A HARD-CORE STOCHASTIC PROCESS
WITH SIMULTANEOUS BIRTHS AND DEATHS

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Abstract. We introduce and examine a class of stochastic spatial point processes with births and deaths, related to spatial loss networks introduced in [6]. In these processes, a point stays in the system until it is removed due to interaction with a conflicting new arrival. In particular, we consider an interaction scheme where two points are conflicting if closed balls of radius $\frac{1}{2}$ around them overlap and a new arriving point "kills" each conflicting point independently with probability $\rho$. The new point is accepted if all conflicting points are killed. We construct this process on the whole Euclidean space $\mathbb{R}^d$. If $\rho$ is large enough, we show existence of a stationary regime and exponential convergence to the stationary distribution. Such stochastic models have been studied earlier as models for populations of interacting individuals or as spatial queuing and resource sharing networks.

1. Introduction

Spatial birth-death (SBD) processes [15] are stochastic models for modeling populations of interacting individuals, where interactions between individuals depend on their location in space and where the ambient space could for instance be a subset of a lattice or Euclidean space. They appear in literature in a variety of applications including statistical mechanics, scheduling, packing and resource allocation, some of which are outlined below. In this paper, we develop a class of spatial stochastic process with births and deaths. Two key distinguishing properties of this process are that it is irreversible and hard-core. These processes can be seen as a modification of well known Gibbsian reversible processes with hard core and Random Sequential Adsorption (RSA).

In [6, 10], the authors study a modification of the RSA scheme, where points arrive according to a Poisson rain process of intensity $\lambda > 0$, it is accepted if there are no conflicting points, and every accepted point departs at a constant rate $1$. The model is also referred to as Continuous Loss Network model. This process is reversible, and the stationary distribution can be shown to be Hard-core Gibbs distribution, specifically with Papangelou conditional intensity

$$\lambda^*(x, \eta) = \lambda \cdot 1(\forall y \in \eta, \|x - y\| > 1).$$

1 A related class of processes is the Glauber dynamics corresponding to the hard-core Gibbs distribution on a graph $G$ [13]. The graph $G$ can be either finite or infinite. Here, the points arrive according to a Poisson rain process on the vertices of $G$ and a new point is accepted if there is no point adjacent to it already in the system. When the subset $G$ is finite, the stationary distribution is the hard-core Gibbs distribution $\mu_{G,\lambda}$, where the probability of observing an independent set $|I|$ is proportional to $\mu_{G,\lambda}(I) \propto \lambda^{|I|}$. The hard-core Gibbs distribution on a lattice is a Statistical Mechanics model for gases in space, and it is a common practice in Statistical Mechanics to consider a model in infinite lattices for extracting intrinsic macroscopic properties of extremely large systems.

For infinite graphs, the Gibbs distribution is defined as a suitable weak limit of appropriate conditional distributions. Every Gibbs distribution on the infinite graph can be seen as a stationary distribution of the Glauber dynamics on the infinite graph. As in the classical Ising model, it is conjectured that the infinite system undergoes a phase transition as the parameter $\lambda$ is increased. That is, for small values of $\lambda$ it is known that there is a unique Gibbs distribution, while for large values of $\lambda$ it is conjectured that there are more than one Gibbs distribution, for $d \geq 2$. For Ising models and some other lattice models of interacting particle systems, a computational phase transition has also been observed, namely for some values of the parameter, the mixing time for the Glauber dynamics on finite graphs is polynomial in the size of the graph, while for other values of the parameter, it is exponential. The location of this phase transition is observed to be at transition between uniqueness and non-uniqueness of the stationary measure in infinite graphs. These ideas extended to predict a phase transition in the Continuous Loss Network model as well. The results of [6] imply that for small values of $\lambda$, there is a unique distribution with Papangelou conditional intensity given in eq. (1) in the infinite domain, which is the stationary state

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of the dynamics of the Continuous Loss Network. While the problem of uniqueness for larger values of \( \lambda \) remains open, it has significant mathematical, physical and computational implications (see \cite{21,4}). More general SBD processes whose stationary distributions are Gibbs measures were studied in \cite{11}, where a Semigroup approach is taken to show the existence of the process, under mild conditions on the interaction potentials. Further, the generator of the process is shown to have a spectral gap, which in particular shows that certain functionals of the point process converge exponentially fast to their expectation under the stationary regime.

The Random Sequential Adsorption (RSA) model \cite{19,17} on the other hand is an irreversible process that trivially has multiple stationary states. On \( \mathbb{R}^d \) this process is defined as follows. Suppose \( D \) denotes the closed disk of radius 1/2 centered at the origin. Points arrive according to a Poisson rain process on \( \mathbb{R}^d \). Assume that each point has a translate of the disk \( D \) attached to it. An arriving point gets accepted into the domain if the disk associated with it does not overlap with previously accepted disks. Thus, the RSA model can be seen as a SBD process, with pure births. This model is appropriate for many physical, chemical and biological processes \cite{3}, particularly when irreversible deposition or reactions are involved. RSA models have also been used to study the performance of medium access and scheduling protocols in wireless networks \cite{15}, where a transmitting antenna blocks other antennas near it. In particular, in operations research, this model is known as on-line packing model.

We work on a process on the state space of configurations on \( \mathbb{R}^d \). New points arrive according to a Poisson point process and each point has a disk \( D \) of radius 1/2 attached to it, as in the RSA scheme. There is pairwise competition between a newly arriving point and already accepted points whose disks intersect the disk of the new point. The interaction is such that the new arriving point "kills" an already accepted point with probability \( \rho \), independent of everything else. The new point is then accepted if it manages to kill all the competing points. Once accepted, a point remains in the system until another arrival kills it. So, at any time, the state satisfies the hard-core condition that no two points in it are within a distance 1 from each other. Further, in contrast to a SBD processes, multiple deaths can occur simultaneously. The RSA model can also be seen as a limiting regime of this process as the limit \( \rho \to 0 \). This model can be applicable in the analysis of dynamical packing and scheduling protocols in operations research and wireless networking.

