Lecture 12

17 Oct 2010

Exam solutions now posted @ course web pages

Up to now: we've been studying $\mathbb{R}^n$

Now: a more general/abstract POV on linear algebra

Vector Spaces (Sec 4.1)

A vector space V is a set whose elements ("vectors") can be added to one another and can be multiplied by scalars (constants) obeying these axioms:

- If $\vec{x}, \vec{y}$ are in V then $\vec{x}+\vec{y}$ is in V.
- If $\vec{x}$ is in V then $c\vec{x}$ is in V for any constant c.
- $\vec{x}+\vec{y} = \vec{y}+\vec{x}$.
- $(\vec{x}+\vec{y})+\vec{w} = \vec{x}+(\vec{y}+\vec{w})$
- There is a vector $\vec{0}$ in V such that $\vec{0}+\vec{x}=\vec{x}$ for all $\vec{x}$ in V.
- $c(\vec{x}+\vec{y}) = c\vec{x}+c\vec{y}$.
- $(c+d)\vec{x} = c\vec{x}+d\vec{x}$.
- $c(d\vec{x}) = (cd)\vec{x}$
- $1\vec{x} = \vec{x}$.
- For every $\vec{x}$ in V there is another vector $-\vec{x}$ in V such that $\vec{x}+(-\vec{x}) = \vec{0}$.

Facts: If V is a vector space and $\vec{x} \in V$ then

- $(-1)\vec{x} = -\vec{x}$
- $c\cdot\vec{0} = \vec{0}$
- $0\cdot\vec{x} = \vec{0}$

Ex $V = \mathbb{R}^n$ is a vector space, for any n.
(with our previous def^n of $\vec{x}+\vec{y}$ and $c\vec{x}$)

Ex $V = \{$all doubly-infinite sequences of numbers$\}$

e.g. $\vec{y} = (\ldots, y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3, \ldots)$ each y_i is a constant

e.g. $\vec{y} = (\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots)$

$\vec{y} = (\ldots, 0, 0, 0, 0, 0, 0, 0, \ldots)$

$\vec{y} = (\ldots, 1, 1, 1, 2, 1, 1, 1, \ldots)$

Rule for addition: $\vec{y} = (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots) = (y_k)$

$\vec{x} = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) = (x_k)$

Then we *define* $\vec{x}+\vec{y} = (\ldots, x_{-2}+y_{-2}, x_{-1}+y_{-1}, x_0+y_0, x_1+y_1, x_2+y_2, \ldots)$

$= (x_k + y_k)$

$c\vec{x} = (\ldots, cx_{-2}, cx_{-1}, cx_0, cx_1, cx_2, \ldots) = (cx_k)$

These rules obey all the vector space axioms.

e.g. $c(\vec{x}+\vec{y}) \stackrel{?}{=} c\vec{x} + c\vec{y}$

$\|$

$c(\ldots, x_{-1}+y_{-1}, x_0+y_0, x_1+y_1, \ldots)$

$\|$

$(\ldots, cx_{-1}+cy_{-1}, cx_0+cy_0, cx_1+cy_1, \ldots)$

$\|$

$(\ldots, cx_{-1}, cx_0, cx_1, \ldots) + (\ldots, cy_{-1}, cy_0, cy_1, \ldots)$

$\|$

$c\vec{x} + c\vec{y}$ ✓

Ex $V = \mathbb{P}_n = \{$all polynomials with real coefficients, of degree $\leq n\}$

e.g. if $n=3$,
$$f = x^3 - 3x^2 + 4x - 7 \in \mathbb{P}_3$$
$$g = \quad 2x^2 - 3x + 9 \in \mathbb{P}_3$$

Define addition, mult. in the "obvious" way.

e.g.
$$f+g = x^3 - x^2 + x + 2 \in \mathbb{P}_3$$
$$2f = 2x^3 - 6x^2 + 8x - 14 \in \mathbb{P}_3$$

We could check that V obeys all the vector space axioms.
(But I won't do it here.)

[Why not take $\mathbb{P}_n' = \{$all poly. of degree exactly $n\}$?
Then if $f, g \in \mathbb{P}_n'$ $f+g$ might not be: e.g.
$$f = x^2 + 3 \in \mathbb{P}_2'$$
$$g = -x^2 + 7x \in \mathbb{P}_2'$$
$$f+g = 7x + 3 \notin \mathbb{P}_2'$$
So $\mathbb{P}_n'$ is not a vector space!]

Ex $V = \mathcal{F} = \{$all real-valued continuous functions of one variable$\}$

e.g.
$$f(x) = \sin x \in V$$
$$f(x) = x^7 - \cos x + \sin\left(\frac{x^2+1}{17}\right) \in V$$

Additive law: Given $f \in V$ and $g \in V$ define $f+g \in V$
by $(f+g)(x) = f(x) + g(x)$

Scalar mult: Given $f \in V$ and constant c define $cf \in V$
by $(cf)(x) = c \cdot f(x)$

Checking the vector space axioms:

- One of the axioms is that there is a vector $\vec{0} \in V$ such that $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in V$.

