Lecture 3 — 2 Sep 2010

Last time: vectors and linear combinations

Facts about vector arithmetic:

$\vec{u}+\vec{v} = \vec{v}+\vec{u}$ $\qquad$ $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$

$(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$ $\qquad$ $(c+d)\vec{u} = c\vec{u}+d\vec{u}$

$\vec{u}+\vec{0} = \vec{0}+\vec{u} = \vec{u}$ $\qquad$ $c(d\vec{u}) = (cd)\vec{u}$

$\vec{u}+(-\vec{u}) = (-\vec{u})+\vec{u} = \vec{0}$ $\qquad$ $1\vec{u} = \vec{u}$

Fact: (from last time)

Solutions of linear eq.

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

are the same as sol^{n}s of lin sys w/ aug. mat.

$$\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b} \end{array}\right]$$

[Say $\vec{b}$ is a lin. comb. of $\vec{a}_1,\dots,\vec{a}_n$ if this lin. sys. has a solution]

Define the span of a set of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ to be the collection of all vectors which are linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

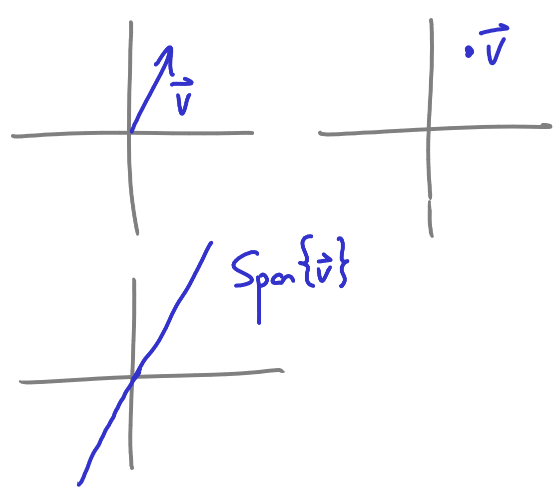
(Notation: $\mathrm{Span}\{\vec{v}_1,\dots,\vec{v}_n\}$)

Ex Say $\vec{v}$ is a nonzero vector in $\mathbb{R}^2$.

What is $\mathrm{Span}\{\vec{v}\}$?

It's all scalar multiples $c\vec{v}$

i.e. a line



Ex Say $\vec{v}_1, \vec{v}_2$ are two nonzero vectors in $\mathbb{R}^3$.

What is $\text{Span}\{\vec{v}_1, \vec{v}_2\}$?

It's the plane through $\vec{v}_1, \vec{v}_2$ and $\vec{0}$.

(Exception: If $\vec{v}_1$ and $\vec{v}_2$ are parallel and nonzero

i.e. $\vec{v}_1 = c\vec{v}_2$

then $\text{Span}\{\vec{v}_1, \vec{v}_2\}$ is a line)

Matrix Equations (Sec 1.4)

If A is an $m \times n$ matrix (m rows, n columns)

and $\vec{x} \in \mathbb{R}^n$

Then we define $A\vec{x}$ to be the linear combination of the columns using the entries of $\vec{x}$ as weights:

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

Ex

$$\begin{bmatrix} 3 & 2 & -1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix} = 6\begin{bmatrix} 3 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 2 \\ 4 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 18-6-2 \\ 0-12+4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ -8 \end{bmatrix}$$

Ex

$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

Rewrite linear systems using this notation:

<u>Ex</u>

$$4x_1 - 6x_2 + x_3 = 0$$
$$x_1 + 2x_2 + 4x_3 = 7$$

$$x_1\begin{bmatrix}4\\1\end{bmatrix} + x_2\begin{bmatrix}-6\\2\end{bmatrix} + x_3\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}0\\7\end{bmatrix}$$

ie $$\underbrace{\begin{bmatrix}4 & -6 & 1\\1 & 2 & 4\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix}0\\7\end{bmatrix}}_{\vec{b}}$$ ie $\underline{A\vec{x} = \vec{b}}$

<u>Fact</u> <u>If</u> A is an $m \times n$ matrix w/ colums $\vec{a}_1, .., \vec{a}_n$ and $\vec{b} \in \mathbb{R}^m$

<u>Then</u> The eq. $A\vec{x} = \vec{b}$

is <u>equivalent to</u>

the sys. of lin. eq. whose aug. mat. is $\left[\vec{a}_1\, \vec{a}_2 \cdots \vec{a}_n \,\middle|\, \vec{b}\right]$

<u>or to</u>

the vector eq. $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$

<u>Fact</u> $A\vec{x} = \vec{b}$ has solution $\Longleftrightarrow$ $\vec{b}$ is a lin. comb. of columns of A

Say $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Is $A\vec{x} = \vec{b}$ consistent for all possible values of $\vec{b}$?

