Lecture 2

31 Aug 2010

Reminder:
- 1^{st} HW due Thursday in class
- Class info at http://www.ma.utexas.edu/users/neitzke

Last time:
- Systems of linear equations, their augmented matrices
- Row equivalence
- Row echelon form (REF), reduced row echelon form (RREF)

Recall Fact: Every matrix is row equivalent to one in RREF.
(The RREF is unique.)

Algorithm for putting a matrix in REF by row operations:

e.g. $\begin{bmatrix} 0 & * & * & * \\ \blacksquare & * & * & * \\ * & * & * & * \end{bmatrix}$

$*$ = anything

$\blacksquare$ = anything not zero

1) By swapping rows, make the leftmost nonzero entry be in the top row.

$$\longrightarrow \begin{bmatrix} \blacksquare & * & * & * \\ 0 & * & * & * \\ * & * & * & * \end{bmatrix}$$

2) "Clean out" the first column containing nonzero entries, by adding multiples of the top row to the other rows.

$$\longrightarrow \begin{bmatrix} \blacksquare & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

3) Set aside the top row, work on the rest of the matrix.

$$\left[\begin{array}{cccc} \blacksquare & * & * & * \\ \hline 0 & * & * & * \\ 0 & * & * & * \end{array}\right]$$

Repeat steps 1-3 until matrix is in REF.

<u>Algorithm for putting matrix in RREF by row ops:</u>

Do 1-3 repeatedly as above. Get a matrix in REF

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

4) Rescale each row to make the pivots 1

$$\longrightarrow \begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

5) Working from R to L, use each pivot to "clean out" the entries above it, by adding multiples of the row containing pivot to other rows

$$\longrightarrow \begin{bmatrix} 0 & 1 & * & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Once we have put the augmented matrix of some linear system in RREF we can easily read off the solutions.

Ex If RREF of aug. matrix is

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 2 & 0 & 7 & 0 & -4 \\ 0 & 1 & 0 & 3 & 0 & 4 & 0 & 8 \\ 0 & 0 & 1 & -1 & 0 & 5 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right] \rightsquigarrow \begin{aligned} x_1 + 2x_4 + 7x_6 &= -4 \\ x_2 + 3x_4 + 4x_6 &= 8 \\ x_3 - x_4 + 5x_6 &= 3 \\ x_5 + 2x_6 &= 7 \\ x_7 &= 3 \end{aligned}$$

Call the variables corresponding to pivot columns "basic variables"
[Here x_1, x_2, x_3, x_5, x_7 are basic]

Write sol^{n}'s as eq for basic vars:

$$\left[\begin{array}{l} x_1 = -4 - 2x_4 - 7x_6 \\ x_2 = 8 - 3x_4 - 4x_6 \\ x_3 = 3 + x_4 - 5x_6 \\ x_5 = 7 - 2x_6 \\ x_7 = 3 \\ \hline x_4 = \text{anything} \\ x_6 = \text{anything} \end{array}\right] \leftarrow \left[\text{Solution set}\right]$$

Call the non-basic variables "free variables."
[Here x_4, x_6 are free]

This system is consistent, has infinitely many solutions
(Any consistent sys. w/ free variables has infinitely many sol^{n}'s)

Ex $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right] \rightsquigarrow \begin{array}{l} x_1 = -2 \\ x_2 = 3 \\ x_3 = 4 \end{array}$

all vars basic — no free vars

Consistent, has unique solution

Ex $\left[\begin{array}{cccc|c} 1 & 0 & 5 & 2 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \begin{array}{r} x_1 + 5x_3 + 2x_4 = 0 \\ x_2 + 3x_3 - x_4 = 0 \\ 0 = 1 \end{array} \longleftarrow !$

x_1, x_2 basic
x_3, x_4 free

Inconsistent - no sol^n (even tho there are free vars)

Whenever the RREF of aug. mat. has pivot in last column the system is inconsistent.

Last note: to see free/basic vars
and consistent/inconsistent

REF is enough — don't need RREF.

Vectors and Vector Equations (Sec 1.3)

(For now): A (column) <u>vector</u> is a set of real numbers arranged in a column:

<u>Ex</u> $\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2} \\ 8 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 6 \\ -2 \\ 0 \end{bmatrix} \in \mathbb{R}^4$

$\in \mathbb{R}^3$ $\quad$ $\in \mathbb{R}^2$

The set of all vectors with n entries is called $\mathbb{R}^n$.

Denote vectors by e.g. $\vec{x}, \vec{y}, \vec{u}, \ldots$ (or bold letters)

Use $\vec{0}$ for zero vector $\qquad$ e.g. $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3$

We can <u>add vectors</u>: by adding their entries

so e.g. if $\vec{x} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

then $\vec{x} + \vec{y} = \begin{bmatrix} -1+4 \\ 7+2 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

But, we can only add vectors of the <u>same length</u>

e.g. $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ <u>doesn't make sense</u> (is <u>undefined</u>)

Scalar multiplication: if c is a constant ("scalar") and $\vec{x}$ is a vector

then $c\vec{x}$ is the vector obtained by multiplying each entry of $\vec{x}$ by c.

