Singularities: from L^2 Hodge theory to Seiberg-Witten geometry

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Between Topology and Quantum Field Theory A conference in celebration of Dan Freed's 60th birthday

Goal: special geometry of singularities at special day.

- 2d Gromov-Witten geometry
- 4d Seiberg-Witten geometry

Special Kähler Manifolds

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Abstract: We give an intrinsic definition of the special geometry which arises in global $N = 2$ supersymmetry in four dimensions. The base of an algebraic integrable system exhibits this geometry, and with an integrality hypothesis any special Kähler manifold is so related to an integrable system. The cotangent bundle of a special Kähler manifold carries a hyperkähler metric. We also define special geometry in supergravity in terms of the special geometry in global supersymmetry.

Landau-Ginzburg (LG) model is a quantum theory of singularities that arises from supersymmetric field theories. It is associated to the data of a complex manifold X with a holomorphic function

$$f : X \to \mathbb{C}$$

called (LG) superpotential.

We start with 2d LG models. Its mathematical origin could be traced back to K. Saito's study (1982) of primitive period maps over the universal unfolding of an isolated hypersurface singularity. It produces a so-called Frobenius manifold in modern terminology which becomes the universal structure in mirror symmetry.

The geometry behind K. Saito's theory is the variation of $\frac{\infty}{2}$-Hodge structures ($\frac{\infty}{2}$-VHS), a notion due to Barannikov-Kontsevich in their study of compact Calabi-Yau geometries.

We will be mainly interested in three classes of LG B-models

1) $X = \mathbb{C}^n$, f is a quasi-homogeneous polynomial;
   - mirror to FJRW theory of Landau-Ginzburg A-model

2) $X = (\mathbb{C}^*)^n$, f is a Laurent polynomial;
   - mirror to Gromov-Witten theory on toric varieties

3) $X = \mathbb{C}^n/G$, f is a G-invariant polynomial on $\mathbb{C}^n$.
   - orbifolds

We illustrate relevant structures by the example of primitive form theory, which is an algebraic model for understanding complex oscillatory integral

$$\int_\Gamma e^{f/t}(\cdots)$$

Let

$$f : X = (\mathbb{C}^{n+1}, 0) \to \Delta = (\mathbb{C}, 0)$$

be the germ of an isolated singularity. We consider the quotient

$$\boxed{\Omega_f := \Omega_X^{n+1} / df \wedge \Omega_X^n}.$$

With a choice of holomorphic volume form $d\mathbf{x} = dx_0 \wedge \cdots \wedge dx_n$

$$\Omega_f = \mathrm{Jac}(f) d\mathbf{x}$$

where $\mathrm{Jac}(f) = \mathbb{C}\{x_0, \cdots, x_n\}/(\partial_i f)$ is the Jacobian algebra of f at 0. There is a non-degenerate pairing on Ω_f given by residue

$$\boxed{\mathrm{Res}_f : \Omega_f \otimes \Omega_f \to \mathbb{C}}.$$

The space Ω_f is the leading part of the Brieskorn lattice

$$\boxed{\mathcal{H}_f^{(0)} := \Omega_X^{n+1} / df \wedge d\Omega_X^{n-1}}.$$

There is a well-defined operator, denoted by a formal variable t,

$$t : \mathcal{H}_f^{(0)} \to \mathcal{H}_f^{(0)}.$$

Given $\alpha \in \mathcal{H}_f^{(0)}$, there exists n-form β such that $\alpha = d\beta$, then

$$t \cdot \alpha := -df \wedge \beta \in \mathcal{H}_f^{(0)}.$$

Symbolically,

$$\boxed{t = -\frac{df}{d}} : \alpha \to -df \wedge d^{-1}\alpha.$$

Descendant forms

Given $\omega \in \mathcal{H}_f^{(0)}$ and $k \geq 0$, we define its k-th descendant form

$$\boxed{\omega^{(-k)} := (-t)^k \omega} \in \mathcal{H}_f^{(0)}.$$

Descendant forms give natural semi-infinite filtrations

$$\cdots \subset \mathcal{H}_f^{(-k)} \subset \mathcal{H}_f^{(-k+1)} \subset \cdots \subset \mathcal{H}_f^{(-1)} \subset \mathcal{H}_f^{(0)}$$

