Bosonization of lattice fermions in higher dimensions

Anton Kapustin

California Institute of Technology

January 15, 2019

Outline

- Bosons, fermions, and spin structures
- Review of the 1d Jordan-Wigner transformation
- Review of $\mathbb{Z}_2$ gauge theories
- Bosonization in 2d
- Some examples
- Euclidean viewpoint
- Bosonization in 3d
- Bosonization and generalized global symmetries

Based on: Yu-An Chen (Caltech), AK, Djordje Radicevic (Perimeter Institute), arXiv: 1711.00515 and Yu-An Chen and AK, arXiv:1807.0781. Also prior work with D. Gaiotto and L. Bhardwaj.

Fermion-boson correspondence

Fermion-boson correspondence in 1+1d:

- Free massless Dirac fermion $\leftrightarrow$ free massless boson
- Massive Thirring $\leftrightarrow$ sine-Gordon
- Free Majorana fermion $\leftrightarrow$ quantum Ising chain/2d Ising model
- Jordan-Wigner transformation for general fermionic systems on a 1d lattice

Fermions from bosons

A related phenomenon: emergent fermions.

- Nucleons as solitons in the 3+1d WZW model of pions and kaons
- Statistics transmutation in 2+1d Chern-Simons-matter theories (both relativistic and non-relativistic)

Spin structures

Puzzle:

Fermions on a general space-time need spin structure, bosons do not.

Possible resolutions:

- The bosonic action secretly depends on spin structure
- Bosonization requires summing over spin structures
- Fermionic theory itself involves summing over spin structures (thus fermions are coupled to a dynamical gauge field).

Nucleons and spin structures

Dan Freed (2006) formulated and resolved this puzzle for the case of the 3+1d WZW model.

Claim:

The WZW topological action for odd N subtly depends on spin structure.

That is, the microscopic QCD action depends on spin structure, and this dependence does not go away even when one focuses on the low-energy effective theory describing pions, kaons, and nucleons.

Spin chains

The Hilbert space of a spin chain is $V = \otimes_{j=1}^{N} V_j$, where $V_j \simeq \mathbb{C}^2$.

The algebra of observables is generated by Pauli matrices acting on each site. I will denote them X_j, Y_j, Z_j. They commute for different j.

The Hamiltonian is

$$H^B = \sum_{j=1}^{N} H_j^B,$$

where H_j^B is an observable which has finite range ($[H_j^B, O_k] = 0$ for $|j-k| \gg 0$ for any observable O_k localized on site k).

We will assume that $S = \prod_j Z_j$ commutes with H^B. Then the spin chain has a $\mathbb{Z}_2$ symmetry.

Fermionic chains

The Hilbert space of a fermionic chain is $W = \widehat{\otimes} W_i$, where $W \simeq \mathbb{C}^2$.

The algebra of observables is generated by $c_j, c_j^\dagger$, the fermionic creation-annihilation operators acting on site j. They anti-commute for different j.

The Hamiltonian is

$$H^F = \sum_{j=1}^{N} H_j^F,$$

where each H_j^F is an even (bosonic) observable which has a finite range. The fermion parity operator is

$$\prod_j (-1)^{c_j^\dagger c_j}.$$

Jordan-Wigner transformation

The algebras of observables of the two models are abstractly isomorphic, and there are many different isomorphisms. The Jordan-Wigner transformation is a special isomorphism which maps even local observables of the fermionic chain to local observables of the spin chain commuting with S:

$$c_j \mapsto \frac{1}{2}(X_j + iY_j)\prod_{k=1}^{j-1} Z_k, \quad c_j^\dagger \mapsto \frac{1}{2}(X_j - iY_j)\prod_{k=1}^{j-1} Z_k$$

In particular, this implies:

$$c_j^\dagger c_j \mapsto \frac{1}{2}\left(1 - Z_j\right), \quad (-1)^{c_j^\dagger c_j} \mapsto Z_j.$$

The inverse transformation is also easily written.