The existence of this process does not follow from the classical theory of jump Markov processes since there is no "first" jump beyond any given time. That is, although locally the process looks like a jump process, in any small time there is an event occurring somewhere in space almost surely. Once the existence is established, several interesting questions arise regarding the evolution of the process. The existence and uniqueness of a stationary distribution is an important question explored for any Markov process. Further, it is useful to establish temporal ergodicity and to obtain bounds on the speed of convergence to the stationary measures.

For point processes, moment measures give important average structural characteristic of the process, such as level of clustering or repulsion. A common approach used in the literature for the study of SBD processes is to look at the differential equations for moment measures \( \{p_n\} \) obtained from the generator of the process. This typically yields an infinite family of coupled differential equations, for moment measures of all orders. In the steady state, equating the time derivative to zero yields a hierarchical system of equations satisfied by the moment measures in this regime. The properties of the steady state can in principle be gleaned from these system of equations. While we refrain from following this approach completely, our analysis relies on the writing differential equations for moment measures of first order.

In this paper, we show the existence of a process satisfying the conditions of the above informal discussion. We give an explicit construction on a probability space containing a marked Poisson point process on \( \mathbb{R}^d \times \mathbb{R}^+ \). The construction involves the thinning of the marked Poisson point process, using a backward investigation algorithm. In other related work, Penrose \cite{16} gives a framework for showing the existence problem for interacting particle systems on an infinite domain. This also encompasses the process studied in this paper. However, the representation developed there is not suitable to create the stationary regime.

We then prove the existence of a stationary regime when \( \rho \) is large enough, with a lower bound depending on the dimension of the space. The technique utilized for this is based on a coupling from the past scheme inspired from that in \cite{2} which deals with a particular class of SBD processes on the infinite Euclidean space. This scheme bears resemblance to the Loynes’ scheme for queuing systems \( \{\Pi\} \), where one constructs a stationary regime using a coupling of an infinite family of identical queuing systems, run using a common driving process, but starting with empty initial conditions from increasing negative times, \( -T \). The limiting state at time 0, as \( T \to \infty \), if it exists, is a stationary state. The key feature in our analysis is the use of differential equations for the density of discrepancies between two processes.
starting at two different times. We bound the rate of growth of the process and show that if \( \rho \) is large enough, these densities decay to zero exponentially with time. This is then shown to be sufficient for the coupling from the past algorithm to work. The exponential decay of the density of discrepancies can also be used to run the coupling from the past argument. This establishes that for large enough values of \( \rho < 1 \), the system has a unique stationary distribution. We conjecture that a \( \rho \to 0 \), the system has multiple stationary states, as in the RSA scheme, which corresponds to \( \rho = 0 \).

In the following, we begin with a formal definition of our model. Existence of this process for a constant rate of arrivals is discussed in Section 2. We then provide sufficient conditions for the coupling from the past argument to work in Section 3. The detailed construction of a stationary regime for the process using the method of coupling from the past is presented in Section 3.1. Finally, we conclude with the result showing that this process with arbitrary initial condition converges in distribution to the distribution of the process is stationary regime in Section 3.3 when \( \rho \) is large enough.

1.1. Notation. For any set \( E \subset \mathbb{R}^d \), let \( M(E) \) denote the space of all simple locally finite counting measures \( \{ \mathbb{R} \} \) on \( (E, \mathcal{B}(E)) \), where \( \mathcal{B}(E) \) is the Borel \( \sigma \)-algebra on \( E \). For \( E' \subset E \) and \( \gamma \in M(E) \) we denote the restriction of \( \gamma \) to \( E' \) by \( |\gamma|_{E'} \). There is a one-to-one correspondence between simple locally finite counting measures and locally finite subsets on a domain \( E \), the correspondence given by the support of a counting measure. In the following we often abuse notation and make this correspondence implicit. Thus, \( \gamma \in M(E) \) also denotes a set consisting of the points in its support. The \( \sigma \)-algebra, \( M(E), \) on \( M(E) \) is generated by the maps \( \gamma \to \gamma(B), B \in \mathcal{B}(E) \).

For any \( x \in \mathbb{R}^d \) and \( \gamma \in M(\mathbb{R}^d) \), we denote by \( \theta_x \gamma \) the translation of \( \gamma \) by \(-x \), i.e., the simple counting measure supported on points \( \{ y - x : y \in \gamma \} \).

Let \( BM(\mathbb{R}^d) \) denote the space of bounded measurable functions that vanish outside a bounded set. Let \( BM_+(\mathbb{R}^d) \subset BM(\mathbb{R}^d) \) denote the subset containing non-negative functions.

2. Hard-core birth-death type processes

We begin with a general description of a class of processes we call hard-core birth-death processes on the Euclidean space \( \mathbb{R}^d \). A hard-core birth and death process is a right continuous Markov process with state space \( \{ \gamma \in M(\mathbb{R}^d) : \forall x, y \in \gamma, \| x - y \| > 1 \} \), and is associated with an arrival rate function \( a : \mathbb{R}^d \times M(\mathbb{R}^d) \to \mathbb{R}^+ \) and a probability transition kernel \( k(\cdot, \cdot) \). The arrival rate function determines the rate of arrival of new points at a specific location, given the current state. In particular, for any bounded measurable region \( B \subset \mathbb{R}^d \), \( \int_B a(x, \eta)dx \) is the rate of arrival of a point in this region at time \( t \), when the state of the system is \( \eta \). We assume that the arrival rate function is local, i.e., there exists an \( R > 0 \), such that \( a(x, \eta) = a(x, \eta \cap B(x, R)) \). The transition kernel \( k \) governs the transitions of state when a new point arrives. If a point arrives at the origin when the state of the system is \( \eta \), the state is transformed to a hard-core subset of \( \eta \cup \{0\} \) using probability distribution \( k(\eta, \cdot) \), independent of everything else. We assume that the kernel is local in that

\[
k(\eta, (\gamma \in M(\mathbb{R}^d) : \gamma|_{B(0,1)} \neq \eta|_{B(0,1)}) = 0.
\]

The transitions to the state when a point arrives at a location \( x \) is obtained by applying the kernel \( k \) after translating the point \( x \) to origin, i.e., the new state is \( \theta^{-1}_x \gamma \), where \( \gamma \) is a sample from \( k(\theta_x \eta, \cdot) \).