The formal completion of $\mathcal{H}_f^{(0)}$ w.r.t. this t-adic topology identifies

$$\boxed{\hat{\mathcal{H}}_f^{(0)} = \Omega_X^{n+1}[[t]]/(td + df)\Omega_X^n[[t]]}.$$

Higher residue

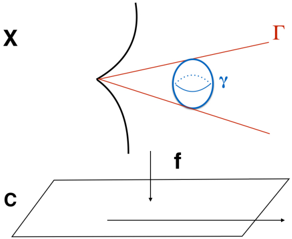
K. Saito defines a sesqui-linear higher residue pairing

$$\boxed{K_f : \hat{\mathcal{H}}_f^{(0)} \times \hat{\mathcal{H}}_f^{(0)} \to t^n \mathbb{C}[[t]]}.$$

whose leading coefficient in t coincides with the residue pairing on $\Omega_f = \mathcal{H}_f^{(0)}/t\mathcal{H}_f^{(0)}$. In other words

$$K_f = t^n(\mathrm{Res}_f + \text{higher orders in } t).$$

Oscillatory integral and Period



Given an element $\omega \in \mathcal{H}_f^{(0)}$, we can consider the oscillatory integral

$$\int_{\Gamma} e^{f/t} \omega$$

Under Laplace transformation, this is related to the period map

$$\boxed{\int_{\gamma} \frac{\omega}{df}}.$$

In Seiberg-Witten curve geometry, we have a 2-form ω with $\omega = d\lambda$. Then the SW period map is related to the period map of the first descendant of ω

$$\int_\gamma \lambda = \int_\gamma \frac{df \wedge d^{-1}\omega}{df} = -\int_\gamma \frac{\omega^{(-1)}}{df}.$$

As we will see, this observation allows us to obtain SW differential arising from higher dimensional geometry (in particular 3-fold fibration) via different choices of the descendants.

Primitive form

Let $\mathcal{M}$ be the miniversal deformation space of $f(x^i)$, represented by $F(x^i, \lambda^\alpha), \lambda \in \mathcal{M}$. Using higher residues, K. Saito [1982] constructed a special family of holomorphic volume forms $\xi(x, \lambda) = \varphi(x, \lambda) d\mathbf{x}$, called primitive form. It determines a set of flat coordinates $\{\tau^\alpha\}$ on $\mathcal{M}$ such that

$$\left(t \frac{\partial}{\partial \tau^\alpha} \frac{\partial}{\partial \tau^\beta} - A_{\alpha\beta}^{\gamma}(\tau) \frac{\partial}{\partial \tau^\gamma} \right) \int e^{F/t} \xi = 0.$$

$A_{\alpha\beta}^{\gamma}$ is nowadays the Yukawa coupling of 2d LG B-models.

On Calabi-Yau geometries, the analogue of primitive form is called $\frac{\infty}{2}$-period map in Barannikov-Kontsevich construction. In Gromov-Witten theory, this is another face of Givental's J-function.

We would like to extend this theory to the case when

$$\mathrm{Crit}(f) = \text{compact}$$

Our construction is based on developing a general L^2-Hodge theory for LG model that allows us to perform the analogue of Barannikov-Kontsevich construction on compact Calabi-Yau's.

This part is joint work in progress with Hao Wen.

Barannikov-Kontsevich construction

Barannikov and Kontsevich's approach to deformation of compact Calabi-Yau is to study dGBV algebra

$$(\mathrm{PV}(X), \bar{\partial}, \partial, [,]).$$

Here $\mathrm{PV}(X) = \Omega^{0,*}(X, \wedge^* T_X)$, ∂ is the divergence operator w.r.t. CY volume form, $[-,-]$ is the Schouten-Nijenhuis bracket.