Ex

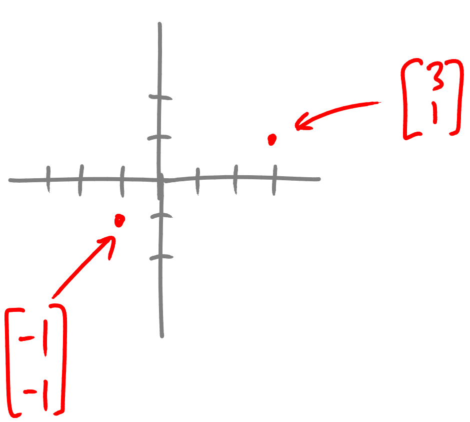
$$4\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}4\\12\end{bmatrix} \qquad -3\begin{bmatrix}5\\0\\-2\end{bmatrix}=\begin{bmatrix}-15\\0\\6\end{bmatrix}$$

$$0\begin{bmatrix}-4\\7\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}=\vec{0}$$

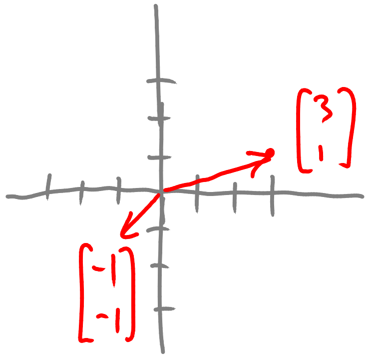
Notation: also write $-\vec{x}$ for the vector $(-1)\vec{x}$.

Picturing $\mathbb{R}^2$

We identify a vector $\begin{bmatrix}x\\y\end{bmatrix}$ with a point (x,y) of the plane.

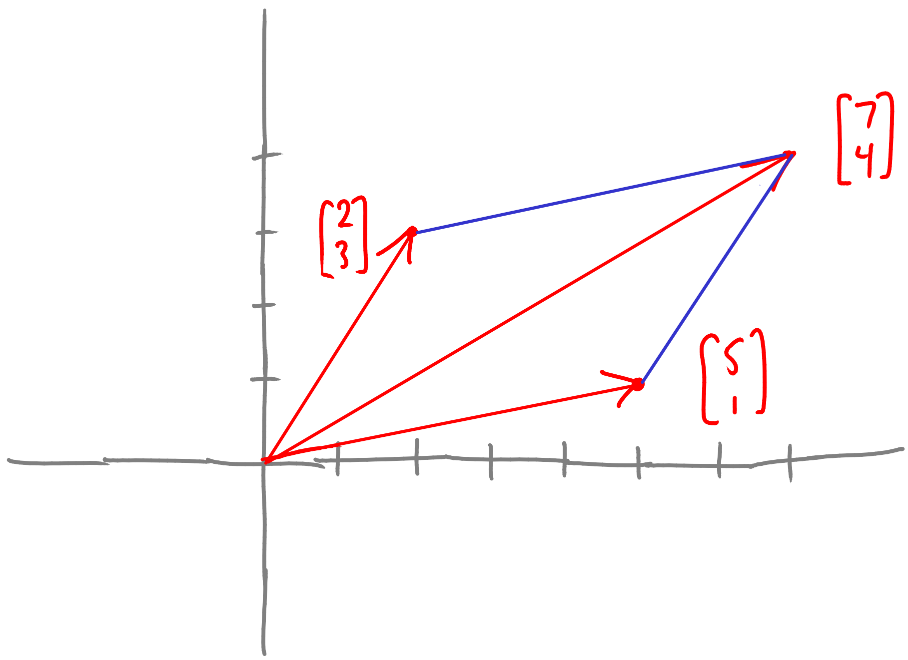


Sometimes also represent them as <u>arrows</u> from the origin:

