, and p connected by arrows]

MARKET CLEARING

- The demand functions:
  \[ \Delta^i(p) = \arg\max_q U^i(\mathcal{E}^i + q(S_1 - p)) \]

- Equilibrium conditions:
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What happens when markets are incomplete and trading is dynamic?

- Instead of one price $p^*$, we need to determine the whole price process $(p_0, (p_1, p_2, p_3))$.

- In the IC&mp case, the equilibrium conditions determine both prices and the geometry (degree of incompleteness) of the market.

- Another complication: no representative-agent analysis. The First Welfare Theorem does not hold anymore.
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Financial Frameworks

Information
A filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})\), where \(\mathbb{P}\) is used only to determine the null-sets.

Agents
A number \(I\) (finite or infinite) of economic agents, each of which is characterized by
- a random endowment \(E^i \in \mathcal{F}_T\),
- a utility function \(U : \text{Dom}(U) \to \mathbb{R}\),
- a subjective probability measure \(\mathbb{P}^i \sim \mathbb{P}\).

Completeness Constraints
A set \(S\) of \(\{\mathcal{F}_t\}_{t \in [0,T]}\)-semimartingales (possibly several-dimensional) - the allowed asset-price dynamics.
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**THE EQUILIBRIUM PROBLEM**

**Problem**

Does there exist \( S \in S \) such that

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\sum_{i \in I} \hat{\pi}_t^i(S) = 0, \quad \text{for all } t \in [0, T], \ \text{a.s},
\]

where

\[
\hat{\pi}_t^i(S) = \arg\max_{\pi} \mathbb{E}^{\pi^i} \left[ U^i \left( \mathcal{E}^i + \int_0^T \pi_u \, dS_u \right) \right]
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denotes the optimal trading strategy for the agent \( i \) when the market dynamics is given by \( S \).

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If such an \( S \) exists, is it unique?

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The equilibrium problem

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Examples of Completeness Constraints

- **Complete markets.** $S$ contains all $\{\mathcal{F}_t\}_{t \in [0, T]}$-semimartingales. (If an equilibrium exists, a complete one will exist).

- **Constraints on the number of assets.** $S$ is the set of all $d$-dimensional $\{\mathcal{F}_t\}_{t \in [0, T]}$-semimartingales. If $d < n$, where $n$ is the spanning number of the filtration, no complete markets are allowed.

- **Information-constrained markets.** Let $\{\mathcal{G}_t\}_{t \in [0, T]}$ be a sub-filtration of $\{\mathcal{F}_t\}_{t \in [0, T]}$, and let $S$ be the class of all $\{\mathcal{G}_t\}_{t \in [0, T]}$-semimartingales.

- **Partial-equilibrium models.** Let $\{S^0_t\}_{t \in [0, T]}$ be a $d$-dimensional semimartingale. $S$ is the collection of all $m$-dimensional $\{\mathcal{F}_t\}_{t \in [0, T]}$-semimartingales such that its first $d < m$ components coincide with $S^0$.

- **“Marketed-Set Constrained” markets** Let $V$ be a subspace of $L^\infty(\mathcal{F}_T)$, satisfying an appropriate set of regularity conditions. Let $S$ be the collection of all finite dimensional semimartingales $\{S_t\}_{t \in [0, T]}$ such that

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\{x + \int_0^T \pi_t dS_t : x \in \mathbb{R}, \pi \in \mathcal{A}\} \cap L^\infty(\mathcal{F}_T) = V,
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- **Markets with “fast-and-slow” information.** Let $\{F_t\}_{t \in [0,T]}$ be generated by two orthogonal martingales $M^1$ and $M^2$, and let $S$ be the collection of all processes of the form

$$S_t = D_t + M^1_t,$$

where $D$ is any predictable process of finite variation. For example, $M^1 = B$ (Brownian motion), $M^2 = N_t - t$ (compensated Poisson process) so that a “typical” element of $S$ is given by

$$S_t = \int \lambda(u, B_u, N_u) \, du + dB_u.$$

The information in $B$ is “fast”, and that in $N$ is “slow”.

Another interesting situation: $M^1 = B$, $M^2 = W$, where $B$ and $W$ independent Brownian motions.
Two paths to existence

- **Representative agents.** Uses the fact that equilibrium allocations are Pareto optimal; works (essentially) only for complete markets.

  Literature in continuous time:
  - **Complete markets:** Bank, Dana, Duffie, Huang, Karatzas, Lakner, Lehoczky, Riedel, Shreve, Ž., etc.
  - **Incomplete markets:** Basak and Cuoco ’98 (incompleteness from restrictions in stock-market participation, logarithmic utility)

- **Excess-demand approach.** Introduced by Walras (1874):
  1. Establish good topological/convexity properties of the excess demand \( \hat{\pi}(S) \), and then
  2. use a suitable fixed-point-type theorem to show existence (Brouwer, KKM, degree-based, etc.)