The Bogomolov-Tian-Todorov lemma, L^2 pairing, and Hodge-to-de Rham degeneration leads to a smooth moduli of deformations that carries a Frobenius manifold structure.

In the Landau-Ginzburg case, X is non-compact. There are two natural dGBV algebras

$$(\mathrm{PV}_c(X), \bar{\partial}_f, \partial, [,]) \quad \text{and} \quad (\mathrm{PV}(X), \bar{\partial}_f, \partial, [,])$$

Here $\mathrm{PV}_c(X)$ is the subspace of compactly supported polyvector fields. They are used to compute the correlation functions of LG B-models for isolated quasi-homogeneous singularities (L-Li-Saito).

However, for general f with compact critical set

1. $\mathrm{PV}(X)$ is too big for integration, or for higher residue theory.
2. $\mathrm{PV}_c(X)$ is too small for Hodge decompositions.

Idea: to find a 'nice' space in between $\mathrm{PV}_c(X)$ and $\mathrm{PV}(X)$ which enjoys both higher residue theory and Hodge decompositions such that the Barannikov-Kontsevich construction applies directly.

We will be interested in the data (X, Ω_X, g, f) where

- X is a non-compact complex manifold equipped with a holomorphic volume form Ω_X
- g is a complete Kähler metric
- $f : X \to \mathbb{C}$ holomorphic with $\mathrm{Crit}(f)$ compact.

We say (X, g) have a bounded geometry if:

1) the injective radius is positive;
2) there exists a uniform bound for each order of covariant derivative of the Riemannian curvature tensor.

Furthermore, if all covariant derivatives of Ω_X and Ω_X^{-1} are bounded on X, we say (X, Ω_X, g) has a bounded CY geometry.

Let $\mathcal{A}(X) = \oplus \mathcal{A}^{i,j}(X)$ be the space of differential forms and

$$\bar{\partial}_f := \bar{\partial} + \partial f \wedge$$

be the twisted Cauchy-Riemann operator. We have densely defined operators $\bar{\partial}_f, \bar{\partial}_f^*, \partial$ on $L^2(X)$ and especially the twisted Laplacian

$$\Delta_f := [\bar{\partial}_f, \bar{\partial}_f^*] = \bar{\partial}_f \bar{\partial}_f^* + \bar{\partial}_f^* \bar{\partial}_f$$

which is an elliptic operator of order 2.

A holomorphic function f on X is said to be strongly elliptic if for $\forall \epsilon > 0, k \geq 2$,

$$\epsilon |\nabla f(z)|^k - |\nabla^k f(z)| \to +\infty \text{ as } z \to \infty.$$

Remark: There are several different tameness conditions in literature, such as elliptic condition, tame condition, M-tame condition, strongly tame condition. Our strongly elliptic assumption is stronger than the above conditions, but on the other hand applies to general interesting cases as we will see.

The following Hodge decomposition was proved by Klimek-Lesniewski (1991) and later also studied by Fan (2011).

Theorem

Let (X, g) be a bounded geometry, and f be strongly elliptic. Then the twisted Laplacian Δ_f has purely discrete spectrum.

We have the Hodge decomposition:

$$L^2_{\mathcal{A}}(X) = \mathrm{Ker}(\Delta_f) \oplus \mathrm{Im}(\bar{\partial}_f) \oplus \mathrm{Im}(\bar{\partial}_f^*).$$

Kähler-Hodge identity

Since g is Kähler, Δ_f behaves much alike the usual Laplacian $\Delta_{\bar\partial}$ on compact Kähler manifolds. In particular, the following generalized Kähler-Hodge identities are known to hold and play important roles in supersymmetric quantum mechanical models:

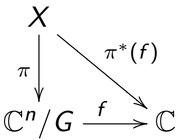
$$[\partial_f, \Lambda] = -i\bar\partial_f^*, \quad [\bar\partial_f, \Lambda] = i\partial_f^*,$$
$$[\partial_f^*, L] = -i\bar\partial_f, \quad [\bar\partial_f^*, L] = i\partial_f.$$

As a consequence, we have another Hodge decomposition

$$L^2_{\mathcal{A}}(X) = \mathrm{Ker}(\Delta_f) \oplus \mathrm{Im}(\partial_f) \oplus \mathrm{Im}(\partial_f^*).$$

The following three classical examples of Landau-Ginzburg models satisfy the strongly elliptic condition:

1) $X = \mathbb{C}^n$, g the standard flat metric, f is a quasi-homogeneous polynomial with isolated singularity;
2) $X = (\mathbb{C}^*)^n$, $g := \frac{1}{2}\sum_i (\frac{dz_i}{z_i} \otimes \frac{d\bar{z}_i}{\bar{z}_i} + \frac{d\bar{z}_i}{\bar{z}_i} \otimes \frac{dz_i}{z_i})$, f is a convenient non-degenerate polynomial;
3) $\pi : X \to \mathbb{C}^n/G$ be a crepant resolution, g be an asymptotic locally Euclidean metric on X. Let f be a G-invariant polynominal on $\mathbb{C}^n$ with an isolated singularity



Then $(X, g, \pi^*(f))$ is a LG model with $\mathrm{Crit}(\pi^*(f))$ compact.

We define the f-twisted Sobolev spaces $\mathrm{PV}_{f,k}(X)$ by

$$\mathrm{PV}_{f,k}(X) := \{\alpha | |\nabla f|^i \nabla^j \alpha \in L^2(X), \forall i + j \leq k\},$$

The spaces $\mathrm{PV}_{f,\infty}(X)$ is defined as the infinite intersection

$$\mathrm{PV}_{f,\infty}(X) := \cap_{k\geq 0} \mathrm{PV}_{f,k}(X).$$

Theorem (H.Wen, -)

$\mathrm{PV}_{f,\infty}(X)$ *is closed under wedge product,* $\bar{\partial}_f, \partial$ *and decomposition into components of Hodge degrees. In particular,* $(\mathrm{PV}_{f,\infty}(X), \bar{\partial}_f, \partial, [,])$ *forms a dGBV algebra. The embeddings*

$$(\mathrm{PV}_c(X), \bar{\partial}_f,) \subset (\mathrm{PV}_{f,\infty}(X), \bar{\partial}_f) \subset (\mathrm{PV}(X), \bar{\partial}_f)$$

are all quasi-isomorphisms.

Hodge-to-de Rham Degeneration

Comparing to the compact CY case, we do not have the $\partial\bar{\partial}_f$-Lemma. We have a similar, though weaker, version in our LG model that suffices to establish the following.

Theorem (H.Wen, -)

Let (X, g, Ω_X) *be a bounded CY geometry and* f *be strongly elliptic. Then Hodge-to-de Rham degeneration holds for the complex* $(\mathrm{PV}_{f,\infty}(X)[[t]], \bar{\partial}_f + t\partial)$*.*

In particular, we know that $H(\mathrm{PV}_{f,\infty}(X)[[t]], \bar{\partial}_f + t\partial)$ is a free $\mathbb{C}[[t]]$-module of finite rank, and the dgla $(\mathrm{PV}_{f,\infty}(X)[[t]], \bar{\partial}_f + t\partial, [-,-])$ is smooth formal

Higher residue pairing

We define the sesquilinear pairing

$$\mathsf{K}_f : \mathrm{PV}_{f,\infty}(X)[[t]] \times \mathrm{PV}_{f,\infty}(X)[[-t]] \to \mathbb{C}[[t]]$$

by

$$\mathsf{K}_f(f(t)\alpha, g(t)\beta) = f(t)g(-t)\int_X (\alpha\beta \vdash \Omega_X) \wedge \Omega_X.$$

- K_f descends to a sesquilinear paring on the cohomology $H(\mathrm{PV}_{f,\infty}(X)[[t]], \bar{\partial}_f + t\partial)$. It generalizes K.Saito's higher residue pairing for isolated singularities.
- Modulo t, K_f defines a non-degenerate pairing on $H(\mathrm{PV}_{f,\infty}(X), \bar{\partial}_f)$. This is the analogue of residue pairing.