The generator of this process has the form:

\[
LF(\eta) = \int_{\mathbb{R}^d} \int_{M(\mathbb{R}^d)} [f(\eta') - f(\eta)]a(x, \eta)k(\theta_x \eta, \theta^{-1}_x dy')dx.
\]

We refer to the acceptance of a point as a birth and the deletion of a point as a death. Hence, the birth of a point occurs when all points in a radius 1 around it are killed.

We focus on a specific model, with arrival rate function \( a \), and transition kernel \( k \), given as follows. Firstly, we assume that the arrivals occur at rate 1 uniformly over \( \mathbb{R}^d \), i.e., \( a(x, \eta) = 1 \). For any point \( p \) in the system, let \( x_p \) denote the location of the point in space, and let \( t_p \) denote the time of it’s arrival.

The interaction between an arriving point and an existing points is pairwise and independent. For any incoming point \( p \) and every point \( q \) in the system at \( t_p \), i.e., \( x_q \in \eta_{t_p-} \cap B(x_p, 1) \), \( q \) is deleted with probability \( \rho \) or \( q \) kills \( p \) instead, independent of everything else. We further assume that the order of these pairwise interactions is not important, so the arriving point \( p \) is deleted if any \( q \) above kills \( p \). The generator of this process defined on \( \mathcal{F} = \{ F(\eta) = \sum_{x \in \eta} f(x) : f \in C_c(\mathbb{R}^d) \} \) is then the following:

\[
LF(\eta) = \int_{\mathbb{R}^d} \left( \rho \eta(B(x,1)) f(x) - \rho \sum_{y \in \eta \cap B(x,1)} f(y) \right) dx.
\]
2.1. Existence. In this section, we give an explicit construction of a process that has the above generator in eq. (3). The approach is similar to the one in [2], where the authors consider an SBD process with death by random connections model and the construction is based on a backward investigation algorithm.

We assume that the initial condition $\eta_0$ is a stationary and ergodic point process. For the construction of the process, we assume that arrival events occur according to a homogeneous Poisson point process $N$ on $\mathbb{R}^d \times \mathbb{R}^+$ with parameter $\lambda = 1$. For any point $p = (x_p, t_p) \in N$, the first coordinate, $x_p$, denotes the location of the arrival and the second coordinate, $t_p$, denotes the time of arrival. The arrival time for points in $\eta_0$ is set to 0, so with an abuse of notation we may assume $\eta_0 \in M(\mathbb{R}^d \times \mathbb{R}^+)$. We assume the existence of marks $I_{p,q}$ for the point $p$, $p,q \in N \cup \eta_0$, which are independent Bernoulli random variables with parameter $\rho$, as described in Section 2. A simple use of the consistency theorem shows that one can define a probability space containing such a marked Poisson point processes and the initial point process $\eta_0$.

We build a process $\eta_t \in M(\mathbb{R}^d)$ that satisfies the following properties almost surely, for any bounded set $K \subset \mathbb{R}^d$:

(a) The process $\eta_t|_K$ is a right-continuous jump process with values in $M(K)$.

(b) If a point $p = (x_p, t_p)$ arrives at time $t = t_p$ and at location $x_p \in K \oplus B(0, 1)$, then

$$
\eta_t|_{K \setminus B(x_p, 1)} = \eta_{t-|_{K \setminus B(x_p, 1)},}
$$

and

$$
\eta_t|_{K \cap B(x_p, 1)} = 1_{x_p \in K} \cdot \delta_{x_p} \prod_{x_q \in \eta_{t-|_{K \cap B(x_p, 1)}}} I_{p,q} + \sum_{x_q \in \eta_{t-|_{K \cap B(x_p, 1)}}} \delta_{x_q} (1 - I_{p,q}).
$$

We have the following lemma:

**Lemma 2.1.** Let $N$ be as described above and let $\eta_0$ be a hard-core point process. There is an $\epsilon > 0$ such that for any hard-core point process $\eta_0$, with probability one, there exists a hard-core birth-death process $\eta_t$, $t \in [0, \epsilon]$, with arrivals from $N$ and the transitions described by eqs. (4) and (5).

**Proof.** We give an algorithm to construct the process for times $t \in [0, \epsilon]$, where $\epsilon > 0$ is fixed later. Let $N_{[0,\epsilon]}$ denote the point process $N$ restricted to the set $\mathbb{R}^d \times [0, \epsilon]$.

To construct $\eta_t$, it is enough to compute whether a point $p \in N_{[0,\epsilon]}$ is accepted when it arrives. We construct a directed dependency graph $G = (V, E)$, with $V = N_{[0,\epsilon]}$ and $(p, q) \in E$ if and only if $x_q \in B(x_p, 2)$ and $t_q < t_p$. So that $(p, q) \in E$ implies that whether the point $p$ is accepted at time $t_p$ depends on whether $q$ was accepted. Note that the radius of influence above of a point in $N_{[0,\epsilon]}$ is set to 2, instead of 1, since a point can interact with a point of $\eta_0$ within a distance 1 from it, which in turn influences another point of $N$ within a distance 1.

The projection of $G$ (not considering directions) onto the spatial dimension is a Random geometric graph (9) on the projection of $N_{[0,\epsilon]}$. That is it is a Random geometric graph on a homogeneous Poisson point process with intensity $\lambda \epsilon$. From theorem 2.6.1 from [9], if $\epsilon$ is small enough so that $\lambda \epsilon$ is less than the critical value, then this graph does not percolate almost surely.

When every component of the graph is finite, the acceptance of any point $p \in N_{[0,\epsilon]}$ can be calculated in finite time using a backward investigation. The backward investigation algorithm sorts the points of the component of $p$ according to their arrival times. Then using the points of $\eta_0$ and the marks of the relevant points, it can calculate the status of the points sequentially in the sorted list of arrivals. Hence, there is an $\epsilon > 0$ small enough so that, almost surely, for every point in $N_{[0,\epsilon]}$, it may be evaluated whether the point is accepted when it arrives. This completes the proof.

Notice that in the proof above $\epsilon$ is independent of the initial conditions. Hence, using the above lemma, we can construct the required process successively on time intervals $[n\epsilon, (n+1)\epsilon], n \in \mathbb{N}$, starting with any initial condition. This result is summarized in the following corollary.

