A wall-crossing formula for 2d-4d DT invariants

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In the last few years there has been a lot of progress in the theory of **generalized Donaldson-Thomas invariants**.

[Kontsevich-Soibelman, Joyce-Song, ...]

In many (all?) cases where they can be defined, these invariants have a physical meaning: **BPS state index in 4d $\mathcal{N} = 2$ SUSY quantum systems**.

[Ooguri-Strominger-Vafa, Denef, Denef-Moore, Gaiotto-Moore-N, Dimofte-Gukov-Soibelman, Cecotti-Vafa, ...]
This talk is motivated by some new progress in physics, involving new “2d-4d” BPS state indices, which can be defined when we add a 2d surface operator to our 4d quantum system.

We have learned a few basic facts about these 2d-4d indices — in particular we have learned what their wall-crossing formula is. The main aim of this talk is to explain these facts.

We conjecture that the 2d-4d indices should be part of a not-yet-formulated extension of Donaldson-Thomas theory.

As Davide explained in his talk, these 2d-4d indices seem to be useful tools even if your ultimate interest is only in the original generalized DT invariants.
I’ll focus on a specific class of situations where both the generalized DT invariants and the new 2d-4d invariants can be defined relatively easily, and the new 2d-4d wall-crossing formula can be directly checked.

In these examples, the generalized DT invariants (as well as their 2d-4d extensions) are counting special trajectories associated with a quadratic differential on a Riemann surface $C$.

First, I’ll review the “old” story; then I’ll give its 2d-4d extension.
Fix compact Riemann surface $C$, with $n > 0$ marked points $z_i$, $i = 1, \ldots, \ell$. Let $C'$ be $C$ with the marked points deleted. Fix (generic) parameters $m_i \in \mathbb{C}$ for each $i$.

Let $B$ be the space of meromorphic quadratic differentials $\varphi_2$ on $C$, with double pole at each $z_i$, residue $m_i^2$:

$$\varphi_2 = \frac{m_i^2}{(z - z_i)^2} dz^2 + \cdots$$
In what follows, if you want to know what is really proven, stick to $C$ of genus zero. All the phenomena I’ll discuss occur already in that case.
Invariants of a quadratic differential

Fix a point of $B$, i.e. fix a meromorphic quadratic differential $\varphi_2$ on $C$ with double pole at each $z_i$, residue $m_i$.

This determines a metric $h$ on $C$, in a simple way:

$$h = |\varphi_2|$$

(so if $\varphi_2 = P(z) \, dz^2$ then $h = |P(z)| \, dz \, d\bar{z}$.)

More precisely, $h$ is a metric on only an open subset of $C$, where we delete both the poles of $\varphi_2$ (the $z_i$) and also the zeroes of $\varphi_2$. $h$ is flat on this open subset.
Invariants of a quadratic differential

Now we can consider finite length inextendible geodesics on $C'$ in the metric $h$. These come in two types:

- **Saddle connections**: geodesics running between two zeroes of $\varphi_2$. These are rigid (don’t come in families).

- **Closed geodesics**. When they exist, these come in 1-parameter families, sweeping out annuli on $C'$. 
Invariants of a quadratic differential

To “classify” these finite length geodesics, introduce a little more technology: $\varphi_2$ determines a branched double cover $\Sigma \to C'$,

$$\Sigma = \{ x : x^2 = \varphi_2 \} \subset T^* C.$$  

Each finite length geodesic can be lifted to a union of closed curves in $\Sigma$, representing some homology class $\gamma \in H_1(\Sigma, \mathbb{Z})^{odd}$.  

We define an invariant

$$\Omega(\gamma) = \begin{cases} 
1 & \text{if there is a saddle connection w/ lift } \gamma \\
-2 & \text{if there is a closed loop w/ lift } \gamma \\
0 & \text{if neither} 
\end{cases}$$
As we vary \( \varphi_2 \), \( \Omega(\gamma) \) can jump, when some finite-length geodesics appear or disappear. This occurs at some real-codimension-1 loci \( \mathcal{W} \subset \mathcal{B} \) ("walls").
Walls

As we vary $\varphi_2$, $\Omega(\gamma)$ can jump, when some finite-length geodesics appear or disappear.
Basic mechanism: decay/formation of bound states.