Harmonic forms give a splitting of the Hodge filtration. As a consequence, we obtain our main theorem.

Theorem (H.Wen, -)

Let (X, g, Ω_X) be a bounded Calabi-Yau geometry, f be a strongly elliptic holomorphic function. Then there exists a Frobenius manifold structure on the cohomology $H(\mathrm{PV}(X), \bar{\partial}_f)$.

Remarks:

- Douai and Sabbah constructed Frobenius manifolds for certain class of LG models on $(\mathbb{C}^*)^n$.
- Kazarkov, Kontsevich and Pantev proved the Hodge-to-de Rham degeneration for a class of LG models that allows certain good compactification $\bar{f} : \bar{X} \to \mathbb{P}^1$.
- Kaledin proved a noncommutative version of Hodge-to-de Rham degeneration on general smooth and proper dg-categories.

Seiberg-Witten geometry

We are interested in 4D N $=2$ SCFT. Seiberg-Witten discovered that for many theories the low energy effective theory on the Coulomb branch could be described by a Seiberg-Witten curve fibered over the moduli space:

$$F(x, z; \lambda_\alpha) = 0$$

Here λ_α's are the parameters including coupling constants, mass parameters, and expectation values for Coulomb branch operators. The period integral of an appropriate 1-form over the Riemann surface $F(x, z; \lambda_\alpha) = 0$ determines the low energy photon coupling.

Singularity and 4d $N = 2$ SCFT

We consider type IIB string theory on the following background:

$$\mathbb{R}^{3,1} \times X$$

Here X is a 3-fold weighted homogeneous isolated singularity. It is argued to define 4d $N = 2$ SCFT (Shapere, Vafa, 99; D. Xie, Yau, 15). The SW solution is associated to a three-fold fibration

$$F(x_1, x_2, x_3, x_4; \lambda_\alpha) = 0$$

which may or may not be reduced to the curve geometry. This suggests that the more general SW solution could be three-fold fibrations rather than curves. Our goal is to figure out the corresponding SW differential.

Consider an isolated weighted homogeneous 3-fold singularity:

$$f : \mathbb{C}^4 \to \mathbb{C}, \quad f(\lambda^{q_i} x^i) = \lambda f(x^i), \quad q_i > 0, \quad \lambda \in \mathbb{C}^*.$$

The rational number

$$\hat{c}_f = \sum_i (1 - 2q_i)$$

is the central charge of the 2d (2,2) SCFT defined by LG model with superpotential f. To define a 4d SCFT we require

$$\sum_i q_i > 1 \Longleftrightarrow \hat{c}_f < 2.$$

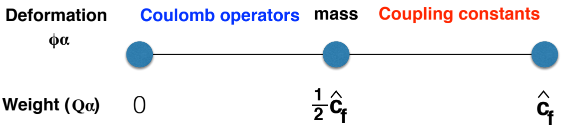
The Coulomb branch of the associated N=2 SCFT is described by the local moduli of miniversal deformations of f:

$$F(x_i, \lambda_\alpha) = f(x_i) + \sum_{\alpha=1}^{\mu} \lambda_\alpha \phi_\alpha$$

where $\{\phi_\alpha\}$ is a basis of the Jacobi algebra Jac(f). Define

$$\boxed{[\lambda_\alpha] = \frac{1 - Q_\alpha}{\sum_i q_i - 1}}, \qquad Q_\alpha = \text{homogeneous weight of } \phi_\alpha.$$