Locality of the Jordan-Wigner map

Let us work with Majorana fermions $\gamma_j = \frac{1}{2}(c_j^\dagger + c_j)$ and $\tilde{\gamma}_j = \frac{i}{2}(c_j^\dagger - c_j)$. Then

$$\gamma_j \mapsto X_j \prod_{k=1}^{j-1} Z_k, \quad \tilde{\gamma}_j \mapsto Y_j \prod_{k=1}^{j-1} Z_k.$$

The algebra of even observables is generated by $(-1)^{F_j} = -i\gamma_j\tilde{\gamma}_j$ and the "hopping operators"

$$S_{j+1/2} = i\gamma_j\tilde{\gamma}_{j+1}.$$

They are all idempotent (square to 1). All $S_{j+1/2}$ commute between each other. $S_{j+1/2}$ anti-commutes with $(-1)^{F_j}$ and $(-1)^{F_{j+1}}$ and commutes with $(-1)^{F_k}$ for all other k.

The JW map acts as follows:

$$(-1)^{F_j} \mapsto Z_j, \quad S_{j+1/2} \mapsto X_j X_{j+1}.$$

Higher-dimensional analogs

Prior work:

- Bravyi, Kitaev (2000)
- Levin, Wen (2003)
- R. C. Ball (2005)
- Kitaev (2005)
- Verstraete, Cirac

In all these works, local fermionic systems are shown to be equivalent to locally-constrained bosonic systems. The constraints are similar to Gauss law constraints.

Our approach allows us to identify which bosonic systems are dual to fermionic systems.

JW map in 2d

Consider now a fermionic system on an arbitrary 2d lattice L, with a fermion on each face of a lattice.

The main claim: There exists a local isomorphism between the algebra of even fermionic observables and the algebra of gauge-invariant observables in a $\mathbb{Z}_2$ gauge theory on L with a modified Gauss law.

The modified Gauss law ensures that a fluxon on a face f is accompanied by electric charge on a vertex $v \in f$.

The construction of this map depends on how one chooses v for each f. I will outline the construction for two cases: the square lattice and an arbitrary triangulation. The latter includes the case of fermions on a honeycomb lattice.

Reminder on $\mathbb{Z}_2$ gauge theories

A $\mathbb{Z}_2$ gauge theory has a spin on every edge e. Let X_e, Y_e, Z_e be the corresponding Pauli matrices.

The Hilbert space of a $\mathbb{Z}_2$ gauge theory consists of spin states satisfying the constraints

$$G_v|\Psi\rangle = |\Psi\rangle, \quad \forall v. \tag{1}$$

Here v is an vertex of L. The operators G_v are idempotent, $G_v^2 = 1$, and satisfy $G_v G_{v'} = G_{v'} G_v$. G_v is a generator of the $\mathbb{Z}_2$ gauge transformation supported at v.

Usually one takes

$$G_v = \prod_{e \supset v} X_e,$$

so that G_v flips the eigenvalue of Z_e for all $e \supset v$. Then (1) means that there are no electric charges anywhere.

Electric charges and magnetic fluxes

Inserting electric charge at v_0 means changing the constraint to $G_{v_0} = -1$, and leaving all other constraints unchanged. Conversely, the Gauss law constraint $G_v = 1$ for all v means that $\mathbb{Z}_2$ gauge theory does not have dynamical electric charges.

A magnetic flux at face f is an idempotent operator

$$W_f = \prod_{e \subset f} Z_e.$$

If $W_f|\Psi\rangle = -|\Psi\rangle$, we say that there a fluxon at f. If $W_f|\Psi\rangle = +|\Psi\rangle$, we say there is no fluxon at f.

Gauge-invariant observables

Any $\mathbb{Z}_2$ gauge theory (with the standard Gauss law) can be rewritten as a theory of fluxons whose number is conserved modulo 2.

Mathematically, this is reflected in the fact that the algebra of gauge-invariant observables is generated by W_f for all f and X_e for all e.

