Lecture 16 | 21 Oct 2010

HW6 will be graded by this afternoon — pick up in box outside my office (<u>anytime</u>)

Midterm 2: 2 weeks from today

covers Ch 3 (determinants)

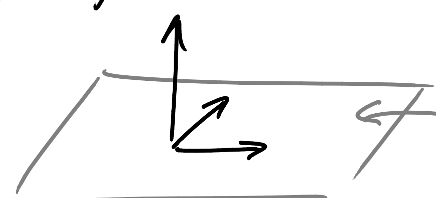
Ch 4 (vector spaces, subspaces, etc)

a little of Ch 5 (eigenvalues/eigenvectors)

Last time: <u>dimension</u> of a vector space V = # vectors in a <u>basis</u> for V

– $\dim \mathbb{R}^5 = 5$ ($\mathbb{R}^5$ has standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5\}$)

– $\dim \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} = 2$



$\longleftarrow \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

Why? Write $H = \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ span H (by definition: this <u>means</u> $H = \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$)

<u>and</u> $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is lin. indep.

<u>So</u> $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is a <u>basis</u> for H. So $\dim H = 2$

– $\mathbb{P}_4$ is 5-dimensional (basis $\mathcal{B} = \{1, t, t^2, t^3, t^4\}$)

Coordinate mapping

$$\mathbb{P}_4 \to \mathbb{R}^5$$

$$p \mapsto [p]_\beta$$

$$t^4 - t^2 + 3t \mapsto \begin{bmatrix} 0 \\ 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

– $H = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_3} \right\}$ dim H?

Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for H?

- They span H
- But they're <u>not</u> lin indep, so <u>not</u> a basis: $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$

Throw $\vec{v}_3$ away to get $\{\vec{v}_1, \vec{v}_2\}$.

- $\{\vec{v}_1, \vec{v}_2\}$ still span H.
- And $\{\vec{v}_1, \vec{v}_2\}$ <u>is</u> lin indep. So $\{\vec{v}_1, \vec{v}_2\}$ <u>is</u> a basis.

So dim $H = 2$.

– $H = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_3}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_4} \right\}$

Find dim H
and find a basis for H.

- $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ span H but are lin dep: $\vec{v}_3 = \vec{v}_2 + \vec{v}_4$

So throw away one vector which is a lin comb. of the others: any of $\vec{v}_2, \vec{v}_3$, or $\vec{v}_4$ will work

e.g. take $\vec{v}_2$:

- $\{\vec{v}_1, \vec{v}_3, \vec{v}_4\}$ still spans H
- and $\{\vec{v}_1, \vec{v}_3, \vec{v}_4\}$ is lin indep

So $\dim H = 3$, H has basis $\{\vec{v}_1, \vec{v}_3, \vec{v}_4\}$

Rank (Sec 4.6)

Suppose A $m \times n$ matrix.

Define **rank A = # pivots in REF of A**

Fact: rank A = dim Col A

[Why? The pivot columns of A form a basis of Col A]

Ex

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 3 & 1 & 3 & -2 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for Col A.

$$A \underset{\times\frac{3}{5}}{\sim} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & -5 & 0 & 1 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & -5 & 0 & 1 \\ 0 & 0 & 4 & 8/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \\ 0 \end{bmatrix} \right\}$ is basis for Col A.

Here rank $A = 3$.

So far, to an $m \times n$ matrix A we associated two subspaces:

$\text{Col } A$ (subsp. of $\mathbb{R}^m$)

$\text{Nul } A$ (subsp. of $\mathbb{R}^n$)

Now introduce one more:

$\text{Row } A = \text{Span}\{\vec{a}_1, \dots, \vec{a}_m\}$ where $\vec{a}_1, \dots, \vec{a}_m$ are the rows of A

Ex $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 7 \end{bmatrix}$ $\quad \text{Row } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 7 \end{bmatrix} \right\}$

$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \end{bmatrix} \right\}$

$\dim \text{Row } A = 2$: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 7 \end{bmatrix} \right\}$ lin indep.

$\dim \text{Col } A = 2$: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \end{bmatrix} \right\}$ is not lin. indep.

but $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ is lin. indep.

