Lecture 1

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Welcome!

All course info is at http://www.ma.utexas.edu/users/neitzke

Systems of Linear Equations (Sec 1.1)

A <u>linear equation</u> is an equation of the form

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_n x_n = b$$

where $a_1, a_2, \ldots, a_n$ are <u>constants</u> ← (real numbers)

b is <u>constant</u>

and $x_1, x_2, \ldots, x_n$ are <u>variables</u>

<u>Ex</u> $3x_1 + 4x_2 = -2$

or $7x_1 + \sqrt[3]{6}\, x_2 + \pi x_3 - 134 x_4 + 12 x_5 = \frac{1}{2}$

or $5x_1 - x_3 = \frac{1}{2}$ ← (here $a_2 = 0$)

but <u>not</u> $x_1^2 + 4x_2 - 7x_3 = 13$

A <u>linear system</u> is a set of linear equations in the same variables $x_1, \ldots, x_n$

<u>Ex</u>
$$\left(\begin{array}{l} -x_1 + 3x_2 = 6 \\ 2x_1 - 2x_2 = -1 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{rcrcrcr} x_1 & & & + & 7x_3 & = & 2 \\ -3x_1 & + & 5x_2 & - & x_3 & = & 0 \end{array}\right)$$

Basic question: Find all the <u>solutions</u> of a linear system
(i.e. find all the values of $x_1, \ldots, x_n$ for which the equations are <u>true</u>)

Ex

$$\begin{aligned} -x_1 + 3x_2 &= 6 \\ 2x_1 - 2x_2 &= 4 \end{aligned}$$

$\Updownarrow$ ← add $2\times$ first eq to second eq

$$\begin{aligned} -x_1 + 3x_2 &= 6 \\ (2+2\cdot -1)x_1 + (-2+2\cdot 3)x_2 &= 4 + 2\cdot 6 \end{aligned}$$

ie

$$\begin{aligned} -x_1 + 3x_2 &= 6 \\ 4x_2 &= 16 \end{aligned}$$

$\vdots$

$$\begin{aligned} x_1 &= 6 \\ x_2 &= 4 \end{aligned}$$

Let's do this more <u>systematically</u>.

Convenient notation: avoid writing the x's over and over —

$$\begin{aligned} 5x_1 \quad\quad\quad + 2x_3 &= 0 \\ -4x_1 + x_2 \quad\quad\quad &= 6 \\ 2x_1 + 3x_2 - 2x_3 &= 4 \end{aligned} \rightsquigarrow \left[\begin{array}{ccc|c} 5 & 0 & 2 & 0 \\ -4 & 1 & 0 & 6 \\ 2 & 3 & -2 & 4 \end{array}\right] \qquad \begin{bmatrix} 5 & 0 & 2 \\ -4 & 1 & 0 \\ 2 & 3 & -2 \end{bmatrix}$$

linear system $\rightsquigarrow$ <u>augmented matrix</u> of the lin. sys. — <u>coefficient matrix</u> of the lin. sys.

Row operations:

1) add a multiple of one row to another row

Ex $\left[\begin{array}{cc|c} 2 & 3 & 4 \\ -1 & 4 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 2+2(-1) & 3+2(4) & 4+2(7) \\ -1 & 4 & 7 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 11 & 18 \\ -1 & 4 & 7 \end{array}\right]$

2) exchange two rows

Ex
$$\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 3 & 8 & 9 \\ 5 & 6 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 4 & 7 \\ 5 & 6 & 1 \\ 3 & 8 & 9 \end{array}\right]$$

3) multiply a row by a nonzero constant

$$\left[\begin{array}{cc|c} 3 & 6 & 9 \\ 5 & -2 & 11 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 5 & -2 & 11 \end{array}\right]$$

If 2 matrices are related by a chain of row operations, call the matrices row equivalent ($\sim$).

Fact: Row operations don't change the set of solutions of the lin. sys.