All operators X_e commute between each other, all operators W_f commute between each other. X_e anti-commutes with W_f iff $e \subset f$, and commutes with it otherwise.

Physical interpretation: X_e is an operator that creates a pair of fluxons on the two faces which share e.

Kinematic Kramers-Wannier duality in 2d

The fact that the algebra of gauge-invariant observables is generated by W_f and X_e can be used to embed it into the algebra of observables in a spin system living on the dual lattice (i.e. spins $\hat{X}_f, \hat{Y}_f, \hat{Z}_f$ live on faces). The map is

$$W_f \mapsto \hat{Z}_f, \quad X_e \mapsto \prod_{f \supset e} \hat{X}_f$$

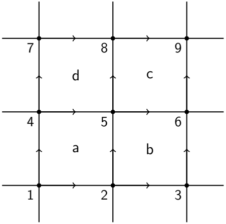
Note that the operators $\hat{Z}_f$ and $\prod_{f \supset e} \hat{X}_f$ generate the algebra of those observables that preserve the net spin parity $\hat{S} = \prod_f \hat{Z}_f$. Thus we can map any $\mathbb{Z}_2$ gauge theory with a local Hamiltonian to a model of spins on the dual lattice whose Hamiltonian is local and preserves $\hat{S}$.

This is a kinematic version of 2d Kramers-Wannier duality (most often discussed in the context of the standard quantum Ising model).

Fermions on a square lattice

Place Majorana fermions $\gamma_f, \tilde{\gamma}_f$ on each face f. We let $(-1)^{F_f} = -i\gamma_f\tilde{\gamma}_f$.

To each edge e we associate a hopping operator S_e which flips the sign of $(-1)^{F_f}$ on the two adjacent faces. This requires choosing an orientation for all e, so let us choose the one in the picture.



Now we can define

$$S_e = i\gamma_{L(e)}\tilde{\gamma}_{R(e)},$$

where $L(e)$ is the face to the left of e, $R(e)$ is the face to the right of e.

The algebra of even observables

The algebra of even observables is generated by the operators $(-1)^{F_f}$ and S_e. The relations are as follows:

- Both S_e and $(-1)^{F_f}$ are idempotent
- $(-1)^{F_f}$ and $(-1)^{F_{f'}}$ commute for all f and f'
- S_e anti-commutes with $(-1)^{F_f}$ if $e \subset f$ and commutes otherwise
- S_e and $S_{e'}$ anti-commute when e and e' share a vertex and are directed E and S or N and W. They commute otherwise.
- $S_{58}S_{56}S_{25}S_{45} = (-1)^{F_a}(-1)^{F_c}$.

Modified $\mathbb{Z}_2$ gauge theory

We place a bosonic spin on every edge, with operators X_e, Y_e, Z_e. As usual, we define $W_f = \prod_{e\subset f} Z_f$.

For every vertex, we impose the Gauss law $G_v|\Psi\rangle = |\Psi\rangle$ where

$$G_v = W_{NE(v)} \prod_{e\supset v} X_e,$$

where $NE(v)$ is the face to the northeast of v.

The condition

$$\prod_{e\supset v} X_e|\Psi\rangle = W_{NE(v)}|\Psi\rangle$$

implements charge-flux attachment: there is electric charge at v iff there is a fluxon at $NE(v)$.

The algebra of gauge-invariant observables

The algebra of gauge-invariant observables is generated by W_f and "fluxon-creation operators"

$$U_e = X_e Z_{r(e)}.$$

Here $r(e)$ is determined by e as follows: translate e by a vector $(-1/2, -1/2)$ to get an edge $\hat{e}$ of the dual lattice, and then let $r(e)$ to be the edge dual to $\hat{e}$ (that is, the unique edge of the original lattice which intersects $\hat{e}$ at midpoint).

It is easy to check that U_e commutes with all G_v and thus is gauge-invariant. The operator U_e is idempotent, just like W_f, and anti-commutes with W_f iff $e \subset f$. In all other cases U_e commutes with W_f.