- $[\lambda_\alpha] < 1$: Coupling constants
- $[\lambda_\alpha] = 1$: Mass parameters
- $[\lambda_\alpha] > 1$: Expectation value of Coulomb branch operators



Period map

Given a family of holomorphic volume forms $\xi(x^i, \lambda_\alpha)$, we consider the period map

$$\mathcal{P} : \mathcal{M} \to H^3, \quad \{\lambda_\alpha\} \to \int_\gamma \frac{\xi}{dF}.$$

Here the integration is over vanishing cycles γ in $F^{-1}(0)$ and $H^3 = H^3(F^{-1}(0), \mathbb{C})$ is the dual space. For simplicity, let us consider the case with no mass parameters. Then H^3 has a natural symplectic structure induced dually from the intersection pairing. This allows choices of an electro-magnetic charge lattice from H_3.

$\mathcal{M}$ is a deformation by Coulomb operators,

$$\dim(\mathcal{M}) = \frac{1}{2} \dim H^3.$$

Seiberg-Witten differential

The low energy effective theory of Coulomb branch is described by Seiberg-Witten (SW) geometry.

Question

What is the Seiberg-Witten differential associated to singularities?

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Naive guess: the SW differential is the family of 3-forms

$$\frac{\xi}{dF}, \qquad \text{where} \qquad \xi = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

The $N = 2$ prepotential comes from the corresponding period map. This is indeed true in the case of ADE singularities ($\hat{c}_f < 1$).

If we go beyond ADE singularities, this no longer holds.

Seiberg-Witten differential

Solution [L-Xie-Yau, 2018]: for 3-fold singularity

$$\boxed{\xi \text{ primitive form} \Longrightarrow \frac{\xi}{dF} \text{ SW differential}}$$

where ξ is K. Saito's primitive form.

The main support for the connection between primitive form and SW differential is about the lagrangian/integrability condition

$$\langle d\mathcal{P}, d\mathcal{P} \rangle = 0$$

for the existence of N $= 2$ prepotential arising from SW period map. The verification of such integrability requires a connection between the intersection pairing for vanishing homology and the period map. Primitive form provides precisely such a relationship.

Curve v.s. 3-fold geometry

This result also generalizes to curve geometry

$$\boxed{\xi \text{ primitive form} \Longrightarrow \frac{\xi^{(-1)}}{dF} \text{ SW differential}}$$

The SW differential for three-fold geometry picks up ξ instead of $\xi^{(-1)}$ for curve geometry.

There is also a 5d hypersurface singularity example. The analogue discussion implies that the SW differential is expected to be

$$\frac{\xi^{(1)}}{dF}.$$

Examples of primitive forms: ADE type ($\hat{c}_f < 1$)

We consider

$$f(x) = x_1^2 + x_2^2 + x_3^k + x_4^N, \quad \frac{1}{k} + \frac{1}{N} > \frac{1}{2}.$$

This example can be reduced to curve geometry. The primitive form is trivial in this case and doesn't depend on the deformation parameter

$$\xi = dx_1 \wedge \cdots \wedge dx_4.$$

The 3-fold SW differential is given by

$$\lambda = \frac{\xi}{dF}$$

Simple elliptic singularity ($\hat{c}_f = 1$)

We consider

$$f(x) = x_1^3 + x_2^3 + x_3^3 + x_4^2$$

The miniversal deformation is

$$F = f + \lambda_1 + \lambda_2 x_1 + \lambda_3 x_2 + \lambda_4 x_3 + \lambda_5 x_1 x_2 + \lambda_6 x_2 x_3 + \lambda_7 x_3 x_1 + \lambda_8 x_1 x_2 x_3.$$

Primitive form of this example is nontrivial and is not unique. They depend only on the marginal parameter λ_8 described as follows:

$$\xi = \frac{dx_1 \wedge \cdots \wedge dx_4}{P(\lambda_8)}$$

where $P(\lambda_8)$ is a period on the cubic elliptic curve

$$\{x_1^3 + x_2^3 + x_3^3 + \lambda_8 x_1 x_2 x_3 = 0\} \subset \mathbb{P}^2.$$

General singularities ($\hat{c}_f > 1$)

