Lecture 13

12 Oct 2010

Last time: vector spaces and subspaces —

If V is a vector space, and H a subset of V, we say H is a subspace if and only if

1) H contains the zero vector $\vec{0}$ of V
2) H is closed under addition
3) H is closed under scalar multiplication

Ex Suppose V is $\mathcal{F}$, the space of all real-valued functions $f(t)$ on the real line.
Suppose H is the set of all functions $f(t)$ obeying $f(3) = 0$.
Is H a subspace of V?

Let's check:

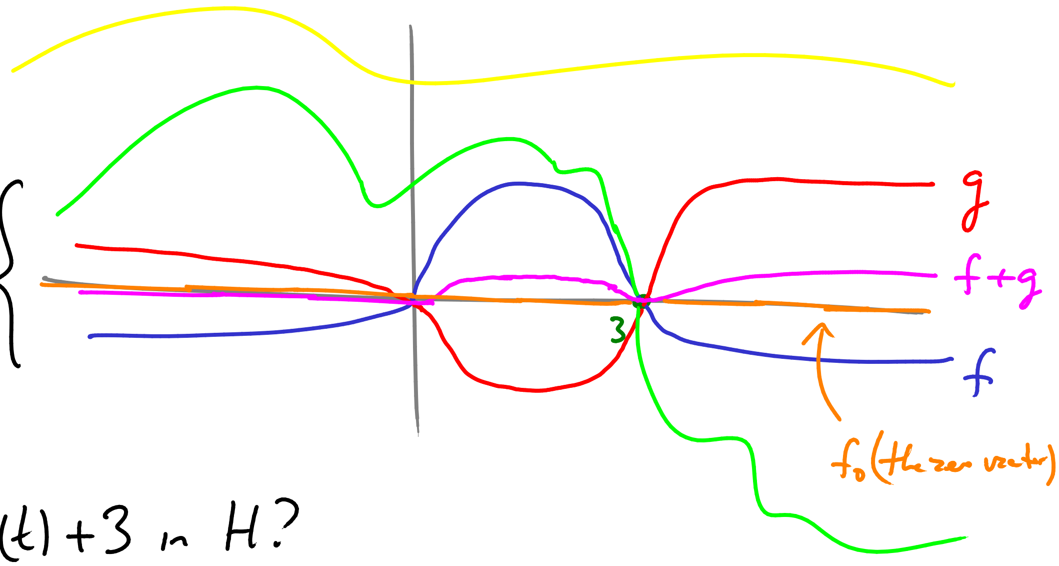
1) In V, the zero vector is the function $f(t) = 0$ (for all t).
This function does belong to H because it obeys $f(3) = 0$. ✓

2) Take two vectors in H: $f(t)$, $g(t)$ with $f(3) = 0$, $g(3) = 0$.
Then look at the vector $j = f + g$
Is j in H?
$j(3) = f(3) + g(3) = 0 + 0 = 0$
So j is in H. ✓

3) Take any vector in H: $f(t)$ with $f(3) = 0$.
And take any scalar c in $\mathbb{R}$.
Then look at $g = c \cdot f$
Is g in H?
$g(3) = c \cdot f(3) = c \cdot 0 = 0$
So g is in H. ✓

All 3 properties satisfied so H is a subspace of V.

Illustration:



Q: Is $f(t) = \sin(t) + 3$ in H?

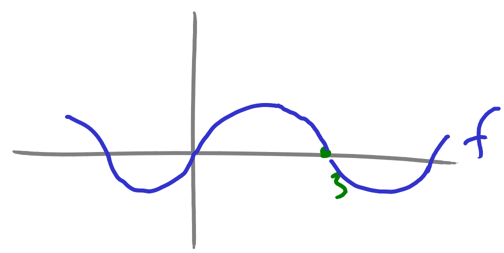
$f(3) = \sin(3) + 3 \neq 0$ so f is not in H.

$f(t) = \sin\left(\frac{t}{3}\right) + 2$?

$f(3) = \sin(1) + 2 \neq 0$ so f is not in H.

$f(t) = \sin\left(\frac{\pi t}{3}\right)$?

$f(3) = \sin(\pi) = 0$ so f is in H.



Null Spaces, Column Spaces and Linear Transformations (Sec 4.2)

Say A is $m \times n$ matrix.

The null space $\mathrm{Nul}\, A$ is the set of all vectors $\vec{x}$ in $\mathbb{R}^n$ obeying the equation $A\vec{x} = \vec{0}$.

Fact: $\mathrm{Nul}\, A$ is a subspace of $\mathbb{R}^n$.

Why? 1) $\vec{0}$ obeys $A \cdot \vec{0} = \vec{0}$, so $\vec{0}$ is in $\mathrm{Nul}\, A$.

2) If $A\vec{x} = 0$ and $A\vec{y} = 0$ then $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
$= \vec{0} + \vec{0} = \vec{0}$.

3) If $A\vec{x}=0$ then $A(c\vec{x}) = c\cdot A\vec{x} = c\cdot\vec{0}=\vec{0}$.

So Nul A obeys the 3 conditions for being a subspace of $\mathbb{R}^n$.

How do we describe Nul A less abstractly?

Ex Find a set of vectors which span Nul A where

$$A=\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Solve $A\vec{x}=\vec{0}$: usual row reduction gives

$$x_1 = 2x_2 + x_4 - 3x_5$$
$$x_3 = -2x_4 + 2x_5$$
x_2, x_4, x_5 free

i.e.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2+x_4-3x_5 \\ x_2 \\ -2x_4+2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

So any vector $\vec{x}$ in Nul A is a linear combination of $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

and vice versa: any linear comb. of those vectors is in Nul A

$$\text{So Nul } A = \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Remarks:

- This procedure always produces a linearly independent spanning set for Nul A.
- The # of vectors in the spanning set we get is = to the # of free variables we get in solving $A\vec{x}=\vec{0}$.