The bosonization map on a square lattice

In fact, one can check that the relations between W_f and U_e are exactly the same as those between $(-1)^{F_f}$ and S_e. Thus we get the bosonization map

$$(-1)^{F_f} \mapsto W_f, \quad S_e \mapsto U_e.$$

Thus fermion number maps to magnetic flux. This is very similar to the kinematic Kramers-Wannier duality.

Note that this map "works" only in a topologically trivial situation. For a torus there are additional "non-local" relations in the algebra of even fermionic observables, which require an additional projection on the bosonic side. The same is true about the 1d Jordan-Wigner map and kinematic Kramers-Wannier.

Fermions on a 2d triangulation

We place a pair of Majorana fermions $\gamma_f, \tilde{\gamma}_f$ on every face. We let $(-1)^{F_f} = -i\gamma_f\tilde{\gamma}_f$ as before. We choose a branching structure on the triangulation (an orientation of every e so that for every f these orientations do not form a loop). We define $S_e = i\gamma_{L(e)}\tilde{\gamma}_{R(e)}$ as before.

The operators $(-1)^{F_f}$ and S_e generate the algebra of even fermionic observables.

- Both S_e and $(-1)^{F_f}$ are idempotent
- $(-1)^{F_f}$ and $(-1)^{F_{f'}}$ commute for all f and f'
- S_e anti-commutes with $(-1)^{F_f}$ if $e \subset f$ and commutes otherwise
- S_e and $S_{e'}$ anti-commute when e and e' share a vertex and have a particular orientation with respect to branching structure. They commute otherwise.
- $\prod_{e \supset v} S_e$ is some function of $(-1)^{F_f}$ for faces f adjacent to v.

The cup product

To describe the last two relations more precisely, we need to recall some useful notions from algebraic topology.

Suppose we are given a triangulation where the vertices around every face are ordered from 0 to 2. Branching structure gives us such an ordering "for free".

A p-cochain on a triangulation is a function on the set of p-simplices (0-simplex is a vertex, 1-simplex is an edge, 2-simplex is a face). We will only need $\mathbb{Z}_2$-valued functions.

There is a bilinear operation on cochains called the cup product, such that the cup product of a p-cochain and q-cochain is a $(p+q)$-cochain. It is defined as follows:

$$(g_1 \cup g_2)(v_0 \ldots v_{p+q}) = g_1(v_0 \ldots v_p) g_2(v_p \ldots v_{p+q})$$

Here a p-simplex is identified with its set of ordered vertices.

The coboundary operator

We also have the coboundary operator δ which maps a p-cochain to a $(p+1)$-cochain:

$$(\delta g)(v_0 \dots v_{p+1}) = \sum_{j=0}^{p+1} g(v_0 \dots v_{j-1} v_{j+1} \dots v_{p+1}).$$

Properties:

- $\delta^2 = 0$
- $\delta(g_1 \cup g_2) = (\delta g_1) \cup g_2 + g_1 \cup (\delta g_2)$.

But note that the cup product is neither commutative nor supercommutative (which is actually the same thing in our case).

The algebra of even observables

The first "nontrivial" relation looks as follows:

$$S_e S_{e'} = S_{e'} S_e (-1)^{\int \Delta_e \cup \Delta_{e'} + \int \Delta_{e'} \cup \Delta_e}.$$

Here Δ_e is a 1-cochain supported at e and the integral of the 2-cochain is simply the sum of its values on all 2-simplices.

The second "nontrivial" relation is

$$\prod_{e \supset v} S_e = c(v) \prod_{f \in I_{02}(v)} W_f,$$

where $I_{02}(v)$ is the set of faces adjacent to v such that v has number 0 or 2 w.r. to f. The factor $c(v) = \pm 1$ is a c-number whose definition I omit.

Modified $\mathbb{Z}_2$ gauge theory

We place a spin on every edge. The Gauss law is given by $G_v|\Psi\rangle = |\Psi\rangle$, where

$$G_v = \prod_{e \supset v} X_e \prod_{f \subset I_0(v)} W_f,$$

where I_0 is the set of faces adjacent to v such that v has number 0 w.r. to f.