Ex Find sol'ns of lin. sys.

$$\begin{aligned} 2x_2 + 4x_3 &= 8 \\ x_1 + 4x_2 - x_3 &= 6 \\ 2x_1 - 3x_2 + x_3 &= -7 \end{aligned}$$

$$\times -2 \quad \left[\begin{array}{ccc|c} 0 & 2 & 4 & 8 \\ 1 & 4 & -1 & 6 \\ 2 & -3 & 1 & -7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & 2 & 4 & 8 \\ 1 & 4 & -1 & 6 \\ 0 & -11 & 3 & -19 \end{array}\right]$$

$$\sim \begin{array}{c} \\ \times\frac{1}{2} \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 2 & 4 & 8 \\ 0 & -11 & 3 & -19 \end{array}\right]$$

$$\sim \begin{array}{c} \\ \\ \times 11 \end{array} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -11 & 3 & -19 \end{array}\right]$$

$$\sim \begin{array}{c} \\ \\ \times \frac{1}{25} \end{array} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 25 & 25 \end{array}\right]$$

$$\sim \begin{array}{c} \\ \\ \times -2 \end{array} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

"row echelon form"

$$\begin{aligned} x_1 + 4x_2 - x_3 &= 6 \\ x_2 + 2x_3 &= 4 \\ x_3 &= 1 \end{aligned}$$

At this pt. can solve the eq. by back-sub.

Or, continue with row operations:

$$\begin{array}{c} \\ \times 1 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \begin{array}{c} \times -4 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 4 & 0 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{aligned} x_1 \qquad\qquad &= -1 \\ x_2 \qquad &= 2 \\ x_3 &= 1 \end{aligned}$$

<u>Solved</u>. (Exactly one solution.)

Ex Find sol^{n}s...

$$\begin{aligned} x_1 + 3x_2 + 4x_3 &= 7 \\ 2x_2 - x_3 &= 6 \\ x_1 + x_2 + 5x_3 &= 3 \end{aligned}$$

$$\times -1 \quad \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 2 & -1 & 6 \\ 1 & 1 & 5 & 3 \end{array}\right]$$

$$\sim \quad \times 1 \quad \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 2 & -1 & 6 \\ 0 & -2 & 1 & -4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 2 & -1 & 6 \\ 0 & 0 & 0 & 2 \end{array}\right] \qquad \begin{aligned} x_1 + 3x_2 + 4x_3 &= 7 \\ 2x_2 - x_3 &= 6 \\ \underline{\underline{0}} &\underline{\underline{= 2}} \end{aligned}$$

So this lin. sys. has <u>no solutions</u>!

Or: this lin. sys. is <u>inconsistent</u>.

[If a lin. sys. has solutions, call it <u>consistent</u>

" " " " no solutions, call it <u>inconsistent</u>.]

Ex Find sol...

$$\begin{aligned} 4x_1 - 8x_2 &= 12 \\ 3x_1 - 6x_2 + 3x_3 &= 24 \\ -x_1 + 2x_2 + 2x_3 &= 7 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 4 & -8 & 0 & 12 \\ 3 & -6 & 3 & 24 \\ -1 & 2 & 2 & 7 \end{array}\right]$$

$$\sim \cdots \sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x_1 - 2x_2 &= 3 \\ x_3 &= 5 \\ 0 &= 0 \end{aligned}$$

ie $\quad x_1 = 3 + 2x_2$

$\quad\;\; x_3 = 5$

$x_2 =$ <u>anything</u>

i.e x_2 is a free variable.

So this lin. sys has <u>infinitely many</u> solutions.

We say a matrix is in Row Echelon Form (REF) if:

- All rows which contain only zeroes are at the <u>bottom</u>
- The leading entry (first nonzero entry) of each row is to the right of the leading entry of the row above

<u>Ex</u>

$$\begin{bmatrix} 2 & -3 & 5 & 7 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 6 & 2 \\ 0 & -2 & -3 \\ 0 & 0 & 5 \end{bmatrix}$$

Ex

$$\begin{bmatrix} 0 & 2 & 1 & 4 & 7 \\ 0 & 0 & 3 & 8 & -1 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

[ie: "each row begins with more zeroes than the row above it"]

Call the leading entries <u>pivots</u> (circled).

We say a matrix is in Reduced Row Echelon Form (RREF) if:

- It is in row echelon form
- The pivots are 1
- All entries above any pivot are 0

Ex

$$\begin{bmatrix} 1 & 6 & 0 & 9 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Fact: Every matrix is row equivalent to one which is in RREF.
The RREF is <u>unique</u>.

From the RREF we can easily read out the <u>solutions</u> of the associated linear system (<u>next time</u>)