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A convex-analytic (sub)approach

A first step towards a solution
Work with random variables instead of processes; for example in the fast-and-slow model with

\[ dS_u^\lambda = \lambda_u \, du + dB_u, \]

we perform the following transformations

\[ \pi \mapsto X_T^{\lambda, \pi} = \int_0^T \pi_u \, dS_u^\lambda, \quad \lambda \mapsto Z_T^\lambda = \mathcal{E}(-\lambda \cdot M), \]

and consider a more tractable version \( \Delta^i \) of the demand function

\[ \Delta^i(Z_T^\lambda) = X_T^{\lambda, \hat{\pi}^i(S^\lambda)}, \]

so that

\[ \Delta^i : E_M \subseteq L^0_+ \to L^0_+ - L^\infty_+. \]

The problem now becomes simple to state:

*Can we solve the equation \( \Delta(Z) = 0 \), a.s. on \( E_M \)?
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Stability of utility maximization in incomplete markets

(Note: fix an agent and drop the index $i$.)

**Theorem** (Larsen and Ž. (2006), to appear in SPA)

Suppose that $\mathcal{E} \equiv 0$. Let $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \Lambda$ be a sequence such that

- $Z^{\lambda_n}$ is a martingale for each $n$,
- the collection $\{V^+(Z^{\lambda_n}_T) : n \in \mathbb{N}\}$ is uniformly integrable, and
- $Z^{\lambda_n}_T \to Z^\lambda_T$ in probability.

Then, for $x_n \to x > 0$ we have

$$u^{\lambda_n}(x_n) \to u^\lambda(x), \text{ and } \hat{X}^{x,\lambda}_T \to \hat{X}^{\lambda,x}_T \text{ in probability.}$$

Here, $V$ is the convex conjugate of the utility function $U$, i.e., $V(y) = \sup_{x>0} (U(x) - xy)$, and $\hat{X}^{x,\lambda}_T$ is the optimal terminal wealth in the market $S^\lambda$ with initial wealth $x$.

**Remarks:**

- The uniform-integrability condition is practically necessary.
- Completes the Hadamard-style analysis of the utility maximization problem - repercussions for estimation.
- Further generalized to the general semimartingale case - under a different perturbation family - including general $\mathcal{E} \in L^\infty$ (Kardaras and Ž. (2007)).
- Therefore, (under suitable conditions) $\Delta$ is $({\mathbb{L}^0, {\mathbb{L}^0})$-continuous.
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- The uniform-integrability condition is practically necessary.
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STABILITY OF UTILITY MAXIMIZATION IN INCOMPLETE MARKETS
(Note: fix an agent and drop the index \(i\).)

**Theorem** (Larsen and Ž. (2006), to appear in SPA)
Suppose that \(\mathcal{E} \equiv 0\). Let \(\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \Lambda\) be a sequence such that

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Some fixed-point theory

The KKM-theorem

Theorem (Knaster, Kuratowski and Mazurkiewicz, 1929) Let $S$ be the unit simplex in $\mathbb{R}^m$, and let $V = \{e_1, \ldots, e_m\}$ be the set of its vertices. A mapping $F : V \to 2^{\mathbb{R}^m}$ is said to be a KKM-map if

$$\text{conv}(e_i, i \in J) \subseteq \bigcup_{i \in J} F(e_i), \ \forall J \subseteq \{1, \ldots, m\}.$$ 

If $F(e_i)$ is a closed subset of $\mathbb{R}^m$ for all $i \in \{1, \ldots, m\}$, then

$$\bigcap_{i \in \{1,\ldots,n\}} F(e_i) \neq \emptyset.$$
The KKM-Theorem can easily be extended to infinite-dimensional vector-spaces as long as mild topological properties are imposed and \textbf{local convexity} is required (Kakutani, Fan, Browder, etc.)

How about \( L^0 \) - the prime example of a non-locally-convex space? Yes, if one can fake compactness there:

\textbf{CONVEX-COMPACTNESS}
(Nikišin, Buhvalov, Lozanovskii, Delbaen, Schahermayer, etc.)

A subset \( B \) of a topological vector space is said to be convex-compact if any family \( (F_\alpha)_{\alpha \in A} \) of closed and convex subsets of \( B \) has the finite-intersection property, i.e.

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\left( \forall D \subseteq_{fin} A \quad \bigcap_{\alpha \in D} F_\alpha \neq 0 \right) \Rightarrow \bigcap_{\alpha \in A} F_\alpha \neq \emptyset.
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A CHARACTERIZATION

Proposition. A closed and convex subset $C$ of a topological vector space $X$ is convex-compact if and only if for any net $(x_{\alpha})_{\alpha \in A}$ in $C$ there exists a subnet $(y_{\beta})_{\beta \in B}$ of convex combinations of $(x_{\alpha})_{\alpha \in A}$ such that $y_{\beta} \to y$ for some $y \in C$.