**Corollary 2.2.** Under the hypothesis of Lemma 2.1, there exists a hard-core birth-death process $\eta_t$, $t \in [0, \infty)$, with arrivals in $N$ and local interactions given by eq. (4) and eq. (5). Further, for any time $t > 0$, $\eta_t$ is spatially stationary and ergodic if $\eta_0$ is spatially stationary and ergodic.

**Proof.** The above discussion gives the construction of the process $\eta_t$ for any finite time interval $t \in [0, T]$.

For any $t > 0$, $\eta_t$ is spatially stationary and ergodic since it is a translation invariant thinning of the spatially stationary and ergodic projection of the process $N_{[0,t]} \cup \eta_0$. □
3. Time Stationarity and Ergodicity

Ergodicity of a Markov process may refer to existence of a unique stationary distribution. Under this condition, the stationary process is ergodic as a dynamical system, i.e. the tail σ-algebra is trivial. In this section, under certain assumptions, we show the existence of a stationary regime for the process constructed in the previous section. While the method is non-constructive, it is inspired from Perfect-simulation or Coupling from the past techniques. A similar technique was used in [2] for proving the existence of a stationary regime in case of their death by random connection model. We show the exponential decay of discrepancies between two coupled processes starting from distinct initial conditions. This exponential decay in turn also proves the uniqueness of the stationary distribution. This argument is spelled out in Section 3.3, where it is also shown that the distribution of the state of Markov process at time $t$ converges to that of the stationary state as $t \to \infty$.

The key ideas of the Coupling from the past technique below are the following: We run the process using a fixed temporally stationary ergodic driving process from time $-T$ until time $t$, call the process $\eta^T_t$. We then show that the limit, as $T \to \infty$, of the random process, $\eta^T_t$, converges almost surely. This is done by showing that any two processes starting from time 0, couple in a time that has finite expectation. The ergodic theorem can then be used to claim that the limiting process $T_t$ exists almost surely. The limiting process $T_t$, is clearly temporally stationary and ergodic since the driving process as it is a factor of the driving process.

Accordingly, in the following section we define a coupling of two hard-core processes and produce sufficient conditions for exponential decay of density of discrepancies between them. Later in section 3.2 we construct the stationary regime using the coupling from the past argument.

3.1. Coupling of two processes; Density of special points. Consider a process $\{\eta^1_t\}_{t \geq 0}$ and $\{\eta^2_t\}_{t \geq 0}$ driven by the same homogeneous Poisson point process, $N \in M(\mathbb{R}^d \times \mathbb{R}^+)$, with $\eta^1_0 = 0$ and $\eta^2_0$ being a spatially stationary and ergodic hard-core point process. At any time $t > 0$, there are some points that are present in both processes. These points are called regular points form a stationary point process $R_t$. The remaining points are present in one of the processes and absent in the other. These points are referred to as special points and the symbol $S_t$ is used to denote the point process formed by these. In particular, the points present in $\eta^1_t$ and absent in $\eta^2_t$ are called antizombies, denoted by $A_t$, and those points alive in $\eta^2_t$ and dead in $\eta^1_t$ are called zombies, denoted by $Z_t$. Thus,

$$R_t = \eta^1_t \cap \eta^2_t, \quad S_t = \eta^1_t \triangle \eta^2_t, \quad A_t = \eta^1_t \setminus \eta^2_t, \quad \text{and} \quad Z_t = \eta^2_t \setminus \eta^1_t.$$  

Now suppose $z \in S_t$ is a special point. Given the realization of $N$, we can build an interaction graph $G$ on the points $S_t \cup N_{(t,\infty)}$, with directed edges $(p, q)$ if and only if $t_p < t_q$, $x_p \in S_{t_q} \cap B(x_q, 1)$ and $I_{p,q} = 1$. Let $G_z$ be the subgraph of $G$ containing $z$, called the family of $z$. Let $M_{z,t,t+\delta}$ denote the elements of $G_z$ alive at time $t + \delta$, with $m_{z,t,t+\delta} = |M_{z,t,t+\delta}|$.

Since at time $t + \delta$ a special point can belong to more than one family, by applying the Mass transport principle (see [3] [12]), we get the following bound:

$$\beta_{S,t+\delta} \leq \beta_{S,t} E_{S_0} \mathbb{E}_{S_0} m_{0,t,t+\delta}.$$  

We have, by superposition principle,

$$\beta_{S,t} E_{S_0} m_{0,t,t+\delta} = \beta_{Z,t} E_{Z_0} m_{0,t,t+\delta} + \beta_{A,t} E_{A_0} m_{0,t,t+\delta}.$$  

Let $i_z = |N_{(t,t+\delta)} \cap [B(z, 1) \times (t, t + \delta)]|$, $j_z = |N_{(t,t+\delta)} \cap [B(z, 2) \times (t, t + \delta)]|$ and $D_z$ be the event that the interaction graph of $z$ between points $t$ and $t + \delta$ contains at least two points. Let $\nu_1$ be the volume of the ball of radius 1. By spatial stationarity we can work with the Palm expectation, assuming that the location of $z$ is at 0.

$$E_{Z_0} m_{0,t,t+\delta} \leq P_{Z_0}(i_0 = 0) + P_{Z_0}(i_0 = 1) E_{Z_0} m_{0,t,t+\delta} |i_0 = 1, D_z| + E_{Z_0} m_{0,t,t+\delta} |1, D_0|.$$  

We note that the last term in the above expression is $o(\delta)$. Indeed, $m_{0,t,t+\delta}$ can be stochastically bounded by a pure birth process with birth rates $\lambda_k = k\nu_1$, $k \geq 1$, as each successive member of a family increases the coverage area of the family by at-most $\nu_1$. Then using the explicit probability distributions for this pure-birth process it is easy to conclude this result (see [20] Chapter 5).