In $V = \mathcal{F}$, $\vec{0}$ is the zero function: $f_0(t) = 0$ for all t

Indeed, if $f \in V$ is any vector (function) then $f + f_0 = f$

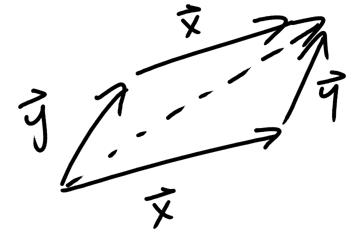
$$\left[\begin{aligned}(f+f_0)(t) &= f(t) + f_0(t) \\ &= f(t) + 0 \\ &= f(t)\end{aligned}\right]$$

- We could also (+ should) check that V satisfies the rest of the vector space axioms, e.g. $c(d\vec{x}) = (cd)\vec{x}$

which here becomes $c(df) = (cd)f$

(I won't check them all now...)

Even though "vector" now means any element of V — not necessarily a column of numbers — can still use some intuition from previous chapters...



But remember that $\vec{x}, \vec{y}$ are elements of V now!

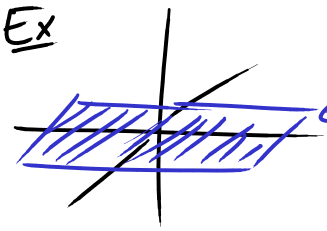
Subspaces

Say V is a vector space.

A <u>subspace</u> H of V is a subset of V with 3 properties:

- The vector $\vec{0} \in V$ is contained in H.
- If $\vec{u}, \vec{v}$ are in H then $\vec{u}+\vec{v}$ is also in H. ("closed under addition")
- If $\vec{u}$ is in H and c is any constant then $c\vec{u}$ is in H. ("closed under mult")

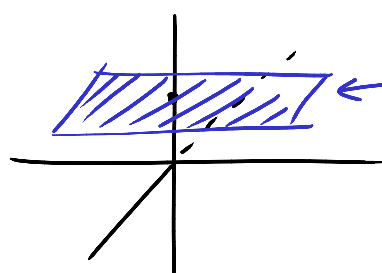
<u>Ex</u> $V = \mathbb{R}^3$



$H =$ xy-plane

$= \left\{ \text{vectors of the form } \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \right\}$

<u>But not</u>



$H' = \left\{ \text{vectors of form } \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \right\}$

because $\underset{\in H'}{\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}} + \underset{\in H'}{\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}} = \underset{\notin H'}{\begin{pmatrix} x+x' \\ y+y' \\ 2 \end{pmatrix}}$ so H' is <u>not</u> closed under addition

ie H' is <u>not</u> <u>a subspace!</u>

<u>Ex</u> For <u>any</u> vector space V, the subset $H = \{\vec{0}\}$ is a subspace.

(Why? $\vec{0}+\vec{0} = \underset{\in H}{\vec{0}}$ and $c \cdot \vec{0} = \underset{\in H}{\vec{0}}$)

Ex Define $H' = \left\{ \begin{array}{l} \text{all vectors in } \mathbb{R}^3 \text{ which have } 0 \text{ as \underline{at least one}} \\ \text{of their entries} \end{array} \right\}$

H' is <u>not</u> a subspace of $\mathbb{R}^3$:

e.g. $$\underset{\substack{\uparrow \\ H'}}{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}} + \underset{\substack{\uparrow \\ H'}}{\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}} = \underset{\substack{\not\uparrow \\ H'}}{\begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}}$$

Ex Say $V = \mathcal{F}$

Then: $H = \{\text{all polynomial functions}\}$ is a subspace of V.

[Because:
- the zero function is a polynomial
- the sum of 2 poly. is a poly.
- a scalar multiple of a poly. is a poly.]

And: $H = \{\text{all } \underline{\text{periodic}} \text{ functions with period } 1\}$ is a subspace of V.

[
- zero f^n is periodic
- sum of 2 periodic f^n's is periodic
- a scaler multiple of periodic f^n is periodic]

Ex If $\vec{v}_1$ and $\vec{v}_2$ are elements of a vector space V

Define $\text{Span}\{\vec{v}_1, \vec{v}_2\}$ to be the set of all lin. comb. of $\vec{v}_1$ and $\vec{v}_2$

i.e. all vectors of the form $\quad x_1\vec{v}_1 + x_2\vec{v}_2 \quad x_1, x_2$ constants

Then $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ is a subspace of V.

Why?
- $\vec{0} \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$
- $(x_1\vec{v}_1 + x_2\vec{v}_2) + (x_1'\vec{v}_1 + x_2'\vec{v}_2) = (x_1 + x_1')\vec{v}_1 + (x_2 + x_2')\vec{v}_2 \in H$

  so H is closed under addition
- $c(x_1\vec{v}_1 + x_2\vec{v}_2) = (cx_1)\vec{v}_1 + (cx_2)\vec{v}_2 \in H$

  so H is closed under scalar mult.

Fact If $\vec{v}_1, \ldots, \vec{v}_p$ are vectors in V then $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a subspace of V.

Why? Just like the above example