For general singularities with $\hat{c}_f > 1$, there exists a highly nontrivial mixing between relevant and irrelevant deformations. The close formula of primitive form is unknown. Here is one example

$$f = x_1^2 + x_2^2 + x_3^3 + x_4^7.$$

This is type E_{12} of the unimodular exceptional singularities. The miniversal deformation is the following

$$F(x,\lambda) = f + \lambda_1 + \lambda_2 x_4 + \lambda_3 x_4^2 + \lambda_4 x_3 + \lambda_5 x_4^3 + \lambda_6 x_3 x_4 + \lambda_7 x_4^4 + \lambda_8 x_3 x_4^2 + \lambda_9 x_4^5 + \lambda_{10} x_3 x_4^3 + \lambda_{11} x_3 x_4^4 + \lambda_{12} x_3 x_4^5.$$

Here $\lambda_1, \cdots, \lambda_{11}$ are relevant deformations, and λ_{12} is an irrelevant deformation.

There is a recursive formula to compute general primitive forms perturbatively [L-Li-Saito, 2015]. For E_{12}, it gives (up to order 10)

$$\zeta = (\varphi(x,\lambda) + O(\lambda^{11}))dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

where

$$\begin{aligned}
\varphi(x,\lambda) = 1 &+ \frac{4}{3\cdot 7^2}\lambda_{11}\lambda_{12}{}^2 - \frac{64}{3\cdot 7^4}\lambda_{11}^2\lambda_{12}{}^4 - \frac{76}{3^2\cdot 7^4}\lambda_{10}\lambda_{12}{}^5 + \frac{937}{3^3\cdot 7^5}\lambda_9\lambda_{12}{}^6 + \frac{218072}{3^4\cdot 5\cdot 7^6}\lambda_{11}^3\lambda_{12}{}^6 \\
&+ \frac{1272169}{3^4\cdot 5\cdot 7^7}\lambda_{10}\lambda_{11}\lambda_{12}{}^7 + \frac{28751}{3^4\cdot 7^7}\lambda_8\lambda_{12}{}^8 - \frac{1212158}{3^4\cdot 7^8}\lambda_9\lambda_{11}\lambda_{12}{}^8 - \frac{38380}{3^3\cdot 7^8}\lambda_7\lambda_{12}{}^9 \\
&+ (\frac{1}{7^2}\lambda_{12}{}^3 - \frac{101}{5\cdot 7^4}\lambda_{11}\lambda_{12}{}^5 + \frac{1588303}{3^4\cdot 5\cdot 7^7}\lambda_{11}^2\lambda_{12}{}^7 + \frac{378083}{3^4\cdot 5\cdot 7^7}\lambda_{10}\lambda_{12}{}^8 - \frac{108144}{3\cdot 7^8}\lambda_9\lambda_{12}{}^9)x_3 \\
&+ (\frac{1447}{3^3\cdot 7^6}\lambda_{12}{}^7 - \frac{71290}{3^3\cdot 7^8}\lambda_{11}\lambda_{12}{}^9)x_4 - \frac{45434}{3^4\cdot 7^8}\lambda_{12}{}^{10}x_3x_4 \\
&- (\frac{53}{3^2\cdot 7^4}\lambda_{12}{}^6 - \frac{46244}{3^3\cdot 7^7}\lambda_{11}\lambda_{12}{}^8)x_3^2 + \frac{22054}{3^4\cdot 7^7}\lambda_{12}{}^9x_3^3.
\end{aligned}$$

In particular, the Seiberg-Witten differential of this example is not given by a rescaling of the trivial form and depends on the coupling constants in a highly nontrivial way.

Happy birthday, Super and Symmetric Dan!