Column Space

Say A an $m \times n$ matrix.

The column space of A, $\mathrm{Col}\, A$, is the set of all linear combinations of the columns of A:

if $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$

$$\mathrm{Col}\, A = \mathrm{Span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$$

Fact: $\mathrm{Col}\, A$ is a subspace of $\mathbb{R}^m$.

[Why? b/c we wrote it as Span of a collection of vectors]

NB: to a matrix A we attached 2 subspaces

$\mathrm{Nul}\, A$, $\mathrm{Col}\, A$

"nota bene"

They are very different!

Linear transformations

Say V and W are two vector spaces.

A function $T : V \to W$ is called a linear transformation if:

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for every $\vec{u}, \vec{v}$ in V
- $T(c\vec{v}) = c\,T(\vec{v})$ for every $\vec{v}$ in V and every constant c

Ex If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ this is just the notion of linear transformation we had before.

Ex If $V = \mathcal{F} = \{\text{smooth functions on the real line}\}$ define a transformation $T: \mathcal{F} \to \mathcal{F}$

by
$$T(f) = f'$$

Is T linear?

- $T(f+g) = (f+g)' = f' + g' = T(f) + T(g)$ ✓
- $T(cf) = (cf)' = c \cdot f' = c \cdot T(f)$ ✓

Ex $T: \mathcal{F} \to \mathcal{F}$

$T(f) = 3f + f''$ is also a linear transformation. (Check it!)

Ex $T: \mathcal{F} \to \mathcal{F}$

$T(f) = ff'$ is *not* a linear transf.

Why? $T(cf) = (cf)(cf)' = c^2 ff'$

But $c \cdot T(f) = c \cdot ff'$

So $T(cf) \neq c \cdot T(f)$ (if $c \neq 1$)

Aside: A linear ODE like
$$f'' + f = 0$$
can be understood as
$$T(f) = 0$$
where $T: \mathcal{F} \to \mathcal{F}$ is a linear transformation $T(f) = f'' + f$

Kernel

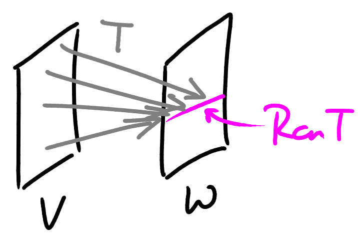
If we have a linear transformation $T: V \to W$ the kernel of T, written $\operatorname{Ker} T$, is the set of all vectors $\vec{v}$ in V with $T(\vec{v}) = \vec{0}$.

Fact: $\operatorname{Ker} T$ is a subspace of V.

Why?
- $\vec{0}$ is in $\operatorname{Ker} T$ because $T(\vec{0}) = \vec{0}$. ✓
- $\operatorname{Ker} T$ is closed under addition: if $\vec{v}$ and $\vec{w}$ are in $\operatorname{Ker} T$ then $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w}) = \vec{0} + \vec{0} = \vec{0}$ so $\vec{v}+\vec{w}$ is in $\operatorname{Ker} T$ ✓
- $\operatorname{Ker} T$ is closed under scalar mult: if $\vec{v}$ is in $\operatorname{Ker} T$ $T(c\vec{v}) = cT(\vec{v}) = c \cdot \vec{0} = \vec{0}$ so $c\vec{v}$ is in $\operatorname{Ker} T$ ✓

Range

If we have a lin. trans. $T: V \to W$ the range of T, $\operatorname{Ran} T$, is the set of all $\vec{w} \in W$ such that $\vec{w} = T(\vec{v})$ for some $\vec{v}$ in V.



Fact: $\operatorname{Ran} T$ is a subspace of W.

Why?
- $\vec{0} = T(\vec{0})$ so $\vec{0}$ is in $\operatorname{Ran} T$ ✓
- If $\vec{w}_1$ and $\vec{w}_2$ are in $\operatorname{Ran} T$ then $T(\vec{v}_1) = \vec{w}_1$, $T(\vec{v}_2) = \vec{w}_2$. Then $T(\vec{v}_1 + \vec{v}_2) = \vec{w}_1 + \vec{w}_2$, so $\vec{w}_1 + \vec{w}_2$ is also in $\operatorname{Ran} T$ ✓
- If $\vec{w}$ is in $\operatorname{Ran} T$ then $T(\vec{v}) = \vec{w}$ for some $\vec{v}$. Then $T(c\vec{v}) = c\vec{w}$ so $c\vec{w}$ is also in $\operatorname{Ran} T$ ✓

So: if T is a lin. trans $\mathbb{R}^n \longrightarrow \mathbb{R}^m$

given by $T(\vec{x}) = A\vec{x}$ $\quad A$ $m \times n$ matrix

then $\operatorname{Ker} T = \operatorname{Nul} A$

$\operatorname{Ran} T = \operatorname{Col} A$

Linearly Independent Sets and Bases

Linear independence for a set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in a vector space V is defined just like before: we say $\{\vec{v}_1, \dots, \vec{v}_p\}$ is lin. indep. if and only if the eq. $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}$ has only the solution $c_1 = 0, c_2 = 0, \dots, c_p = 0$.

Ex The set $\{\sin t, \cos t\}$ is linearly independent in $\mathbb{F}$.