Lecture 15 — 19 Oct 2010

Last time: a set $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ of vectors in V is a <u>basis</u> for V if:

- $\mathcal{B}$ is lin indep
- $\mathrm{Span}(\mathcal{B}) = V$

<u>Coordinate mapping</u> $\vec{x} \longrightarrow [\vec{x}]_{\mathcal{B}}$ or $[\vec{x}]_{\mathcal{B}} \longrightarrow \vec{x}$

$\vec{x} \in V$

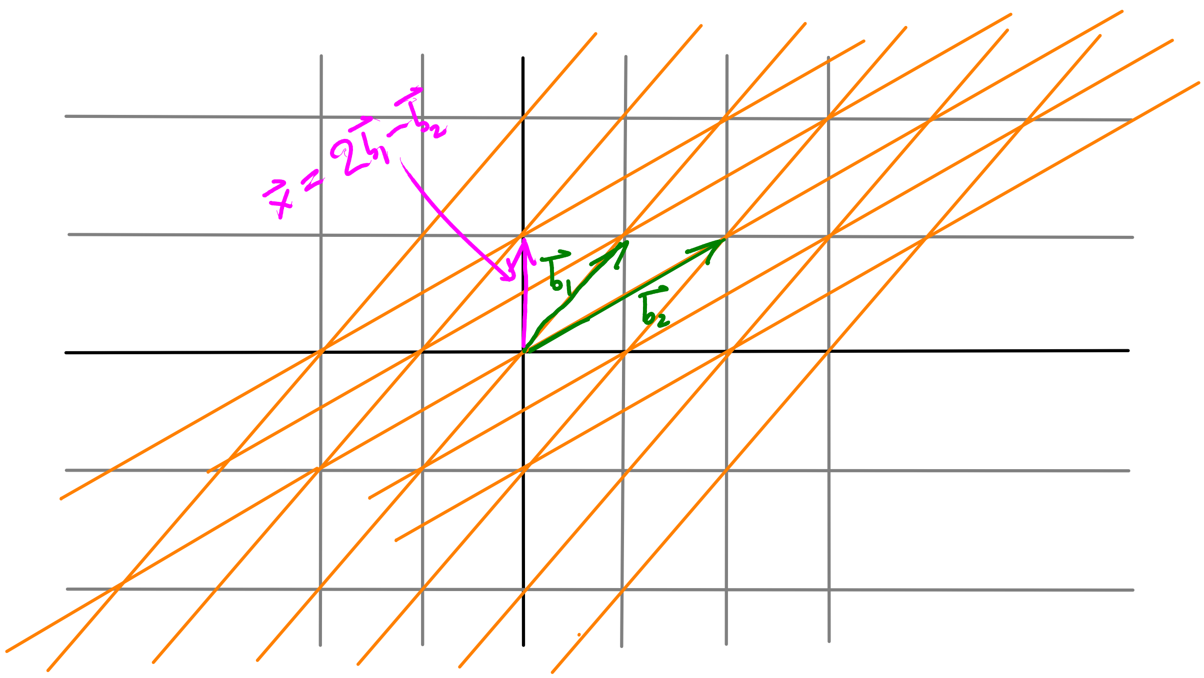
$[\vec{x}]_{\mathcal{B}} \in \mathbb{R}^n$

If $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ then $\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$

<u>e.g.</u> if $V = \mathbb{R}^2$ and $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

here $\vec{x} \in \mathbb{R}^2$

$[\vec{x}]_{\mathcal{B}} \in \mathbb{R}^2$



$\vec{x} = 2\vec{b}_1 - \vec{b}_2$

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Passing between $\vec{x}$ and $[\vec{x}]_{\mathcal{B}}$ or vice versa, when $V = \mathbb{R}^n$:

In this case the coord. mapping $[\vec{x}]_{\mathcal{B}} \longmapsto \vec{x}$ takes $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

And it's a <u>linear</u> xform. So what is its standard matrix?

i.e. what matrix $P_{\mathcal{B}}$ has $P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \vec{x}$?

Remember the recipe for finding standard matrix: apply the coord. mapping to vectors $\vec{e}_1, \dots, \vec{e}_n$ to get the columns of $P_{\mathcal{B}}$.

