A Hilbert bundle description of differential K-theory

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Attribution

Joint work with Alexander Gorokhovsky

Slogans

Differential K-theory is a "K-theory of finite dimensional vector bundles with connections".

It is closely linked to local index theory, as will be described.

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Today: Differential K-theory as a "K-theory of infinite dimensional vector bundles with (super)connections".

Some motivation:

1. It unifies various earlier models for differential K-theory.
2. The analytic index becomes almost tautological.
3. The even and odd cases can be treated similarly.
4. Extension to twisting by H^3.

Structure of the talk

- Summary of differential K-theory
- Superconnections on Hilbert bundles
- Infinite dimensional cycles
- Twisted differential K-theory

K-theory

M is a smooth manifold.

$K^0(M)$ is the free abelian group generated by isomorphism classes of finite dimensional complex vector bundles on M, quotiented by the relations $[E_2] = [E_1] + [E_3]$ if there is a short exact sequence

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0.$$

Generators of differential K-theory

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A generator for $\check{K}^0(M)$ is a quadruple $\mathcal{E} = (E, h^E, \nabla^E, \omega)$, where

- E is a finite dimensional complex vector bundle on M.
- h^E is a Hermitian metric on E.
- ∇^E is a Hermitian connection on E.
- $\omega \in \Omega^{odd}(M)/\operatorname{Im}(d)$.

Relations for $\check{K}^0(M)$

Given three such quadruples, we impose the relation

$$\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$$

if there is a short exact sequence of Hermitian vector bundles

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0,$$

and

$$\omega_2 = \omega_1 + \omega_3 - CS\left(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}\right) \in \Omega^{odd}(M)/\operatorname{Im}(d).$$

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Here the Chern-Simons form CS satisfies

$$dCS\left(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}\right) = \mathrm{ch}\left(\nabla^{E_2}\right) - \mathrm{ch}\left(\nabla^{E_1}\right) - \mathrm{ch}\left(\nabla^{E_3}\right).$$

Exact sequences

Quotienting by the relations defines $\check{K}^0(M)$. There are a forgetful map

$$f : \check{K}^0(M) \to K^0(M),$$

and a Chern character map

$$\mathrm{Ch} : \check{K}^0(M) \to \Omega_K^{even}(M)$$

coming from

$$\mathrm{Ch}(E, h^E, \nabla^E, \omega) = \mathrm{ch}(\nabla^E) + d\omega.$$

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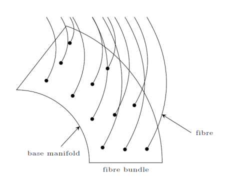
$$0 \longrightarrow K^{-1}(M; \mathbb{R}/\mathbb{Z}) \longrightarrow \check{K}^0(M) \xrightarrow{\mathrm{Ch}} \Omega_K^{even}(M) \longrightarrow 0$$

$$0 \longrightarrow \frac{\Omega^{odd}(M)}{\Omega_K^{odd}(M)} \longrightarrow \check{K}^0(M) \xrightarrow{f} K^0(M) \longrightarrow 0$$

Atiyah-Singer families index theorem



Suppose that $\pi : M \to B$ is a fiber bundle.

Topological assumptions: The fibers are compact and even dimensional. The fiberwise tangent bundle is *spin*^{c}.

Geometric assumptions: Riemannian metrics on the fibers, Hermitian connection on the associated *spin*^{c} line bundle.

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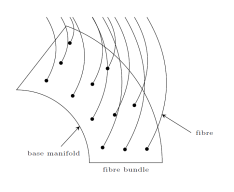

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There are index maps

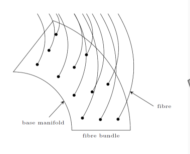
$$\mathrm{ind}_{an}, \mathrm{ind}_{top} : K^0(M) \to K^0(B).$$

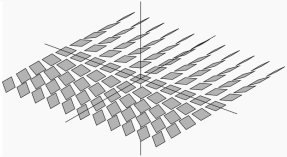
Atiyah-Singer families index theorem

$$\mathrm{ind}_{an} = \mathrm{ind}_{top} \, .$$

Index theorem in differential K-theory

Suppose in addition that there is a horizontal distribution on the fiber bundle.