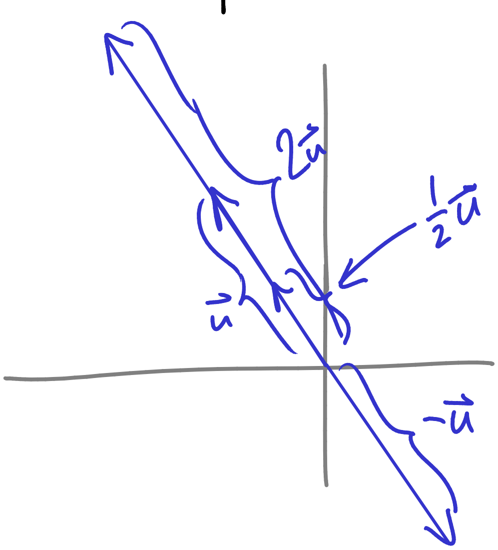


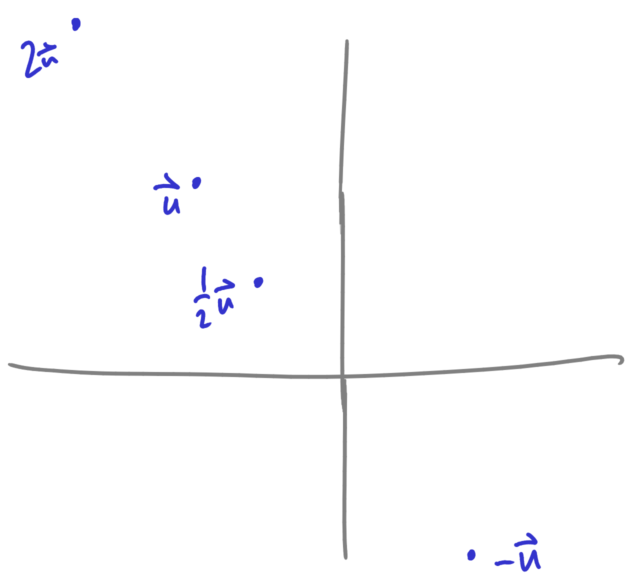
Addition of vectors in $\mathbb{R}^2$ can be pictured via the "parallelogram law":

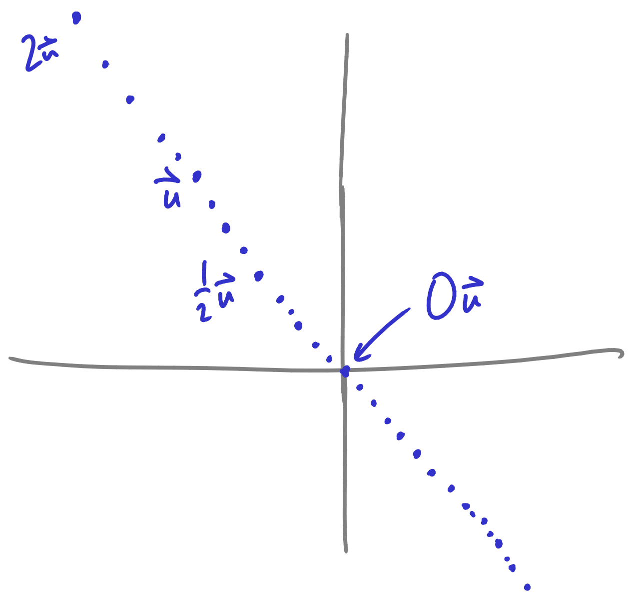
<u>Ex</u> $\begin{bmatrix}5\\1\end{bmatrix}+\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}7\\4\end{bmatrix}$



Scalar multiplication rescales the <u>length</u> of a vector







The set of all scalar multiples $c\vec{u}$ makes up a line. (through the origin)

Can also picture vectors in $\mathbb{R}^3$ similarly: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longleftrightarrow$ point (x,y,z)

Linear combinations

A vector of the form

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$$

is called a linear combination of the vectors $\vec{v}_1, \ldots, \vec{v}_p$.

Ex
$$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

so $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ is a lin. comb. of $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$.

Ex

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a lin. comb. of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(for <u>any</u> x_1, x_2!)

Ex

$$\vec{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} -5 \\ 7 \\ 5 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 10 \\ 17 \end{bmatrix}$$

Is $\vec{b}$ a lin comb of $\vec{a}_1, \vec{a}_2$? (If so, with what weights?)

i.e: Do there exist x_1, x_2 such that $x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}$?
If so, what are they?

i.e. $$\begin{bmatrix} 2x_1 \\ x_1 \\ 4x_1 \end{bmatrix} + \begin{bmatrix} -5x_2 \\ 7x_2 \\ 5x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 17 \end{bmatrix}$$

i.e. $$\begin{bmatrix} 2x_1 - 5x_2 \\ x_1 + 7x_2 \\ 4x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 17 \end{bmatrix}$$

Linear system for the variables x_1, x_2.

Aug. matrix: $$\left[\begin{array}{cc|c} 2 & -5 & 1 \\ 1 & 7 & 10 \\ 4 & 5 & 17 \end{array}\right] = \left[\vec{a}_1 \; \vec{a}_2 \,\middle|\, \vec{b}\right]$$

RREF:

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_1 = 3 \\ x_2 = 1 \end{array}$$

So $3\vec{a}_1 + \vec{a}_2 = \vec{b}$

i.e. $\vec{b}$ is a lin. comb. of $\vec{a}_1$ and $\vec{a}_2$, with weights $c_1 = 3$ and $c_2 = 1$.

<u>Fact</u>: $\vec{b}$ is a lin. comb. of $\vec{a}_1, \ldots, \vec{a}_p$

$\Updownarrow$ (if and only if)

The lin. sys. represented by aug. matrix

$$\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_p & \vec{b} \end{array}\right]$$

is <u>consistent</u>