This "coincidence" always happens: even though $\text{Row } A$, $\text{Col } A$ are very different (even live inside different spaces, e.g. above $\text{Row } A$ is inside $\mathbb{R}^3$ while $\text{Col } A$ inside $\mathbb{R}^2$)

yet, their dimensions are the same!

ie, Fact: $\text{rank } A = \dim \text{Row } A = \dim \text{Col } A$.

Why? See it thru next example:

Ex $A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & -2 & -4 \\ 3 & 0 & -3 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad$ Find bases for $\text{Row } A$, $\text{Col } A$, $\text{Nul } A$.

First, row reduce A:

$$A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B$$

- Basis for Col A: pivot columns of A $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

- Basis for Row A: rows of B containing pivots $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

- To find Nul A: solve $A\vec{x} = \vec{0}$ for $\vec{x}$

  use RREF of A above: $\begin{array}{l} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 \text{ free} \end{array}$ $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

so $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A

Summary

dim Col $A = 2$ dim Row $A = 2$ dim Nul $A = 1$

rank $A = 2$

NB: rank A + dim Nul A = 3

Generally: Fact: rank A + dim Nul A = n (# columns of A)

[Why? rank = # pivot cols
dim Nul A = # free vars = # cols w/o pivots]

Change of Basis (Sec 4.7)

Suppose V vector space

$\mathcal{B}$ a basis for V $\quad \mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$

$\mathcal{C}$ a basis for V $\quad \mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ $\quad n = \dim V$

For any vector $\vec{x}$ in V have 2 coord. vectors $[\vec{x}]_{\mathcal{B}}$, $[\vec{x}]_{\mathcal{C}}$

Ex $V = \mathbb{P}_2$ $\quad \mathcal{B} = \{1, t, t^2\}$

$\mathcal{C} = \{1 - t + t^2,\ t + t^2,\ t\}$

$\vec{x}$ = the polynomial $1 + 2t^2$ $\quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $\quad [\vec{x} = 1 \cdot 1 + 0 \cdot t + 2 \cdot t^2]$

$[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad \left[\begin{array}{l} \vec{x} = 1 \cdot (1 - t + t^2) \\ + 1 \cdot (t + t^2) \\ + 0 \cdot t \end{array}\right]$

How do we relate $[\vec{x}]_{\mathcal{B}}$ to $[\vec{x}]_{\mathcal{C}}$?

(e.g. given one, how do we find the other?)

There is a matrix ("change of basis matrix") $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$

such that $$[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\vec{x}]_{\mathcal{B}}$$

Fact: $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ is $$\begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{bmatrix}$$

How does that help? Need to be able to find the vectors $[\vec{b}_1]_{\mathcal{C}}, \dots, [\vec{b}_n]_{\mathcal{C}}$.

To do that:

$$P_{\mathcal{C}} [\vec{b}_1]_{\mathcal{C}} = \vec{b}_1 \qquad (\text{where } P_{\mathcal{C}} = [\vec{c}_1 \cdots \vec{c}_n])$$

So can solve this for $[\vec{b}_1]_{\mathcal{C}}$

and then do similarly for $[\vec{b}_2]_{\mathcal{C}}, \dots, [\vec{b}_n]_{\mathcal{C}}$

ie: have to do row reduction of $[P_{\mathcal{C}} | \vec{b}_1], [P_{\mathcal{C}} | \vec{b}_2], \dots, [P_{\mathcal{C}} | \vec{b}_n]$

Can do them all at once: take the matrix $[P_{\mathcal{C}} | \vec{b}_1 \cdots \vec{b}_n]$

ie $[P_{\mathcal{C}} | P_{\mathcal{B}}]$

and row reduce that:

we'll get $[I \mid \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}]$

Ex $V = \mathbb{R}^2$ w/ two bases

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} \quad \vec{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

$\mathcal{C} = \{\vec{c}_1, \vec{c}_2\} \quad \vec{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \quad \vec{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

What is the matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$?

Row reduce $[P_{\mathcal{C}} | P_{\mathcal{B}}] = \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right]$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$