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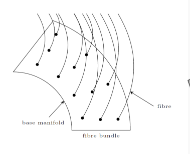

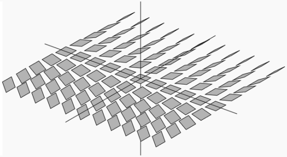

(Freed-L.) There are index maps

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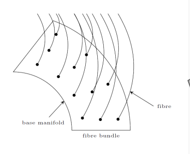

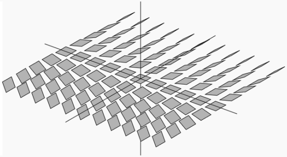

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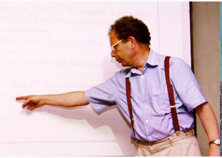
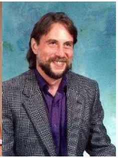
Their construction uses local index theory methods.

Index theorem in differential K-theory

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as maps from $\check{K}^0(M)$ *to* $\check{K}^0(B)$.

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Applying f, one recovers the Atiyah-Singer families index theorem. Applying Ch, one recovers Bismut's local version of the families index theorem.

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- $\mathbb{R}/\mathbb{Z}$-index theorem
- Computation of $\mathbb{R}/\mathbb{Z}$-valued eta invariants.
- Computation of the determinant line bundle, along with its Quillen metric and compatible connection (up to isomorphism).

Motivation

From the viewpoint of analytic index theory, it is more natural to use infinite dimensional vector bundles.

Unbounded Kasparov KK-theory: $K^0(M) \cong KK^0(\mathbb{C}, C(M))$, the latter being given in terms of unbounded Fredholm operators on $\mathbb{Z}_2$-graded Hilbert $C(M)$-modules. Can we give a model for differential K-theory along these lines?

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Problem: This doesn't make sense if E is infinite dimensional.

Solution: Replace the connection ∇ by a superconnection.

Superconnections

E is a finite dimensional $\mathbb{Z}_2$-graded vector bundle on M.

(Quillen) A superconnection on E is a sum

$$A = A_{[0]} + A_{[1]} + A_{[2]} + \dots,$$

where

- $A_{[0]} \in \Omega^0(M; \mathrm{End}_{odd}(E))$
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In the previous description of $\check{K}^0(M)$, you can replace connections by superconnections.

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Say that $\mathcal{H} \to M$ is a $\mathbb{Z}_2$-graded Hilbert bundle. We want to be able to talk about superconnections on $\mathcal{H}$.

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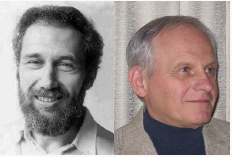
Say that H is a fiber of the bundle. $U(H)$ is too big. We will restrict this using a pseudodifferential calculus based on a "Dirac operator" D.

Abstract Sobolev spaces



Say H is a $\mathbb{Z}_2$-graded Hilbert space,

$D = \begin{pmatrix} 0 & \partial_+^* \\ \partial_+ & 0 \end{pmatrix}$ is a self-adjoint operator.

Assume that $\operatorname{Tr} e^{-\theta D^2} < \infty$ for all $\theta > 0$.

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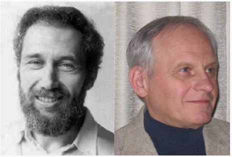

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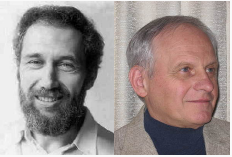

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For $s \in \mathbb{Z}^{<0}$, put $H^s = (H^{-s})^*$.

Put $H^\infty = \bigcap_{s \geq 0} H^s$.

Abstract pseudodifferential operators

Definition

op^k consists of the closed operators F on H so that $F(H^\infty) \subset H^\infty$ and for all $s \in \mathbb{Z}$, F extends to a bounded operator from H^s to H^{s-k}.

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The space of "Dirac-type operators":

$$\mathcal{P} = \left\{ \begin{pmatrix} 0 & P_+^* \\ P_+ & 0 \end{pmatrix} \in op^1 : \frac{1}{\sqrt{P^2+1}} \in op^{-1} \right\}.$$

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Clearly $D \in \mathcal{P}$.

Lemma

$\mathcal{P}$ is closed under order-zero perturbations.

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A map $\mathbb{R}^k \to G$ is declared to be "smooth" if it preserves the smooth maps $\mathbb{R}^k \to H^s$ and $\mathbb{R}^k \to op^k$.

Here H^s and op^k have Fréchet topologies.

