Spectral Networks

Joint work w/ Gaiotto, Moore. In progress.
I'll introduce a new structure:

- $g = A_{K-1}$
- curve $C$ (smooth compact)
- tuple $(\psi_1, \ldots, \psi_K)$
  - $\psi$ mer. section of $K_C$
- $\theta \in \mathbb{R}/2\pi \mathbb{Z}$

(We previously did this for $g = A_1$.)

HK metric on Hitchin system $\mathcal{H}(G, C)$
- IR Lagrangian on $T^3 \times S^1$
- integers $S^\infty(Y)$
  - 4d BPS degeneracies
  - DT invariants for local CY

$\gamma \to$ Framed 2d-4d wall-crossing

Cluster coordinate system on space of
- flat $G$-connections on $C \setminus \{s_a\}$

= reps of line operators on $T^3 \times S^1$

Spectral trajectories

Consider spectral cover
\[ \Sigma = \{ \lambda^K - \sum_{i=2}^{K-1} \lambda^{K-i} \psi_i = 0 \} \subset T^* C \]
Branched, $K$-sheeted covering. Assume $(\psi_i)$ generic $\Rightarrow$ only simple branch pts.

Call the roots (locally) $\lambda_1, \ldots, \lambda_K$. Complex 1-forms.

Then define an $ij$-trajectory to be an oriented path in $C$ along which $e^{i\theta}(\lambda_i - \lambda_j)$ is real and positive.

For fixed $i; j$ these give a (local) foliation of $C$. 

\[ \text{For fixed } i; j \text{ these give a (local)} \]
\[ \text{foliation of } C. \]
It is singular at pts. where $\lambda_i = \lambda_j$ (branch points)

(and at the poles of $\lambda_i - \lambda_j$ — less imp. for what follows)

Spectral network

Network of spectral trajectories on $C$, built up as follows:

Begin at each branch pt. with

Evolve the trajectories for infinite time.
If two cross, they can "give birth" to a new one, which then also evolves for infinite time.

If we have "enough punctures" (e.g. one when $\gamma_i$ has pole of order $i$) then all trajectories eventually asymptote to punctures. (So, the network is not dense on $C$.)

In the special case $\mathcal{O}_1 = \mathbb{A}_2$ this leads to a simple structure: trajectories cannot cross so get a cell decomposition of $C$.

$\Rightarrow$ an ideal triangulation.)
But more generally, these networks may be rather intricate.
(Show some examples, varying with $i$.)

**BPS degeneracies**

At some special values of $i$, spectral network jumps discontinuously.
At these $i$, there exist collisions—sub-networks where trajectories meet head-on:

Whenever we have such a sub-network, it can be canonically lifted to a class $\gamma \in H_1(\Sigma, \mathbb{Z})$.
For any $\gamma \in H_1(\Sigma, \mathbb{Z})$, define $\Omega(\gamma) \in \mathbb{Z}$ to be the # of such collisions w/ lift $\gamma$.

**Conj.**

1) $\Omega(\gamma)$ are BPS degeneracies in the 4d $N=2$ theory $S[A_{k-1}, C]$ obtained by compactifying $(2,0)$ theory on $C$. (String networks)
2) $\Omega(\gamma)$ depend on $(\varphi_1, \ldots, \varphi_k)$ in a way governed by Kontsevich-Soibelman WCF.
3) $\Omega(\gamma)$ are gen DT invariants attached to Fukaya category of a local CY:
   [Diaconescu-Douglas-Parter, Bridgeland-Smith for $A_2$]
   $A_{k-1}$ singularity fibered over $C$.

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**Framed wall sng**

The spectral network consists of walls of marginal stability:

In $S[A_{k-1}, C]$ we have BPS surface operator $S_z$ for $z \in \mathbb{C}$.
A path from $z$ to $z'$ gives an interface between $S_z$ and $S_{z'}$.
BPS states of this interface can appear/decay when $z$ or $z'$ cross the spectral network.
Cluster coordinates (roughly)

\[ M = \{ \text{flat } G\text{-conn. on } C \text{ w/invariant flag } E \text{ each } s_i \} \]

\[ M_{ab} = \{ \text{ } C^X \text{-conn. on } \text{triv. } L \xrightarrow{L} \Sigma \} \]

We'll give a map \( M_{ab} \rightarrow M \) which we conjecture is \( \sim \) onto open dense subset ("big cell"). i.e. \((C)^2 \rightarrow M. \) [For \( A_1 \) shear coords.]

Idea:\ \( T_\ast (L) \) has induced diagonal connection away from branch pts. But it has monodromy around branch pts.

Modify it by cutting and gluing along the edges of the spectral network: along each \( ij \)-edge we have a canonical \( \Delta: \xi_j \rightarrow \xi_i \)
and we glue by \( 1 + \Delta \)

Resulting conn. is no longer diagonal, extends over branch pts.

\[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]

so gives a point of \( M. \)

- The coordinate functions on \( M \) so obtained are views of "IR line operators" in \( S[C, A_{K-1}] \) on \( TR^3 \times S^2. \)
- When the spectral network jumps, the equiv. coordinate xfrm is expected to be a simple "cluster-like" map: of the form:

\[ X_y \rightarrow X_y (1 - X_y) \]

\[ <r, r> \]