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Examples.

• Any convex and compact subset of a TVS is convex-compact.
• A closed and convex subset of a unit ball in a dual $X^*$ of a normed vector space $X$ is convex-compact under any compatible topology (essentially Mazur),
• Any convex, closed and bounded-in-probability subset of $\mathbb{L}^0_+$ is convex-compact (essentially Komlós).
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ATTAINMENT OF MINIMA

Theorem. Let $A$ be a convex-compact subset of $X$, and let $f : A \rightarrow \mathbb{R}$ be a convex lower-semicontinuous function. Then $f$ attains its minimum on $A$.

A MINIMAX-TYPE THEOREM

Theorem. Let $A, B$ be a convex-compact subsets of TVS $X$ and $Y$, respectively. Let $f : A \times B \rightarrow \mathbb{R}$ be a function with the following properties:

- $x \mapsto f(x, y)$ is usc and (quasi)-concave for each $y \in B$,
- $y \mapsto f(x, y)$ is lsc and (quasi)-convex for each $x \in A$.

Then

$$\max_x \min_y f(x, y) = \min_y \max_x f(x, y).$$
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**Generalized KKM theorem**

**Theorem.** Let $A$ be convex-compact subset of a TVS $X$. Let $\{F(x)\}_{x \in A}$ be a family of closed and convex subsets of $A$ such that

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Then

$$\bigcap_{x \in A} F(x) \neq \emptyset.$$ 

**The state of affairs**

Using the generalized KKM theorem, we can show existence of equilibria in many cases of some interest (it works for an infinity of agents, too).

The requirement of (quasi)-convexity it places on the excess-demand function is a serious one. We are trying to sort the situation out (work in progress with Malamud, Anthropelos) . . .

Kardaras (’08) uses convex-compactness to give a general abstract framework for existence of numéraire portfolios.
THE DIRECT (SUB)APPROACH

Let us consider the fast-and-slow model with \( \{ \mathcal{F}_t \}_{t \in [0, T]} \) be generated by a Brownian motion \( B \) and a one-jump-Poisson process \( N \) with intensity \( \mu > 0 \). We let \( S \) be the collection of all processes of the form

\[
S_t = \int_0^t \lambda(u, B_u, N_u) \, du + dB_t,
\]

where \( \lambda : [0, T] \times \mathbb{R} \times \{0, 1\} \to \mathbb{R} \) ranges through bounded measurable functions.

- There is a finite number \( I \) of agents,
- each agent has the exponential utility \( U^i(x) = -\exp(-\gamma_i x) \),
- the random endowments are of the form \( E^i = g^i(B_T, N_T) \).

**Theorem.** Under the assumption that \( g^i \in C^{2+\delta}(\mathbb{R}), i \in I, \delta \in (0, 1) \), there exists \( T_0 > 0 \) such that an equilibrium market, unique in the class \( C^{2+\delta, 1+\delta/2}([0, T] \times \mathbb{R}) \), exists whenever \( T \leq T_0 \).

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**Theorem\(^*\)** The restriction \( T < T_0 \) is superfluous.
Sketch of the proof

• Express the optimal portfolio in the form

$$\pi^i_t = \frac{1}{\gamma_i} \lambda(t, B_T, N_t) - u^i_b(t, B_t, N_t),$$

where solves the semi-linear system of two interacting PDEs

$$\begin{cases}
0 = u^i_t + \frac{1}{2} u^i_{bb} - \lambda u^i_b + \frac{1}{2 \gamma_i} \lambda^2 - \mu \frac{\gamma}{\gamma} (\exp(-\gamma u^i_n) - 1) \\
u^i(T, \cdot, \cdot) = g^i.
\end{cases}$$

where $u^i_n(t, b, 0) = u^i(t, b, 1) - u^i(t, b, 0)$, $u^i_n(t, b, 1) = 0$.

• Write the market-clearing condition

$$0 = \sum_{i=1}^I \pi^i_t(\lambda) = \frac{1}{\gamma} \lambda - \sum_{i=1}^I u^i_b(\lambda),$$

in the form

$$F(\lambda) = \sum_{i=1}^I \gamma u^i_b(\lambda) = \lambda.$$

where $\frac{1}{\gamma} = \sum_{i=1}^I \frac{1}{\gamma_i}$. 
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where \( \frac{1}{\bar{\gamma}} = \sum_{i=1}^{I} \frac{1}{\gamma_i}. \)
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• Show that the mapping
  \[ \lambda \mapsto u^i_b(\lambda) \]
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What next

Some research directions:

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- (Alternative sources of incompleteness) jumps, transactions costs, default, etc.
- (Numerical methods) forward-backward SDEs, iterative approaches
- (Partial equilibria) with application to "pricing" in incomplete markets
- (Statistical issues) calibration, etc.
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