Superconnection

Suppose that $\mathcal{H} \to M$ is a $\mathbb{Z}_2$-graded Hilbert bundle with structure group G. It now makes sense to say that a superconnection on $\mathcal{H}$ is a sum

$$A = A_{[0]} + A_{[1]} + A_{[2]} + \dots,$$

where

- $A_{[0]} \in \Omega^0(M; \mathcal{P})$
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Then

$$\mathrm{ch}(A) = \mathrm{Tr}_s\, e^{-A^2} \in \Omega^{even}(M)$$

makes sense, using a Duhamel expansion of e^{-A^2}.

Relative Chern character

If $A_{[0]} - A'_{[0]} \in \Omega^0(M; op^0)$, put

$$\eta(A, A') = \int_0^1 \mathrm{Tr}_s\left(\frac{dB}{dt} e^{-B^2(t)}\right) dt,$$

where $B(t) = (1 - t)A + tA'$.

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Suppose that $A_{[0]}$ is fiberwise invertible. Put

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Then

$$d\eta(A, \infty) = \mathrm{ch}(A).$$

A Hilbert bundle description of differential K-theory

Generators for $\check{K}^0(M)$

Generators are triples $(\mathcal{H}, A, \omega)$, where

- $\mathcal{H} \to M$ is a $\mathbb{Z}_2$-graded Hilbert bundle with structure group G.
- A is a superconnection on $\mathcal{H}$.
- $\omega \in \Omega^{odd}(M)/\operatorname{Im}(d)$.

Relations for $\check{K}^0(M)$

1. $$[\mathcal{H}, A, \omega] + [\mathcal{H}', A', \omega'] = [\mathcal{H} \oplus \mathcal{H}', A \oplus A', \omega + \omega']$$

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Theorem (Gorokhovsky-L.) The natural map $\check{K}^0_{stan}(M) \to \check{K}^0(M)$ is an isomorphism, where $\check{K}^0_{stan}(M)$ is the "standard" differential K-group defined using finite dimensional vector bundles and connections.

Comparison map

The inverse map $q : \check{K}^0(M) \to \check{K}^0_{stan}(M)$ in a special case:

Suppose that $\mathrm{Ker}(A_{[0]})$ forms a $\mathbb{Z}_2$-graded finite dimensional vector bundle on M.

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Then

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where $B = (I - Q)A(I - Q) + QA_{[1]}Q$.

Unification

The Hilbert bundle version $\check{K}^0(M)$ of differential K-theory unifies some other models. First, the natural map $\check{K}^0_{stan}(M) \to \check{K}^0(M)$ is an isomorphism.

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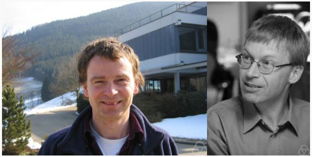

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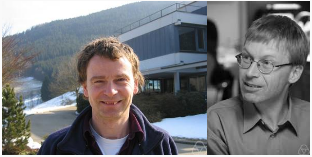

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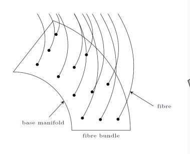
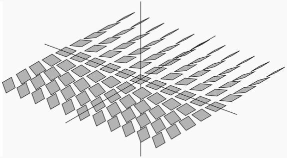
On the other hand, there are no obvious comparison maps with the Hopkins-Singer model for differential K-theory.

Pushforward





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Topological assumptions: The fibers are compact and even dimensional. The fiberwise tangent bundle is *spin*^{c}.

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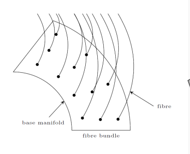

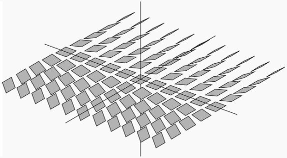

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There was an analytic index map (Freed-L.)

$$\mathrm{ind}_{an} : \check{K}^0_{stan}(M) \to \check{K}^0_{stan}(B).$$

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$$\omega' = \int_{M/B} \mathrm{Td}\left(\nabla^{T^V M}\right) \wedge \omega + \lim_{u \to 0} \eta((\pi_* A)_u, \pi_* A) \in \Omega^{odd}(B) / \operatorname{Im}(d).$$

Pushforward theorem

Theorem (Gorokhovsky-L.)

