The Smooth Homotopy Category

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ultra-violet	infra-red

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$$\mathsf{Mfds} \quad \subset \quad \mathsf{SmHtp} \quad \supset \quad \mathsf{Htp}$$

$$\widehat{\mathrm{Mfds}} \quad \leftarrow \quad \mathsf{SmHtp} \quad \rightarrow \quad \mathsf{Htp}$$

A basic adjunction:

$$\begin{array}{ccc} \text{groups} & \longleftrightarrow & \text{based homotopy types} \\ \pi_1(M,x) & \longleftarrow & (M,x) \\ G & \longmapsto & BG \end{array}$$

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Two ideas:

(i) space as a set with a topology, *vs* space as a set with paths
(ii) the ‘radar’ perspective

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The *local* description of a object of SmHtp is also perturbative in nature: we expand functions in Taylor series at a given point.

The structures of these very different perturbative theories have remarkable similarities.

The vector space of k-fold operations

$$\mathfrak{g} \times \; \ldots \; \times \mathfrak{g} \quad \longrightarrow \quad \mathfrak{g}$$

one can perform in a Lie algebra $\mathfrak{g}$, e.g.

$$(\xi_1, \xi_2, \ldots, \xi_5) \quad \longmapsto \quad [[\xi_1, [\xi_2, \xi_3], [\xi_4, \xi_5]]$$

is $H_{(k-1)(n-1)}(C_k(\mathbb{R}^n); \mathbb{R})$.

The Lie expressions correspond to planetary systems.

Origins of SmHtp

For a Lie group G and an abelian Lie group A with G-action we can define cohomology groups

$$H^*_{\mathrm{alg}}(G; A)$$

with the properties

$$H^0_{\mathrm{alg}}(G; A) \;=\; A^G$$

$$H^1_{\mathrm{alg}}(G; A) \;=\; \mathrm{Hom}_{\mathrm{cr}}(G; A)/\sim$$

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Is there a 'space' $\mathcal{B}G$ such that $H^*_{\mathrm{alg}}(G;A) \;=\; H^*(\mathcal{B}G;A)$?

If G is *discrete* then $H^*_{\mathrm{alg}}(G;A) \;=\; H^*(BG;A)$, where

$$[X;BG] \;=\; \mathrm{Bdl}_G(X)$$

If A is discrete, we still have $H^*_{\mathrm{alg}}(G;A) \;=\; H^*(BG;A)$ for all G, BUT if G is a real vector space V then

$$H^*_{\mathrm{alg}}(V;\mathbb{R}) \;=\; \wedge^*(V^*)$$

and if G is reductive then

$$H^*_{\mathrm{alg}}(G;\mathbb{R}) \;=\; H^*(G_{\mathbb{C}}/G;\mathbb{R}).$$

Thus

$$H^*_{\mathrm{alg}}(G;\mathbb{Z}) \otimes \mathbb{R} \;\to H^*_{\mathrm{alg}}(G;\mathbb{R})$$

is far from an isomorphism.

The van Est spectral sequence suggests there is a fibration

$$\mathcal{B}\mathfrak{g} \to \mathcal{B}G \to BG,$$

where $\mathcal{B}\mathfrak{g}$ is a 'space' such that $H^*(\mathcal{B}\mathfrak{g}; \mathbb{R}) = H^*_{\mathrm{Lie}}(\mathfrak{g}; \mathbb{R})$.

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Question Is *any* object of SmHtp obtained by putting a bundle of infinitesimal structures on an ordinary homotopy type?

Provisional definition:

SmHtp is the category of 'half-exact homotopy functors'

$$\mathrm{Htp}^{\mathrm{opp}} \;\to\; \mathrm{Mfds}.$$

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$\mathcal{B}G$ is the functor $X \mapsto \mathrm{Flat}_G(X)$, where
$\mathsf{Flat}_G(X) \;=\; \{\mathrm{Flat}\ G - \mathrm{bundles\ on}\ X\}/\sim$,

$\mathcal{B}\mathfrak{g}$ is $X \mapsto \mathrm{Flat}_{\mathfrak{g}}(X)$,
the classes of flat connections in the trivial $\mathfrak{g}$-bundle.

The van Est fibration becomes the Puppe sequence

$$\ldots \;\to\; [X; G] \to\; \mathrm{Flat}_{\mathfrak{g}}(X) \;\to\; \mathrm{Flat}_G(X) \;\to\; \mathrm{Bdl}_G(X).$$

Motivation: Quillen's algebraic K-theory

If A is the ring of integers in a number field then

$$A^\times \hookrightarrow (A \otimes \mathbb{R})^\times = (\mathbb{R}^\times)^r \times (\mathbb{C}^\times)^c.$$

$$K_i(A) \to K_i(A \otimes \mathbb{R})$$

The K-theory of A is $K_A^*(\mathrm{point})$, where $X \mapsto K_A(X)$ is the group-completion of the semigroup-valued functor

$$X \mapsto \mathrm{Flat}_{A-\mathsf{Mod}}(X).$$

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For a topological ring, we can perform the group-completion in SmHtp.