Where is the wall? Define a function $Z_\gamma$ on $B$ (the parameter-space of quadratic differentials $\varphi_2$) by

$$Z_\gamma = \oint_\gamma \lambda$$

where $\lambda$ is the canonical 1-form on $T^* C$. Then the wall is the locus in $B$ where

$$Z_{\gamma_1} / Z_{\gamma_2} \in \mathbb{R}.$$
Wall-crossing formula

The jump of the $\Omega(\gamma)$ at the wall is governed by the Kontsevich-Soibelman WCF.

To state that formula (which will take a few slides), we axiomatize our structure a bit: the data are

- Complex manifold $\mathcal{B}$ (space of quadratic differentials on $\mathbb{C}$)
- Lattice $\Gamma$ w/ antisymmetric pairing $\langle,\rangle$ ($H^1(\Sigma, \mathbb{Z})^{odd}$)
- Homomorphism $Z : \Gamma \to \mathbb{C}$ for each point of $\mathcal{B}$, varying holomorphically over $\mathcal{B}$ ($Z_\gamma = \oint_\gamma \lambda$)
- “invariants” $\Omega : \Gamma \to \mathbb{Z}$ for each point of $\mathcal{B}$ (counts of finite length geodesics)
Wall-crossing formula

Attach a “$\mathcal{K}$-ray” in $\mathbb{C}$ to each $\gamma$ with $\Omega(\gamma) \neq 0$.

Slope of the $\mathcal{K}$-ray is given by the argument of $Z_\gamma$.

These rays move around as we vary the quadratic differential $\varphi_2$, i.e. as we move in $\mathcal{B}$.

Walls in $\mathcal{B}$ are loci where some set of rays become aligned:

Near the wall.  On the wall.

Focus on these participating rays.
Wall-crossing formula

Introduce torus $T \simeq (\mathbb{C}^*)^{\text{rank } \Gamma}$ with coordinate functions $X_\gamma : T \to \mathbb{C}^*$ for each $\gamma \in \Gamma$, obeying

$$X_\gamma X_{\gamma'} = (-1)^{\langle \gamma, \gamma' \rangle} X_{\gamma+\gamma'}.$$  

To each $\gamma$, assign a (birational) automorphism $K_\gamma$ of $T$:

$$K_\gamma : X_{\gamma'} \mapsto (1 - X_\gamma)^{\langle \gamma, \gamma' \rangle} X_{\gamma'}$$

Now consider a product over all participating $\gamma$,

$$\prod_{\gamma} K_\gamma^\Omega(\gamma) :$$

where :: means we multiply in order of the phase of $Z_\gamma$.

The Kontsevich-Soibelman WCF is the statement that this automorphism is the same on both sides of the wall.
Wall-crossing formula
For example: if $\langle \gamma_1, \gamma_2 \rangle = 1$,

$$K_{\gamma_1} K_{\gamma_2}$$

equals

$$K_{\gamma_2} \Omega^\prime(\gamma_2) K_{\gamma_1 + \gamma_2} \Omega^\prime(\gamma_1)$$

if and only if

$$\Omega^\prime(\gamma_1) = 1$$
$$\Omega^\prime(\gamma_2) = 1$$
$$\Omega^\prime(\gamma_1 + \gamma_2) = 1$$
Wall-crossing formula

More interesting example: if $\langle \gamma_1, \gamma_2 \rangle = 2$,

$$\mathcal{K}_{\gamma_1} \mathcal{K}_{\gamma_2} = \left( \prod_{n=0}^{\infty} \mathcal{K}_{n \gamma_1 + (n+1) \gamma_2} \right) \mathcal{K}_{\gamma_1 + \gamma_2}^{-2} \left( \prod_{n=\infty}^{0} \mathcal{K}_{(n+1) \gamma_1 + n \gamma_2} \right)$$

So,

- on one side of the wall we have only $\Omega(\gamma_1) = 1$ and $\Omega(\gamma_2) = 1$, all others zero;
- on the other side we have infinitely many $\Omega(\gamma) = 1$, and also $\Omega(\gamma_1 + \gamma_2) = -2$. 
Wall-crossing formula

Key fact: KS WCF holds for our integer invariants $\Omega(\gamma)$!

So e.g.

$$K_{\gamma_1}K_{\gamma_2} = K_{\gamma_2}K_{\gamma_1+\gamma_2}K_{\gamma_1}$$

for $\langle \gamma_1, \gamma_2 \rangle = 1$ says that if we have two saddle connections that intersect at 1 point, then after wall-crossing a third saddle connection will appear.