Hence,

$$E_{Z_0} m_{0,t,t+\delta} \leq e^{-\nu_1 \lambda \delta} + \nu_1 \lambda e^{-\nu_1 \lambda \delta} E_{Z_0} m_{0,t,t+\delta} |i_0 = 1, D_z| + o(\delta).$$  

Define $\mathfrak{g}_t = \bigcup_{x \in \mathfrak{g}_t} B(x, 1)$, $\mathfrak{z}_t = \bigcup_{x \in \mathfrak{z}_t} B(x, 1)$ and $\mathfrak{a}_t = \bigcup_{x \in \mathfrak{a}_t} B(x, 1)$. We have the following possible disjoint events within $\{i_0 = 1\} \cap D_0$:
$E_1$: If the point arrives in $B(0, 1) \cap (R_t \cup R_t)^c$, then

$$m_{0,t,t+\delta} = \begin{cases} 2 & \text{w.p. } 1 - \rho \\ 0 & \text{w.p. } \rho \end{cases}.$$ 

Hence, $E^0_{Z^t}[m_{0,t,t+\delta}\mid i_0 = 1, D_0^c, E_1] = 2(1 - \rho)$.

$E_2$: If the point arrives in $B(0, 1) \cap R_t \cap R_t^c$, then

$$m_{0,t,t+\delta} = \begin{cases} 2 & \text{w.p. } \rho^k(1 - \rho) \\ 1 & \text{w.p. } (1 - \rho)(1 - \rho^k) \\ 0 & \text{w.p. } \rho \end{cases},$$

where $k$ is the number of regular points that interact with the incoming point. Hence,

$$E^0_{Z^t}[m_{0,t,t+\delta}\mid i_0 = 1, D_0^c, E_2] \leq (1 - \rho^2).$$

$E_3$: If the point arrives in $B(0, 1) \cap R_t \cap R_t$, then

$$m_{0,t,t+\delta} = \begin{cases} 2 & \text{w.p. } \rho^k(1 - \rho) \\ 1 & \text{w.p. } (1 - \rho)(1 - \rho^k) \\ 0 & \text{w.p. } \rho \end{cases},$$

where $k$ is the number of antizombies that interact with the incoming point. Hence,

$$E^0_{Z^t}[m_{0,t,t+\delta}\mid i_0 = 1, D_0^c, E_3] \leq (1 - \rho^2).$$

$E_4$: If the point arrives in $B(0, 1) \cap R_t \cap \mathbb{R}_t$, then

$$m_{0,t,t+\delta} = \begin{cases} 2 & \text{w.p. } \rho^k(1 - \rho) \\ 1 & \text{w.p. } (1 - \rho)(1 - \rho^k) \\ 0 & \text{w.p. } \rho \end{cases},$$

where $k$ is the number of regular and antizombies that the point interacts with. Therefore,

$$E^0_{Z^t}[m_{0,t,t+\delta}\mid i_0 = 1, D_0^c, E_4] \leq (1 - \rho^2).$$

Consequently, taking all 4 cases above into account we obtain,

$$E^0_{Z^t}[m_{0,t,t+\delta}\mid i_0 = 1, D_0^c] \leq 1 - \rho^2 + \frac{1}{\nu_1} E^0_{Z^t}((2(1 - \rho) - (1 - \rho^2))\ell(B(0, 1) \cap (R_t \cup R_t)^c))$$

$$\leq 1 - \rho^2 + \left[\frac{(1 - \rho)^2}{\nu_1} E^0_{Z^t}\ell(B(0, 1) \cap (R_t \cup R_t)^c))\right]$$

$$= 1 + \left[1 - 2\rho - \frac{(1 - \rho)^2}{\nu_1} E^0_{Z^t}\ell(B(0, 1) \cap (R_t \cup R_t))\right]$$

$$\leq 1 + \left[1 - 2\rho - \frac{(1 - \rho)^2}{\nu_1} E^0_{Z^t}\ell(B(0, 1) \cap R_t)\right].$$

From eqs. (6)–(8),

$$\beta_{S_{i+k}} - \beta_{S_i} \leq \beta_{S_i} [e^{-\nu_1 \lambda \delta}(1 + \nu_1 \lambda \delta) - 1]$$

$$+ \nu_1 \lambda e^{-\nu_1 \lambda \delta} \left(1 - 2\rho - \frac{1}{\nu_1} (1 - \rho)^2 E^0_{S_t}\ell(B(0, 1) \cap R_t)\right) + o(\delta).$$

Therefore,

$$\frac{1}{\beta_{S_i}} \frac{d\beta_{S_i}}{dt} = \frac{1}{\beta_{S_i}} \limsup_{\delta \to 0} \frac{\beta_{S_{i+k}} - \beta_{S_i}}{\delta}$$

$$\leq \nu_1 \lambda \left(1 - 2\rho - \frac{1}{\nu_1} (1 - \rho)^2 E^0_{S_t}\ell(B(0, 1) \cap R_t)\right)$$

$$\leq \nu_1 \lambda (1 - 2\rho).$$

If $\rho > \frac{1}{2}$, then we see that $\beta_{S_i}$ decreases exponentially to zero, i.e., there exists $c > 0$, such that $\beta_{S_i} \leq \beta_{S_0} e^{-ct}$. Exponential convergence will be useful to prove the existence of a stationary regime. This result can be interpreted as saying that the information about the initial state is erased sufficiently quickly by the incoming points.
In the following we utilize the geometry of the interactions to gain more from the inequality \([11]\). Let \(\kappa\) be the kissing number for balls in \(\mathbb{R}^d\). We now note that
\[
\ell(B(0, 1) \cap \mathcal{R}_t) \geq \frac{\nu_1}{4d} \mathbb{1}(R_t(B(0, \frac{3}{2}) > 0)) \geq \frac{\nu_1}{4d(\kappa - 1)} R_t(B(0, \frac{3}{2})),
\]
where the first inequality, we ignore regular points greater than \(3/2\) and then \(B(0, 1) \cap B(x, 1)\) contains a ball of radius \(1/4\), if \(|x| \leq \frac{3}{2}\). The second inequality is true since \(R_t(B(0, \frac{1}{2}))\) takes only values \(0, 1, \ldots, \kappa - 1\).