$$K_*(\mathbb{C}) = \{\mathbb{Z} \;\; \mathbb{C}^\times \;\; 0 \;\; \mathbb{C}^\times \;\; 0 \;\; \mathbb{C}^\times \;\; 0 \;\; \ldots\}$$

$$K_*(\mathbb{R}) = \{\mathbb{Z} \;\; \mathbb{R}^\times \;\; 0 \;\; \mathbb{T} \;\; 0 \;\; \mathbb{R}^\times \;\; 0 \;\; \ldots\}$$

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SmHtp is a *homotopy* category, and we need a category of 'spaces' from which it arises.

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Two candidates: **simplicial manifolds** and **commutative DGAs**

A simplicial manifold $\mathcal{M}$ defines the manifold-valued homotopy-functor

$$X \ \mapsto \ [X; \mathcal{M}]_{spll},$$

where X is a discrete simplicial space

— in the language of 'spaces with two topologies'

$$X \ \mapsto \ [X; |\mathcal{M}^{discr}|\ ].$$

A smooth manifold M is identified with a *constant* simplicial manifold.

The forgetful functor

$$\mathrm{SmHtp} \;\to\; \mathrm{Htp}$$

is given by $\mathcal{M} \mapsto |\mathcal{M}|$, and the forgetful functor

$$\mathrm{SmHtp} \;\to\; \widehat{\mathrm{Mfds}}$$

is $\mathcal{M} \mapsto \pi_0(\mathcal{M})$.

Think of a simplicial manifold $\mathcal{M} = \{\mathcal{M}_p\}$ as the manifold $\mathcal{M}_0$ equipped with a "generalized equivalence relation".

Examples

(i) If $\mathcal{U} = \{U_\alpha\}$ is an open covering of a manifold M we have a simplicial manifold

$$M^{\mathcal{U}} \;=\; \{\bigsqcup_{\alpha} U_\alpha \quad \bigsqcup_{\alpha,\beta} U_\alpha \cap U_\beta \;\; \dots \}.$$

The inclusion $M^{\mathcal{U}} \to M$ is an equivalence in SmHtp.

(ii) If we choose a metric on M, and a small $\varepsilon > 0$, we have the *thickening* M_Δ of M, with

$$M_{\Delta,p} \;=\; \{(x_0, \ldots, x_p) \in M^{p+1} : d(x_i, x_j) < \varepsilon\}.$$

We should think of this as "M with nearby points identified": it represents the homotopy type of $|M|$ of M as an object of SmHtp.

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We should think of this as "M with nearby points identified": it represents the homotopy type of $|M|$ of M as an object of SmHtp.

(iii) $\mathcal{S}_M$, for which $\mathcal{S}_{M,p}$ is the manifold of smooth singular simplexes in M.

M_Δ and $\mathcal{S}_M$ are equivalent in SmHtp.

(iv) Any smooth groupoid or smooth category is a simplicial manifold, for example $\mathcal{B}G$, with $\mathcal{B}G_p = G^p$.

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$$[M; \mathcal{B}G]_{sm} = \mathrm{Bdl}_G(M)$$

$$[M_\Delta^{(1)}; \mathcal{B}G]_{sm} = \mathrm{Bdl}_G^{conn}(M)$$

$$[M_\Delta^{(p)}; \mathcal{B}G]_{sm} = \mathrm{Flat}_G(M) \quad \text{for } p \geq 2,$$

where $M_\Delta^{(p)}$ denotes the p-skeleton of M_Δ.

In particular we get the ‘differential’ or Deligne-Cheeger-Simons cohomology:

$$[M_{\Delta}^{(1)}; \mathcal{B}\mathbb{T}]_{sm} \;=\; H_{sm}^{2}(M),$$

and, more generally,

$$[M_{\Delta}^{(p)}; \mathcal{K}(\mathbb{T}, p)]_{sm} \;=\; H_{sm}^{p+1}(M),$$

where $\mathcal{K}(\mathbb{T}, p) \;=\; \mathcal{B} \ldots \mathcal{B}\mathbb{T}$ is the Eilenberg-Maclane object:

$$\mathcal{K}(\mathbb{T}, p) \;=\; Z^{p}(\Delta; \mathbb{T}).$$

(Here Δ is the cosimplicial object formed by the standard simplexes.)

Commutative DGAs/$\mathbb{R}$ (with nuclear topology) are related to simplicial manifolds by contravariant adjoint functors defined in terms of the simplicial DGA $\Omega^\bullet(\Delta)$.