This means that a fluxon on face v is accompanied by electric charge on the vertex 0 of this face.

The bosonization map on a triangulation

Gauge-invariant observables are generated by W_f and fluxon-creating operators U_e. U_e creates two fluxons on the faces adjacent to e and an even number of electric charges:

$$U_e = X_e \prod_{e' \in J(e)} Z_{e'},$$

where $J(e)$ is defined as follows: we first find all faces for which e is the edge 12, and for each such face take its 01 edge.

The bosonization map is

$$(-1)^{F_f} \mapsto W_f, \quad S_e \mapsto U_e.$$

This gives an isomorphism provided the space is simply-connected.

Example 1: fermions hopping on a square lattice

Fermionic Hamiltonian (nearest-neighbor hopping):

$$H^F = t \sum_e (c^\dagger_{L(e)} c_{R(e)} + c^\dagger_{R(e)} c_{L(e)}) + \mu \sum_f c^\dagger_f c_f .$$

Bosonic Hamiltonian:

$$H^B = \frac{t}{2} \sum_e X_e Z_{r(e)} (1 - W_{L(e)} W_{R(e)}) + \frac{\mu}{2} \sum_f (1 - W_f)$$

Note that the "kinetic" term which creates pairs of fluxons is rather elaborate and is non-zero only if the flux is different on the two faces. This reflects that the fermionic Hamiltonian preserves fermion number, not just fermion parity $(-1)^F$.

Interlude: the standard $\mathbb{Z}_2$ gauge theory

The standard $\mathbb{Z}_2$ gauge theory has the standard Gauss law

$$\prod_{e \supset v} X_e = 1$$

and the Hamiltonian:

$$H = g^2 \sum_e X_e + \frac{1}{g^2} \sum_f (1 - W_f).$$

The operator X_e creates a pair of fluxons on the faces adjacent to e.

When we modify the Gauss law, W_f is still gauge-invariant, but X_e is not. One can regard U_e (which on a square lattice is given by $X_e Z_{r(e)}$) as a gauge-invariant version of X_e.

Example 2: a gauge theory dual to a free fermion

Consider now a gauge theory with a modified Gauss law and the Hamiltonian

$$H^B = g^2 \sum_e U_e + \frac{1}{g^2} \sum_f (1 - W_f).$$

The corresponding fermionic Hamiltonian is

$$H^F = g^2 \sum_e i\gamma_{L(e)}\tilde{\gamma}_{R(e)} + \frac{1}{g^2} \sum_f (1 + i\gamma_f \tilde{\gamma}_f).$$

This a free fermion Hamiltonian with nearest neighbor hopping. It does not preserve F, but can be easily diagonalized.

For example, if we start with gauge theory on a regular triangular lattice, the dual fermions live on vertices of the honeycomb lattice. The dispersion law has two Dirac points in the Brillouin zone.

Euclidean gauge theories

Every Hamiltonian lattice gauge theory can be re-written as a Euclidean lattice gauge theory in one dimension higher. The 3d lattice is 2d lattice times discretized time.

For example, the standard $\mathbb{Z}_2$ gauge theory is equivalent to a Euclidean theory with an action

$$S = \sum_f J_f |\delta a(f)|,$$

where a is a $\mathbb{Z}_2$-valued function on edges (i.e. a 1-cochain), and $|\delta a(f)|$ is 1 if the flux of a through f is nontrivial, and 0 otherwise. The couplings J_f are different for space-like and time-like faces.

The theory has a gauge-invariance:

$$a \mapsto a + \delta f,$$

where f is a 0-cochain.

A Chern-Simons-like term

The Euclidean action encodes both the Hamiltonian and the Gauss law. To get a modified Gauss law one needs a term in the action which is gauge-invariant only up to total derivatives.