Hence, to bound \(\mathbb{E}_t^{S_0} \ell(B(0, 1) \cap \mathcal{R}_t)\) from below we calculate bounds on the derivative of \(\beta_{S_t} \mathbb{E}_t^{S_0} R_t(B(0, \frac{3}{2})\). If \(C \subset \mathbb{R}^d\) is a set of measure 1, then
\[
\beta_{S_t} \mathbb{E}_t^{S_0} R_t(B(0, \frac{3}{2})) = \mathbb{E} \left[ \sum_{x \in S_t \cap C} \sum_{y \in R_t \cap B(x, \frac{1}{2})} 1 \right].
\]
The derivative of the above expression depends on rates of increase and decrease of both regular and special points. We now give a lower bound on the derivative by accounting for various types of interactions.

- We first consider the killings (rate of decrease).
  - For each point \(x \in S_t \cap C\), a new point could arrive from the Poisson rain and kill \(x\) with probability \(\rho\). This type of interaction results in a rate equal to
    \[-\nu_1 \lambda \rho \times \mathbb{E} \left[ \sum_{x \in S_t \cap C} \sum_{y \in R_t \cap B(x, \frac{1}{2})} 1 \right] = -\nu_1 \lambda \rho \beta_{S_t} \mathbb{E}_t^{S_0} R_t(B(0, \frac{3}{2})).\]
  - For each point \(x \in S_t \cap C\) and \(y \in R_t \cap B(x, \frac{3}{2})\), a new point could arrive in \(B(x, 1) \cap \mathcal{R}_t\), the point \(x\) survives, but \(y\) is killed. This results in a rate equal to
    \[-\lambda \rho (1 - \rho) \times \mathbb{E} \left[ \sum_{x \in S_t \cap C} \sum_{y \in R_t \cap B(x, \frac{1}{2})} \ell(B(x, 1) \cap B(y, 1)) \right] \geq -\lambda \rho (1 - \rho) \mathbb{E} \left[ \sum_{x \in S_t \cap C} \sum_{y \in R_t \cap B(x, \frac{3}{2})} \nu_1 \right] = -\nu_1 \lambda \rho (1 - \rho) \beta_{S_t} \mathbb{E}_t^{S_0} R_t(B(0, \frac{3}{2})).\]
  - For each point \(x \in S_t \cap C\) and \(y \in R_t \cap B(x, \frac{3}{2})\), a new point could arrive in the region \(B(y, 1) \cap B(x, 1)^c\) and kill \(y\). This results in a rate of change equal to
    \[-\lambda \rho \times \mathbb{E} \left[ \sum_{x \in S_t \cap C} \sum_{y \in R_t \cap B(x, \frac{1}{2})} \ell(B(y, 1) \cap B(x, 1)^c) \right] \geq -\lambda \rho \mathbb{E} \left[ \sum_{x \in S_t \cap C} \sum_{y \in R_t \cap B(x, \frac{3}{2})} \nu_1 \right] = -\nu_1 \lambda \rho \beta_{S_t} \mathbb{E}_t^{S_0} R_t(B(0, \frac{3}{2})).\]

- We now consider the rate of increase:
  - For each \(x \in S_t \cap C\), a new point could arrive at \(B(x, \frac{3}{2}) \setminus B(x, 1)\) and compete with other points. It survives and becomes a regular with a probability at least \(\rho^{\kappa}\). This type of interaction results in a rate at least equal to \(\rho^{\kappa} \left(\left(\frac{3}{2}\right)^d - 1\right) \nu_1 \lambda \beta_{S_t}\). Let \(c_0 := \rho^{\kappa} \left(\left(\frac{3}{2}\right)^d - 1\right)\).
  - We ignore the rate at which new special points are created.

Consequently,
\[
\frac{d}{dt} \beta_{S_t} \mathbb{E}_t^{S_0} R_t(B(0, \frac{3}{2})) \geq c_0 \nu_1 \lambda \beta_{S_t} - (\rho - \rho^2) \nu_1 \lambda \beta_{S_t} \mathbb{E}_t^{S_0} R_t(B(0, \frac{3}{2})).
\]

Using the product rule and eq. \([10]\), we obtain:
\[
\frac{d}{dt} \mathbb{E}_t^{S_0} R_t(B(0, \frac{3}{2})) \geq c_0 \nu_1 \lambda - \nu_1 \lambda \rho (3 - \rho) \mathbb{E}_t^{S_0} R_t(B(0, \frac{3}{2})) - \frac{1}{\beta_{S_t}} \frac{d}{dt} \beta_{S_t} \mathbb{E}_t^{S_0} R_t(B(0, \frac{3}{2}))
\]
\[
\geq c_0 \nu_1 \lambda - (1 + \rho - \rho^2) \nu_1 \lambda \mathbb{E}_t^{S_0} R_t(B(0, \frac{3}{2})).
\]
This shows that
\[
\liminf_{t \to \infty} \mathbb{E}_t^{S_0} R_t(B(0, \frac{3}{2})) \geq \frac{c_0}{1 + \rho - \rho^2} = \frac{\rho^{\kappa} \left(\left(\frac{3}{2}\right)^d - 1\right)}{1 + \rho - \rho^2}.
\]
From eqs. (11) and (9), it then follows that
\[
\limsup_{t \to \infty} \frac{d\beta_{S_t}}{dt} \leq \nu_1 \lambda \left( (1 - 2\rho) - \frac{\rho^c(1 - \rho)^2((3d)^d - 1)}{4d(\alpha - 1)(1 + \rho - \rho^c)} \right).
\]

Thus, using Gronwall’s theorem we have the following result:

**Theorem 3.1.** If
\[
(1 - 2\rho) - \frac{\rho^c(1 - \rho)^2((3d)^d - 1)}{4d(\alpha - 1)(1 + \rho - \rho^c)} < 0,
\]
then there exist constants \( \alpha, c > 0 \) such that for all \( t > 0 \), \( \beta_{S_t} \leq c e^{-ct} \).

This gives a better range of \( \rho \), for which coupling from the past argument can be performed. For \( d = 1 \), for example, the above sufficient condition is satisfied when \( \rho > 0.497 \).