Ex $V = \mathbb{R}^2$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\} = \{\vec{b}_1, \vec{b}_2\}$

Then if $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then $\vec{x} = 1 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 0 \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

if $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $\vec{x} = 0 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

So $\vec{x} = P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}}$ where $P_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 4 & 3 \end{bmatrix}$

Fact $P_{\mathcal{B}} = \begin{bmatrix} \vec{b}_1 \cdots \vec{b}_n \end{bmatrix}$

So finding $\vec{x}$ from $[\vec{x}]_{\mathcal{B}}$ is easy.

Finding $[\vec{x}]_{\mathcal{B}}$ from $\vec{x}$: have to solve $\vec{x} = P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}}$ for $[\vec{x}]_{\mathcal{B}}$

Could do this by row red, or, find the inverse matrix $P_{\mathcal{B}}^{-1}$ and use $P_{\mathcal{B}}^{-1} \vec{x} = [\vec{x}]_{\mathcal{B}}$.

The Dimension of a Vector Space (Sec 4.5)

Fact If V has a basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ then any set of vectors in V containing more than n vectors is linearly dependent.

Why? This is a slight extension of something we already know for $V = \mathbb{R}^n$: there we have the basis $\mathcal{B} = \{\vec{e}_1, \dots, \vec{e}_n\}$ (standard basis) and indeed, every set of $> n$ vectors in $\mathbb{R}^n$ is linearly dependent.

To see it's true for any V, use the coordinate mapping $V \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^n$

$\Rightarrow$ Fact If V has a basis of n vectors, then every basis of V has n vectors.

[Why? If have 2 bases with n and m vectors, and $m > n$, then we have m lin. indep. vectors in V $\Rightarrow$ ~~impossible~~ by previous fact]

So: If V is spanned by some finite set of vectors, then it has a basis — call V finite-dimensional, and define

$$\dim V = \#\text{ of vectors in a basis of } V$$

If V is not spanned by any finite set of vectors, then it has no basis — call V infinite-dimensional

Ex $V = \mathbb{R}^n$: has std basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$ $\Rightarrow$ $\dim \mathbb{R}^n = n$

Ex $V = \mathbb{P}_n$: has basis $\{1, t, t^2, \ldots, t^n\}$ $\Rightarrow$ $\dim \mathbb{P}_n = n+1$

Ex $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ has basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ $\Rightarrow$ $\dim H = 2$

Ex Subspaces of $\mathbb{R}^3$:

point (origin)	$\dim = 0$
line	$\dim = 1$
plane	$\dim = 2$
whole $\mathbb{R}^3$	$\dim = 3$

Ex $V = \mathcal{F} = \{\text{all real-valued functions on the real line}\}$ is infinite-dimensional

Ex $H = \{\text{all functions of the form } f(t) = at\}$ is a subspace of $\mathcal{F}$

What is a basis for H? Take the function $f(t) = t$. It's an element of H. Look at $\mathcal{B} = \{f(t)\}$.

$\mathcal{B}$ is linearly indep. (it's a set containing one vector which is not $\vec{0}$)
$\mathcal{B}$ spans H. (any element of H is a scalar multiple of $f(t)=t$)
So $\mathcal{B}$ is a basis for H. So $\dim H = 1$.

Fact If V is finite-dimensional
and H is a subspace of V
then H is finite-dimensional and $\dim H \leq \dim V$.

Ex What are the possible dim's of subspaces of $\mathbb{P}_5$?

$6, 5, 4, 3, 2, 1, 0$.

Fact Say $\dim V = n$.
Any set of n vectors which is lin indep is a basis for V.
Any set of n vectors which span V is a basis for V.

Fact Suppose A is an $m \times n$ matrix.
$\dim(\text{Nul } A) = $ # free variables in eq. $A\vec{x} = \vec{0}$.
$\dim(\text{Col } A) = $ # pivot columns in A.

Ex If $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

Nul A is subspace of $\mathbb{R}^5$
Col A is " " $\mathbb{R}^3$

What are $\dim \text{Nul } A$, $\dim \text{Col } A$?

Row reduce:

$$A \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 cols w/o pivots $\Rightarrow$ $A\vec{x}=\vec{0}$ has 3 free var $\Rightarrow$ $\dim \operatorname{Nul} A = 3$

2 cols w/ pivots $\Rightarrow$ $\dim \operatorname{Col} A = 2$

$2+3=5$

Fact If A is $m \times n$ matrix

$$\dim \operatorname{Nul} A + \dim \operatorname{Col} A = n$$

Ex If columns of A are lin indep then $\dim \operatorname{Nul} A = 0$

$\dim \operatorname{Col} A = n$

$$\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$$