If (E, ∇^E, ω) is a generator of $\check{K}^0_{stan}(M)$ then

$$\mathrm{ind}_{an}([E, \nabla^E, \omega]) = [\mathcal{H}, \pi_* A, \omega']$$

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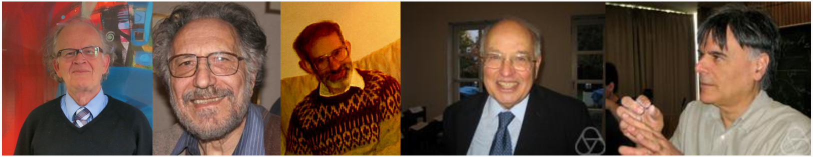
Can one also push forward infinite dimensional cycles?
Formally yes, but there are some technical questions.

A Hilbert bundle description of differential K-theory

Twisted K-theory



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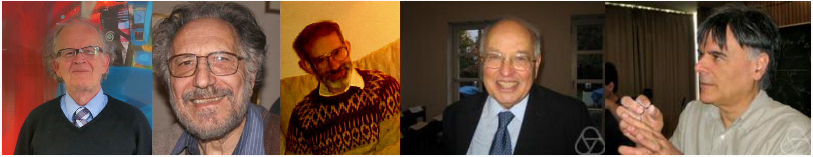

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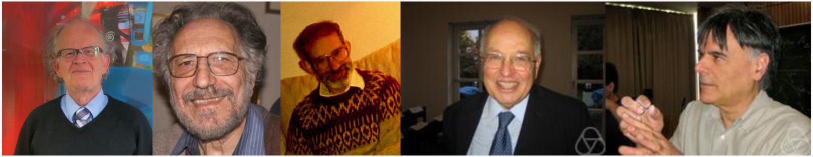

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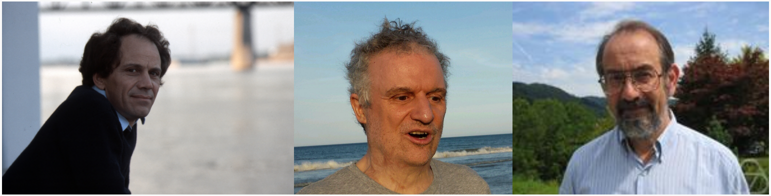
Can one extend the previous model from differential K-theory to twisted differential K-theory?

Gerbes



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Data for a gerbe:

- An open cover $\{U_\alpha\}$ of M.
- A complex line bundle $\mathcal{L}_{\alpha\beta}$ on $U_\alpha \cap U_\beta$.
- An isomophism $\mu_{\alpha\beta\gamma} : \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \to \mathcal{L}_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

$U(1)$-connection on a gerbe

We have line bundles $\mathcal{L}_{\alpha\beta}$ on overlaps. A $U(1)$-connection on the gerbe consists of

- A Hermitian metric on $\mathcal{L}_{\alpha\beta}$.
- Connective structure: A Hermitan connection $\nabla_{\alpha\beta}$ on $\mathcal{L}_{\alpha\beta}$ so
$$\mu^*_{\alpha\beta\gamma}\nabla_{\alpha\gamma} = (\nabla_{\alpha\beta} \otimes I) + (I \otimes \nabla_{\beta\gamma}).$$
- Curving: $\kappa_\alpha \in \Omega^2(U_\alpha)$ so
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Then $H = d\kappa_\alpha$ is a globally defined closed 3-form on M, the de Rham representative of the gerbe's class in $\mathrm{H}^3(M; \mathbb{Z})$.

Twisted differential K-theory

A twisted Hilbert bundle $\mathcal{H}$ is given by Hilbert bundles $\mathcal{H}_\alpha$ over the U_α's, with isomorphisms $\phi_{\alpha\beta} : \mathcal{H}_\alpha \otimes \mathcal{L}_{\alpha\beta} \to \mathcal{H}_\beta$ on $U_\alpha \cap U_\beta$.

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A superconnection on $\mathcal{H}$ is given by superconnections A_α on the $\mathcal{H}_\alpha$'s so $\phi^*_{\alpha\beta} A_\beta = (A_\alpha \otimes I) + (I \otimes \nabla_{\alpha\beta})$ on $U_\alpha \cap U_\beta$.

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The generators for twisted differential K-theory are now triples $(\mathcal{H}, A, \omega)$ as before. Quotienting by the relations, one gets the twisted differential K-theory group.

Model for twisted differential K-theory

Theorem (Gorokhovsky-L.): Up to isomorphism, the twisted differential K-group only depends on the gerbe through its isomorphism class. It is independent of the choices of connective structure and curving.

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This gives an explicit model for twisted differential K-theory. It remains to show that it agrees with other models (Bunke-Nikolaus).

Happy Birthday!