### 3.2. Coupling from the Past

In this section we construct a stationary regime for the hard-core process using the method of coupling from the past. Consider a doubly infinite Poisson point process \( N \) on \( \mathbb{R}^d \times (-\infty, \infty) \) with mean measure \( \lambda(d\mathbf{x}) \times dt \). Let \( \{ \theta_t \}_{t \in \mathbb{R}} \) be a group of time-shift operators under which the point process \( N \) is ergodic. Let \( \eta_t \) be the process starting at time 0 with empty initial condition. Now, consider the sequence of processes \( \{ \eta_t^T, t > -T \}_{T \in \mathbb{N}} \), obtained with empty initial condition from time \(-T\) by using \( N \), for \( T \in \mathbb{N} \). We have \( \eta_t^T = \eta_{t+T} \circ \theta_{-T} \).

The processes \( \eta_t^T \) and \( \eta_0^T \) are driven by the same Poisson process beyond time 0. Treating the points in \( \eta_0^T \) as initial conditions for the augmented process \( \{ \zeta_t \} \), if the sufficient conditions of Theorem 3.1 hold then the density of special points in the coupling of \( \eta_t^1 \) and \( \eta_0^1 \) goes to zero exponentially quickly. The following theorem shows that the two processes coincide on a compact time in finite time with finite expectation. For a compact set \( C \subset \mathbb{R}^d \), let
\[
\tau(C) := \inf\{ t > 0 : \eta_t^1|_C = \eta_0^1|_C, \quad s \geq t \}.
\]

Note that \( \tau(C) \) is not a stopping time.

In the following lemma, recall that \( S_t \) is the set of discrepancies, \( \eta_t^0 \triangleleft \eta_t^1 \).

**Lemma 3.2.** If for some \( c > 0 \), \( \beta_{S_t} \leq c e^{-ct} \), then the minimum time \( \tau(C) \) beyond which \( \eta_t \) and \( \hat{\eta}_t \) to coincide on the set \( C \) has bounded expectation

**Proof.** We view \( S_t(C), t > 0 \), as a simple birth-death process. Let \( S_t(C) = S_0(C) + S^+(0, t] - S^-(0, t] \)

where \( S^+ \) and \( S^- \) are point processes on \( \mathbb{R}^+ \) with the following properties:

1. \( S^+ \) is a simple counting process, with a jump indicating the arrival of a new special point in \( C \).
2. \( S^- \) is a counting process, with a jump indicating the departure of corresponding number of special points from \( C \).

Since special points result from interaction of arriving points with existing special points, the rate of increase in \( S^+ \) is bounded above by \( S_t(C \oplus B(0, 1)) \times \lambda_1 \). Hence,
\[
E S^+(0, \infty) \leq \lambda_1 \int_0^\infty E S_t(C \oplus B(0, 1)) dt = \lambda_1 \ell(C \oplus B(0, 1)) \int_0^\infty \beta_{S_t} dt < \infty.
\]

This also shows that \( S^+(0, \infty) \) and \( S^-(0, \infty) \) exist and are finite a.s. Thus \( \lim_{t \to \infty} S_t(C) \) also exists and are finite. From the fact that \( \lim_{t \to \infty} E S_t(C) = \lim_{t \to \infty} \beta_{S_t} \ell(C) = 0 \), by dominated convergence theorem, we have \( E \lim_{t \to \infty} S_t(C) = 0 \). Thus, \( \lim_{t \to \infty} S_t(C) = 0 \) a.s. This also shows that \( \tau(C) < \infty \) a.s.

Let \( S \) be the random measure on \( \mathbb{R}^+ \), with \( S[0, t] = S_t(C) \) for all \( t \geq 0 \). We have
\[
Er(C) \leq E \int_0^\infty tS^-(dt) = E \int_0^\infty tS^+(dt) - E \int_0^\infty tS(dt) \leq \lambda_1 \ell(C \oplus B(0, 1)) \int_0^\infty t\beta_{S_t} dt + E \int_0^\infty S[0, t]dt = \lambda_1 \ell(C \oplus B(0, 1)) \int_0^\infty t\beta_{S_t} dt + t(C) \int_0^\infty \beta_{S_t} dt < \infty.
\]

□
Now, let $V^T_y$ denotes the time at which the executions of processes $\eta^T_t$ and $\eta^{T+1}_t$ coincide in $B(y,1)$, i.e., $V^T_y = \tau(B(y,1)) \circ \theta_{-T} - T$. Then we have:

\[(12)\]
\[
V^T_y + T = \tau(B(y,1)) \circ \theta_{-T} = V^0_y \circ \theta_{-T}.
\]

Consequently we have the following lemma

**Lemma 3.3.** If $\mathbb{E}\tau(B(y,1)) < \infty$, then

\[(13)\]
\[
\lim_{T \to \infty} V^T_y = -\infty.
\]

**Proof.** By eq. \[(12)\], $V^T_y + T$ is a stationary and ergodic sequence. Hence, by Birkhoff’s pointwise ergodic theorem,

\[
\lim_{T \to \infty} \sum_{i=0}^{T} \frac{|V^i_y + i|}{T} = \mathbb{E}\tau(B(y,1)) < \infty, \text{ a.s.}
\]

Therefore the last term in the summation, $\frac{V^T_y + T}{T} \to 0$ a.s., as $T \to \infty$. This implies the desired result that

\[
\lim_{T \to \infty} V^T_y = -\infty, \text{ a.s.}
\]

Thus, for every realization of $N$, any compact set $C$ and $t \in \mathbb{R}$, there exists a $k \in \mathbb{N}$ such that for all $T > k$, $\tau(C) \circ \theta_{-T} - T < t$. That is, for $T > k$, the execution of all the processes starting at $-T$ coincide at time $t$ on the compact set $C$. The limit

\[(14)\]
\[
\Upsilon_t := \lim_{T \to \infty} \eta_{t+T} \circ \theta_{-T}
\]

may now be defined as the weak limit of restrictions to compact sets, $\Upsilon_t|_C = \lim_{T \to \infty} \eta_{t+T} \circ \theta_{-T}|_C$. Further,

\[
\Upsilon_t \circ \theta_t = \lim_{T \to \infty} \eta_{t+T} \circ \theta_{-T+1} = \lim_{T \to \infty} \eta_{t+1+T-1} \circ \theta_{-T+1} = \Upsilon_{t+1}.
\]

So, the process $\Upsilon$ is $\{\theta_t\}_{t \in \mathbb{R}}$ compatible. In fact, $\Upsilon$ is also $\{\theta_t\}_{s \in \mathbb{R}}$ compatible and temporally ergodic, since it is a factor of the driving process $N$. Thus, $\Upsilon$ is the stationary regime for this process.