Similarly in the formula

$$K_{\gamma_1}K_{\gamma_2} = \left( \prod_{n=0}^{\infty} K_{n\gamma_1+(n+1)\gamma_2} \right)K_{\gamma_1+\gamma_2}^{-2} \left( \prod_{n=\infty}^{0} K_{(n+1)\gamma_1+n\gamma_2} \right)$$

for $\langle \gamma_1, \gamma_2 \rangle = 2$, on one side we have two saddle connections intersecting at two points; on the other side we have infinitely many saddle connections plus a single closed geodesic.
So far, so good: we described a simple class of enumerative invariants $\Omega(\gamma) \in \mathbb{Z}$ attached to a punctured curve $C$, and explained that they give examples of the general wall-crossing formula of Kontsevich-Soibelman.

Now, we consider our “2d-4d” extension. $\Omega(\gamma) \in \mathbb{Z}$ will be slightly refined to some new objects $\omega(\gamma, \gamma_{ij}) \in \mathbb{Z}$, and we will also introduce new $\mu(\gamma_{ij}) \in \mathbb{Z}$. 
Invariants of a quadratic differential plus a point

As before: Fix compact Riemann surface $C$, with $n > 0$ marked points $z_i$, $i = 1, \ldots, n$. Let $C'$ be $C$ with the marked points deleted. Fix (generic) parameters $m_i \in \mathbb{C}$ for each $i$.

New 2d-4d datum: Fix a point $z \in C$.

As before: let $\mathcal{B}$ be the space of meromorphic quadratic differentials $\varphi_2$ on $C$, with double pole at each $z_i$, residue $m_i^2$:

$$\varphi_2 = \frac{m_i^2}{(z - z_i)^2} dz^2 + \cdots$$
As before: we are interested in counting finite-length geodesics on $C'$, in the flat metric $h$ determined by $\varphi_2$.

However, now we allow them to be open, i.e. to have one end on the point $z$. (And the other end on a zero of $\varphi_2$ as before.)
Invariants of a quadratic differential plus a point

To categorize these open geodesics, we again consider their lifts to the double cover $\Sigma$:

These give 1-chains $\gamma_{ij}$ with $\partial \gamma_{ij} = z_i - z_j$; let $\Gamma_{ij}$ be set of such 1-chains modulo boundaries. $\Gamma_{ij}$ is a torsor over the homology $\Gamma$. For any $\gamma_{ij} \in \Gamma_{ij}$, we define new invariant $\mu(\gamma_{ij})$ by

$$\mu(\gamma_{ij}) = \begin{cases} 1 & \text{if there is an open geodesic w/ lift } \gamma_{ij} \\ 0 & \text{if not} \end{cases}$$
Invariants of a quadratic differential plus a point

In the presence of the extra point $z$, we can also keep track of slightly more information about the ordinary finite geodesics: measure their homology classes on $\Sigma$ punctured at the preimages of $z$.

To encode that information: for any $\gamma \in \Gamma$ and $\gamma_{ij} \in \Gamma_{ij}$, we define new invariant $\omega(\gamma, \gamma_{ij})$, by

$$\omega(\gamma, \gamma_{ij}) = \Omega(\gamma)\langle \gamma, \gamma_{ij} \rangle$$

(To define the intersection number with the open path $\gamma_{ij}$, if $\gamma$ is an isolated geodesic, use the actual geodesic representative for $\gamma$. If $\gamma$ not isolated, use the two ends of the family, take the average.)
As before: as we vary $\varphi_2$ and $z$, $\mu(\gamma)$ and $\omega(\gamma, \gamma_{ij})$ can jump.

Two sample pictures:
2d-4d wall-crossing formula

To state our extended “2d-4d” WCF, axiomatize our new structure: to our old list

- Complex manifold $\mathcal{B}$ (space of quadratic differentials on $C$)
- Lattice $\Gamma$ w/ antisymmetric pairing $\langle \cdot, \cdot \rangle$ ($H^1(\Sigma, \mathbb{Z})^{odd}$)
- Homomorphism $Z : \Gamma \to \mathbb{C}$ for each point of $\mathcal{B}$, varying holomorphically over $\mathcal{B}$ ($Z_\gamma = \oint \gamma \lambda$)
- “invariants” $\Omega : \Gamma \to \mathbb{Z}$ for each point of $\mathcal{B}$ (counts of finite geodesics)