Write $\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right]$ Does it have a pivot in last column?

Row reduce: $\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + b_1 - \frac{1}{2}b_2 \end{array}\right]$?

It's consistent if and only if $b_3 + b_1 - \frac{1}{2}b_2 = 0$.

If we had gotten instead $\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + \frac{1}{4}b_1 \\ 0 & 0 & 3 & b_3 + b_1 - \frac{1}{2}b_2 \end{array}\right]$

Then the system is always consistent no matter what b_1, b_2, b_3 are.

Fact If A is an $m \times n$ matrix
Then the following statements are equiv:

a) For every $\vec{b} \in \mathbb{R}^m$ there's a solution to $A\vec{x} = \vec{b}$
b) Every $\vec{b} \in \mathbb{R}^m$ is a lin. comb. of columns of A
c) The columns of A span $\mathbb{R}^m$
d) A has pivot positions in every row

Faster scheme for calculating $A\vec{x}$:

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 4 & 0 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3x_1 + x_2 - x_3 \\ 0x_1 + 4x_2 + 0x_3 \\ 2x_1 + 5x_2 - x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & -1 \\ -3 & 0 & 4 \end{bmatrix} \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix} = \begin{bmatrix} 2\cdot 6 + 5\cdot 1 + (-1)\cdot 2 \\ -3\cdot 6 + 0\cdot 1 + 4\cdot 2 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \end{bmatrix}$$

A special kind of matrix:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}}$$

$I\vec{x} = \vec{x}$ for every $\vec{x}$

Call I "identity matrix"

(We wrote 3×3 version, but similar matrix exists in $n\times n$ size)

Facts

$$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$$

$$A(c\vec{u}) = c(A\vec{u})$$

e.g. $A = [\vec{a}_1\ \vec{a}_2]$ $\quad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$A(\vec{u}+\vec{v}) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}$$

$$= (u_1+v_1)\vec{a}_1 + (u_2+v_2)\vec{a}_2$$

$$= (u_1\vec{a}_1 + u_2\vec{a}_2) + (v_1\vec{a}_1 + v_2\vec{a}_2)$$

$$= A\vec{u} + A\vec{v} \quad \checkmark$$

Solution sets of Lin Sys (Sec 1.5)

Any lin. sys. is equiv. to a matrix eq $A\vec{x} = \vec{b}$

If it's equiv to $A\vec{x} = \vec{0}$ (i.e. $\vec{b} = \vec{0}$)

then we call it a homogeneous linear system

Homogeneous lin sys. are always consistent, because they always have the solution $\vec{x} = \vec{0}$. Call this the "trivial solution."

We may ask: Does the equation $A\vec{x} = \vec{0}$ have any nontrivial solutions?

Fact: A homogeneous lin. sys has nontrivial solutions

$\Updownarrow$

it has at least 1 free variable.

Ex Describe the sol^n set of

$$2x_1 - x_2 + 6x_3 = 0$$
$$4x_1 - 2x_2 + 13x_3 = 0$$
$$-2x_1 + x_2 - 7x_3 = 0$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 4 & -2 & 13 & 0 \\ -2 & 1 & -7 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - \frac{1}{2}x_2 = 0$
$x_3 = 0$

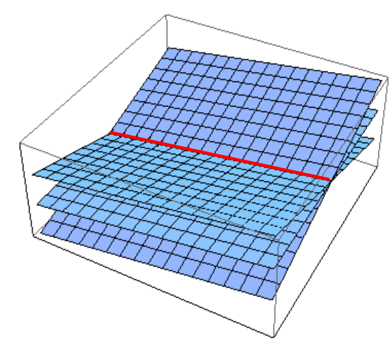
i.e.

$x_1 = \frac{1}{2}x_2$
$x_3 = 0$
x_2 free

ie $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$

So the space of solutions is $\text{Span}\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$

[a line in $\mathbb{R}^3$] $\longrightarrow$



3 linear eq $\longrightarrow$ 3 planes: solution set is the line where all 3 intersect

Ex Describe solutions of

$$4x_1 + 8x_2 + 4x_3 = 0$$
$$-2x_1 - 4x_2 - 2x_3 = 0$$

$$\left[\begin{array}{ccc|c} 4 & 8 & 4 & 0 \\ -2 & -4 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 2x_2 + x_3 = 0$

ie $x_1 = -2x_2 - x_3$
x_2 free
x_3 free

ie $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix}$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So the space of sol$^{\underline{n}}$s is $\mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

[a <u>plane</u> in $\mathbb{R}^3$]