### 3.3. Convergence in Distribution

Let $\{\eta_t^f\}_{t \geq 0}$ be a Hard-core process driven by a homogeneous Poisson point process $N'$ on $\mathbb{R}^2 \times \mathbb{R}^+$, as described in Section 2 with ergodic initial conditions. Let $\Upsilon_0$ be stationary regime of the process at time zero. Consider the process $\hat{\eta}_t$ with initial condition $\hat{\eta}_0 = \Upsilon_0$ and being driven by the point process $N'$. Note that $\hat{\eta}_t \overset{d}{=} \Upsilon_t \overset{d}{=} \Upsilon_0$. If the conditions of Theorem 3.1 are satisfied then we can conclude that the density of the discrepancies between the two processes vanishes exponentially to zero. This gives the following quantitative estimate on the difference of the Laplace functional $L_t$ and $\hat{L}_t$ of $\eta_t$ and $\hat{\eta}_t$ respectively.

**Lemma 3.4.** Let $S_t = \eta_t \triangle \hat{\eta}_t$. If there exists $c > 0$ such that $\beta_{S_t} \leq \beta_{\eta_0} e^{-ct}$, then for any $f \in BM_+(\mathbb{R}^2)$, we have

\[(15)\]
\[
|L_t(f) - \hat{L}_t(f)| \leq \|f\|_{L^1} \beta_{\eta_0} e^{-ct}
\]

**Proof.** Let $f \in BM_+(\mathbb{R}^2)$, Then,

\[
L_t(f) - \hat{L}_t(f) = \mathbb{E} \left[ e^{-\int f(x)\eta_t(dx)} \left( 1 - \prod_{x \in \hat{\eta}_t \setminus \eta_t} e^{-f(x)} \prod_{x \in \eta_t \setminus \hat{\eta}_t} e^{f(x)} \right) \right] \\
\leq \mathbb{E} \left[ e^{-\int f(x)\eta_t(dx)} \left( \int f(x)\eta_t(dx) - \int f(x)\hat{\eta}_t(dx) \right) \right].
\]
\[ \rho = 0.8, \text{ PF} = 0.247 \quad (\text{a}) \]
\[ \rho = 0.5, \text{ PF} = 0.291 \quad (\text{b}) \]
\[ \rho = 0.2, \text{ PF} = 0.327 \quad (\text{c}) \]
\[ \rho = 0, \text{ PF} = 0.542 \quad (\text{d}) \]

Figure 1. Samples from the stationary state on a finite window. The packing fraction is observed to increase as \( \rho \to 0 \).

Similarly,
\[
\begin{align*}
L_t(f) - \hat{L}_t(f) &= \mathbb{E} \left[ e^{-\int f(x) \hat{\eta}_t(dx)} \left( \prod_{x \in \hat{\eta}_t \setminus \eta_t} e^{f(x)} \prod_{x \in \eta_t \setminus \hat{\eta}_t} e^{-f(x)} - 1 \right) \right] \\
&\geq \mathbb{E} \left[ e^{-\int f(x) \hat{\eta}_t(dx)} \left( \int f(x) \eta_t \setminus \hat{\eta}_t(dx) - \int f(x) \hat{\eta}_t \setminus \eta_t(dx) \right) \right].
\end{align*}
\]

Hence,
\[
\begin{align*}
|L_t(f) - \hat{L}_t(f)| &\leq \mathbb{E} \left[ \max \left\{ e^{-\int f(x) \hat{\eta}_t(dx)}, e^{-\int f(x) \eta_t(dx)} \right\} \left| \int f(x) \eta_t \setminus \hat{\eta}_t(dx) - \int f(x) \hat{\eta}_t \setminus \eta_t(dx) \right| \right] \\
&\leq \mathbb{E} \int f(x) \eta_t \setminus \hat{\eta}_t(dx) - \int f(x) \hat{\eta}_t \setminus \eta_t(dx) \\
&\leq \mathbb{E} \int f(x) S_t(dx) \\
&= \beta_S \| f \|_{L^1} \\
&\leq \beta_S \| f \|_{L^1} e^{-ct}.
\end{align*}
\]

Since point-wise convergence of Laplace functional also implies convergence in distribution, we can conclude that \( \tau_t \) converges weakly to \( \Upsilon_0 \) as \( t \to \infty \).

4. Concluding Remarks

In this paper we focused on the Hard-core model on an infinite domain where the interactions are pairwise. It was shown that under the conditions of Theorem 3.1, a stationary regime exists. It remains to be seen if exponential convergence as above can be shown for all value of \( \rho > 0 \). Differential equations of higher order moment measures might be necessary for controlling the decay in density of special points in this case. The RSA scheme \( (\rho = 0) \), for example is a monotonic process, and
thus has a stationary distribution that depends heavily on the initial condition. As in lattice models of interacting particle systems, we suspect that there exists a critical value, \( \rho_c > 0 \), such that for all \( \rho < \rho_c \) the stationary distribution is non-unique, while for \( \rho > \rho_c \) the stationary distribution is unique.

Further, it would also be useful to obtain quantitative bounds on the packing efficiencies of the hard-core processes considered here, in their stationary regimes. For \( \rho = 0.5 \), simulation indicated that this value is approximately 0.32 on the plane. While this is much less than the RSA scheme, whose packing fraction in the jamming limit is predicted to be around 0.54, the existence of a unique stationary regime means that the process meets the primary requirement for fairness of a resource allocation scheme. The dependence of the packing efficiency on the parameter \( \rho \) is also unclear. Further, packing efficiencies of other hard-core point processes need to be compared with the hard-core spatial birth-death processes. In particular, one class of processes where the hard-core structure shows up are the Matérn type-I and type-II processes [14]. These processes can be considered as a dependent thinning of Poisson point processes, based on a retention rule. The packing efficiencies of these Matérn processes are known, and they form an interesting class for comparison of packing efficiencies.

Acknowledgments

The author would like to thank his PhD advisor, François Baccelli, for many valuable discussions on this problem. This work is supported by award from the Simons Foundation (award number 197982) to the University of Texas at Austin

References