we now add

- $\Gamma$-torsors $\Gamma_{ij}$ for $i, j = 1, \ldots, n$, with addition operations $\Gamma_{ij} \times \Gamma_{jk} \to \Gamma_{ik}$, satisfying associativity ($n = 2$, spaces of 1-chains with boundary $z_i - z_j$)
- Maps $Z : \Gamma_{ij} \to \mathbb{C}$ obeying additivity ($Z_{\gamma_{ij}} = \int_{\gamma_{ij}} \lambda$)
- “invariants” $\omega : \Gamma \times \Gamma_{ij} \to \mathbb{Z}$, satisfying
  $\omega(\gamma, \gamma' + \gamma_{ij}) = \omega(\gamma, \gamma_{ij}) + \Omega(\gamma)\langle \gamma, \gamma' \rangle$ (refined counts of finite geodesics)
- “invariants” $\mu : \Gamma_{ij} \to \mathbb{Z}$ for each $i, j$ with $i \neq j$, and each point of $\mathcal{B}$ (counts of open geodesics)
2d-4d wall-crossing formula

As before, attach a ray in $\mathbb{C}$ to each nonzero invariant: “$K$-ray” for each $\gamma$ with $\omega(\gamma, \cdot) \neq 0$, “$S$-ray” for each $\gamma_{ij}$ with $\mu(\gamma_{ij}) \neq 0$.

Slope of the rays given by the argument of $Z_\gamma$ or $Z_{\gamma_{ij}}$.

The rays move around as we vary the quadratic differential $\varphi_2$ and the point $z$, i.e. as we move in $B \times C$.

As before, walls in $B \times C$ are loci where some set of rays become aligned:

- **Near the wall.**
- **On the wall.**

Focus on these participating rays.
2d-4d wall-crossing formula

To formulate the KS WCF, we used an auxiliary gadget, the torus $T \simeq (\mathbb{C}^\times)^{\text{rank } \Gamma}$. The $\Omega(\gamma)$ got encoded into automorphisms $\mathcal{K}_\gamma^{\Omega(\gamma)}$ of $T$.

For the 2d-4d WCF, we decorate that story a bit: add a trivializable holomorphic rank-$n$ vector bundle $V$ over $T$. The 2d-4d invariants get encoded into automorphisms of that object:

▶ The $\omega(\gamma, \cdot)$ contain slightly more information than $\Omega(\gamma)$; correspondingly, they determine an object $\mathcal{K}_\gamma^\omega$, which lifts $\mathcal{K}_\gamma^{\Omega(\gamma)}$ to act on $V$.

▶ The new invariants $\mu(\gamma_{ij})$ give new automorphisms $\mathcal{S}_{\gamma_{ij}}^\mu$ which leave points of $T$ fixed, act only on the fiber of $V$. (Unipotent matrices with one off-diagonal entry, in the $ij$ place.)
2d-4d wall-crossing formula

Just to show you that the formulas are concrete:

$K^\omega_\gamma$ and $S_{\gamma ij}$ are automorphisms of a vector bundle $V$ over $T$. I’ll give their action on a basis of sections of $\text{End}(V)$: so along with the $X_\gamma$ we had before (functions on $T$), now also have sections $X_{\gamma ij}$ (“elementary matrix” sections of $\text{End}(V)$), and the automorphisms act by

\[
K^\omega_\gamma : X_{\gamma ij} \mapsto (1 - X_\gamma)^{\omega(\gamma, \gamma)_{ij}} X_{\gamma ij}
\]

\[
S^\mu_{\gamma kl} : X_{\gamma ij} \mapsto (1 + \mu X_{\gamma kl}) X_{\gamma ij} (1 - \mu X_{\gamma kl})
\]

This is enough for our purposes.
Now consider a product over all participating rays

\[ \prod_{\gamma, \gamma_{ij}} K_{\gamma}^\omega S_{\gamma_{ij}}^\mu : \]

where :: means we multiply in order of the phase of $Z_\gamma$, $Z_{\gamma_{ij}}$.

This object is an automorphism of the torus $T$, lifted to act on the vector bundle $V$.

The 2d-4d WCF is the statement that this automorphism is the same on both sides of the wall.
Networks

Story so far might seem a little lame: 2d-4d WCF involves an integer $n$, but my only example of 2d-4d invariants had $n = 2$. There is a more general version: the quadratic differential $\varphi_2$ is replaced by a tuple of $k$-differentials $(\varphi_k)_{2 \leq k \leq n}$, and instead of counting geodesics on $C$, we count certain networks:

Each leg is labeled by a pair of sheets $(x_i, x_j)$ of the $n$-fold covering

$$
\Sigma = \{x^n + \sum_{k=0}^{n-2} x^k \varphi_{n-k} = 0\} \subset T^* C
$$

and is a straight line in the coordinate $\int x_i - x_j$, with inclination $\vartheta$, the same for all legs.
We don’t know anywhere that these networks have been studied before (would be very curious to learn a reference!)