In the continuum, we can use the Chern-Simons term $S_{CS} = \frac{ik}{4\pi} \int AdA$, since under $A \mapsto A + df$ we have

$$S_{CS}(A + df) - S_{CS}(A) = \frac{ik}{4\pi} \int_{\partial X} fdA.$$

We can write a similar term for a $\mathbb{Z}_2$ gauge theory on a triangulation:

$$S'(a) = \pi i \int_X a \cup \delta a.$$

It leads to the correct modified Gauss law for wavefunctions.

Euclidean gauge theories dual to fermionic systems

Natural guess: the simplest Hamiltonian gauge theory is equivalent to the simplest Euclidean gauge theory with a Chern-Simons-like term:

$$S = \frac{1}{g^2} \sum_f J_f |\delta a(f)| + i\pi \int_X a \cup \delta a.$$

This turns out to be essentially correct, except one needs to come up with a definition of the Chern-Simons-like term for a lattice which is not a 3d triangulation, but is a product of a 2d triangulation and discrete time, or a cubic lattice.

It is remarkable that such simple Euclidean bosonic theories can be equivalent to theories of free fermions. Removing the CS-like term leads to a non-integrable theory dual to the 3d Ising model.

Fermions on a 3d cubic lattice

Place a pair of Majorana fermions $\gamma_c, \tilde{\gamma}_c$ at each cube of a cubic lattice. Choose an orientation for every face f and let

$$(-1)^{F_c} = -i\gamma_c\tilde{\gamma}_c, \quad S_f = i\gamma_{L(f)}\tilde{\gamma}_{R(f)},$$

where $L(f)$ and $R(f)$ are cubes to the left and to the right of the oriented face f.

The algebra of even observables is generated by idempotent operators $(-1)^{F_c}$ and S_f. S_f flips the sign of $(-1)^{F_c}$ for cubes adjacent to f.

There is a constraint for every edge e: $\prod_{f\supset e} S_f$ is some function of the operators $(-1)^{F_c}$ for the cubes adjacent to e.

All this suggests that the bosonic description will involve spins on faces and a Gauss law constraint on each e.

A 2-form gauge theory

In other words, 3d bosonization requires a $\mathbb{Z}_2$-valued 2-form gauge field B, with a 1-form gauge symmetry

$$B \mapsto B + \delta\lambda.$$

With the usual Gauss law constraint, such a system is Kramers-Wannier-dual to ordinary bosonic spins on the dual lattice.

Fermions lead to a modified Gauss law instead. I will simply write down the Euclidean action which gives the right Gauss law:

$$S = \frac{1}{g^2}\sum_c |\delta B(c)| + i\pi \int_X (B \cup B + B \cup_1 \delta B)$$

In the limit $g \to 0$ this reduces to a Walker-Wang model whose input is the category of supervector spaces (the general Walker-Wang model is a 4d TQFT constructed from a braided fusion category).

Bosonization and generalized symmetries

The 2+1d and 3+1d examples illustrate a general rule (Gaiotto, A. K., 2015): in d dimensions, the bosonic dual of a fermionic system has a $(d-2)$-form $\mathbb{Z}_2$ global symmetry with a particular 't Hooft anomaly.

The anomaly can be described by means of a $(d+1)$-dimensional action depending on a $(d-1)$-form $\mathbb{Z}_2$ gauge field C:

$$S_{d+1} = \int_M Sq^2 C.$$

Here $C \in Z^{d-1}(M, \mathbb{Z}_2)$.

For example, for $d=4$, the symmetry acts by $B \mapsto B + \beta$, where β is a 2-cocycle. The action is invariant, but the symmetry cannot be gauged, thanks to the topological term $\int (B \cup B + B \cup_1 \delta B)$.

This anomaly appeared first in Dan's 2006 paper on the WZW model. We now understand its importance.

Topology, higher structures and physics

Bosonization in higher dimensions illustrates nicely the importance of both topology and higher structures (generalized symmetries) in physics.

Both are central themes in Dan's work. I am confident there will be many more applications of these ideas in physics, and Dan will play a major role in discovering them.

Happy birthday, Dan!