For a DGA $\mathcal{A}$ we write

$$\mathrm{Spec}(\mathcal{A}) \ = \ \mathsf{Hom}_{DGA}(\mathcal{A}; \Omega^\bullet(\Delta)),$$

and, for a simplicial manifold $\mathcal{M}$,

$$\mathcal{A}_{\mathcal{M}} \ = \ \mathrm{Map}_{spll}(\mathcal{M}; \Omega^\bullet(\Delta)).$$

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But simplicial manifolds give us only **positively-graded** DGAs.

Examples

(i) $\mathcal{A}_M \ = \ \mathrm{C}^\infty(M)$

(ii) $\mathrm{Spec}(\Omega^\bullet(M)) \ = \ \mathcal{S}_M$

(iii) We can now define $\mathcal{B}\mathfrak{g}$ by means of the usual Lie algebra cochain algebra

$$\mathcal{A}_{\mathcal{B}\mathfrak{g}} \ = \ \wedge^\bullet(\mathfrak{g}^*)$$

A homomorphism $\wedge^\bullet(\mathfrak{g}^*) \to \Omega^\bullet(M)$ is just a $\mathfrak{g}$-valued 1-form on M which satisfies the Maurer-Cartan equation.

If $\mathfrak{g} = \mathrm{Lie}(G)$ with dim $G < \infty$, then $\mathrm{Spec}(\mathcal{A}_{\mathcal{B}\mathfrak{g}}) \ = \ \mathfrak{S}_G/G$.
In SmHtp this is equivalent to the *group-germ* of G, i.e.

$$\mathcal{B}\mathfrak{g}_p \ = \ \{(g_1, \ldots, g_p) \in U^p : g_i g_{i+1} \ldots g_j \in U\}$$

for a convex neighbourhood of $1 \in G$.

The based loop space

The loop space $\Omega_x M$ at a point x of a manifold M is a *group* in SmHtp.

Any group in Htp can be represented by an honest topological group.

Milnor's construction of $\check{\Omega}_x M$ as a group:
achieve associativity by using unparametrized paths
obtain inverses by cancelling return paths.

$$\check{\Omega}_x M \;=\; \pi_1(M_\Delta^{(1)}, x)$$

This is an infinite-dimensional Lie group.

Kapranov's description of $\omega_x M = \mathrm{Lie}(\check{\Omega}_x M)$:

choose a coordinate chart at x

$\omega_x M = FL^{\geq 2}(T_x M)$,

i.e. the sub-Lie-algebra of the free Lie algebra $FL(T_x M)$ consisting of terms of degree ≥ 2.

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$$\mathrm{Hom}(\check{\Omega}_x M; G) \;=\; \mathrm{Bdl}_G^{conn}(M)$$

The vector fields ∂_i of the chart give covariant derivatives D_i in a bundle with connection. Define $FL^{\geq 2}(T_x) \to \mathfrak{g}$ by

$$[\partial_i, \partial_j] \;\mapsto\; [D_i, D_j] = K_{ij}$$

$$[\partial_i, [\partial_j, \partial_k]] \;\mapsto\; [D_i, [D_j, D_k]] = D_i K_{jk}$$

Jacobi identity $\quad\longleftrightarrow\quad$ Bianchi identity

Better:

$[\Delta^1; M_\Delta^{(1)}]_{sm}$ is the smooth groupoid of unparametrized paths in M. Its infinitesimal version is the Lie algebroid $\{FL(T_x)\}$.

This presents the object $M_\Delta^{(1)}$ as a bundle of infinitesimal structures over $|M| \;=\; M_\Delta^{(1)}$.

M as an object of SmHtp:

$$M \;\to\; |M| \quad \text{corresponds to} \quad \Omega^0(M) \;\leftarrow\; \Omega^\bullet(M)$$

To make this into a fibration we replace $\Omega^0(M)$ by the equivalent DGA $\Omega^\bullet(M;\mathcal{F})$, where:
$\mathcal{F} \;=\; \{\mathcal{F}_x\}$ is a bundle of algebras
$\mathcal{F}_x \;=\;$ infinite jets of smooth functions at x
$\mathcal{F}$ has a flat connection.

The fibre of $M \to |M|$ at x is represented by the algebra $\mathcal{F}_x$ of formal power series, whose spectrum is a point.
The information contained in $\mathcal{F}_x$ is precisely the choice of the tangent space T_x.

The replacement $\Omega^\bullet(M;\mathcal{F})$ for $\Omega^0(M)$
arises in Fedosov's deformation quantization.

$\mathcal{F}_x \;\cong\; \hat{S}(T_x^*) \;=\; \hat{S}_x$ by the choice of an exponential map.
This isomorphism gives us a flat connection in the bundle $\{\hat{S}_x\}$.

$p \in \wedge^2(T_x)$ defines a *quantization* $\hat{S}_x[[\hbar]]$ of $\hat{S}_x$.

The induced connection in the quantization is flat only mod $\hbar$, but if $[p,p]=0$ it can be altered power-by-power in $\hbar$ to make it flat.

$H^0(\Omega^\bullet(M;\hat{S}[[\hbar]]))$ is the quantization of $\Omega^0(M)$.