They are a natural generalization of the trajectories of quadratic differentials, which are somewhat better-studied. \[\text{[Strebel, Masur, ...]}\]

We expect that the counting of these objects — and their open analogues, with one leg ending on a marked point \(z\) — is governed by our 2d-4d WCF (details still being checked).
The 2d-4d WCF does not come out of nowhere.

Cecotti-Vafa previously studied purely 2d field theories and gave a WCF with structure similar to the KS WCF and the 2d-4d WCF.

In that WCF, the relevant group of “automorphisms” which appears is just $GL(n)$.

The 2d-4d WCF is a kind of hybrid between that WCF and the KS WCF.

(It fits into a general formalism where the “invariants” are allowed to belong to a general graded Lie algebra.)
Applications

What are the 2d-4d invariants good for?

As mentioned in Davide’s talk, if you are in a situation where the 2d-4d invariants can be defined, knowing $\mu$ is enough to give information about the original generalized DT invariants $\Omega$.

Indeed in many cases $\mu$ actually determines $\Omega$! And $\mu$ seem to be somewhat more easily computable...
A geometric application

One application of the KS WCF: a construction of a family of hyperkähler metrics on the total space $\mathcal{M}$ of a torus bundle over $\mathcal{B}$.

Very roughly, construction proceeds by gluing together patches which look like the torus $T$, with the gluing maps given by the automorphisms $\mathcal{K}_{\gamma}^{\Omega(\gamma)}$. [Strominger-Yau-Zaslow, Kontsevich-Soibelman, Auroux, Gross-Siebert, ...]

In the examples I have been discussing here, $\mathcal{M}$ is a moduli space of solutions of Hitchin equations over $C$ with gauge group $SU(n)$. 
A geometric application

The 2d-4d WCF gives a new construction of a hyperholomorphic bundle $\mathcal{V}$ over this hyperkähler $\mathcal{M}$ (think of this as a generalization of a Yang-Mills instanton.) Very roughly, construction proceeds by gluing together patches which look like the vector bundle $V$ over $T$, with gluing maps given by $\mathcal{K}_\gamma$ and $S_{\gamma ij}^\mu$.

In the examples I have been discussing here, $\mathcal{M}$ is a moduli space of solutions of Hitchin equations over $C$ with gauge group $SU(k)$, $\mathcal{V}$ is the universal bundle restricted to $z \in C$. 
We conjecture that 2d-4d invariants obeying the 2d-4d WCF exist in other “Donaldson-Thomas situations” as well.

Roughly speaking, whenever we have a \((A \text{ or } B)\) brane whose moduli space is 0-dimensional, we should be able to use it to define a 2d-4d version of the counting of \((B \text{ or } A)\) branes.

(A funny-looking mixing of the two mirror-dual categories!)
2d-4d invariants in general

For example, suppose we have a Calabi-Yau threefold $X$. The standard dogma is that there should be generalized
Donaldson-Thomas invariants $\Omega(\gamma)$ which “count” special
Lagrangian cycles (A branes) on $X$, for $\gamma \in H_3(X, \mathbb{Z})$.

Now fix a class in $H_2(X, \mathbb{Z})$ supporting finitely many holomorphic
curves $Y_1, \ldots, Y_n$ (B branes).

In this situation, with luck, there should be 2d-4d invariants
$\omega(\gamma, \gamma_{ij}) \text{ and } \mu(\gamma_{ij})$, with $i, j = 1, \ldots, n$, and $\gamma_{ij}$ lying in torsors $\Gamma_{ij}$
for $H_3(X, \mathbb{Z})$ (roughly 3-chains with boundary $Y_i - Y_j$).

I am not bold enough to try to say today how they should be defined.
Motivated by physics, we propose that there should exist a new “2d-4d” extension of the usual theory of generalized Donaldson-Thomas invariants. $\Omega(\gamma)$ replaced by $\mu(\gamma_{ij})$ and $\omega(\gamma, \gamma_{ij})$.

In particular, we claim this new theory governs the counting of certain open and closed networks of trajectories on Riemann surfaces.

While we don’t know how to define this theory in general, we do know what its wall-crossing behavior should be.