Lecture 7 | 16 Sep 2010

Comments on HW:

- Mostly good!
- Grading – mostly by completion — 2pts each
  only a few (6) checked for correctness — 5pts each
- On T/F questions – please write 1 line of explanation
- Sometimes if it looks easy it is easy...

Fast review: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 9 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix} \right\}$ are lin dep

b/c it's 4 vectors in $\mathbb{R}^3$ and $4 > 3$

Last time: Matrix operations

$$\begin{matrix} A & m \times n \\ B & n \times p \end{matrix} \longrightarrow AB \quad m \times p$$

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "\text{multiply by } B"] \arrow[rr, bend right, "\text{mult. by } AB"'] & \mathbb{R}^n \arrow[r, "\text{mult. by } A"] & \mathbb{R}^m \end{tikzcd}$$

$$AB(\vec{x}) = A(B(\vec{x}))$$

Matrix powers A^n is just A multiplied by itself n times (if A is square)

Ex $A^4 = A \cdot A \cdot A \cdot A$

(Not the same as raising every entry of A to the n^{th} power!)

Ex $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

If $T(\vec{x}) = A\vec{x}$ then $\underbrace{T \circ T \circ \cdots \circ T}_{n \text{ times}}(\vec{x}) = A^n \vec{x}$

Ex Rotation of $\mathbb{R}^2$ by an angle θ is represented by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

Transposition: If we have any matrix A

Define a matrix A^T ("transpose of A")

by $(A^T)_{ij} = A_{ji}$ i.e. swap the rows and columns

Ex If $A = \begin{bmatrix} 1 & 3 \\ 0 & 4 \\ 2 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 6 \end{bmatrix}$

Facts
- $(A^T)^T = A$
- $(rA)^T = rA^T$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

(and $(ABC)^T = C^T B^T A^T$, etc...)

The Inverse Of A Matrix (Sec 2.2)

If we have the equation $ax = b$
and want to solve for x, it's easy (if $a \neq 0$):
multiply both sides by a^{-1}. Then the eq becomes

$$a^{-1} \cdot ax = a^{-1} b$$
$$1 \cdot x = a^{-1} b$$
$$\underline{\underline{x = a^{-1} b}}$$

What's the matrix version of that?

Say A is a square matrix. $(n \times n)$
We say C (another $n \times n$ matrix) is the inverse of A if

$$AC = I \quad \text{and} \quad CA = I$$

If C is the inverse of A we often write $C = A^{-1}$.

Ex If $A = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}$.

Why? $A \cdot A^{-1} = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$A^{-1} \cdot A = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Not every square matrix has an inverse. If A has an inverse, call it invertible, or nonsingular. If A doesn't have an inverse, call it non-invertible, or singular.

Fact. Say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

If $ad-bc \neq 0$ then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

If $ad-bc = 0$ then A is non-invertible (singular).

Ex $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ is non-invertible b/c $1\cdot 8 - 2\cdot 4 = 0$.

Ex Find the inverse of $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$.

Just use the above formula: $A^{-1} = \frac{1}{12-10}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{5}{2} \\ -1 & \frac{3}{2} \end{bmatrix}$

So: If A is invertible, then to solve

$$A\vec{x} = \vec{b}$$

we can just multiply both sides by A^{-1}:

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

So — **Fact:** If A is invertible then

$$A\vec{x} = \vec{b}$$

has a <u>unique</u> solution, given by $\vec{x} = A^{-1}\vec{b}$

Ex Solve the equations

$$3x_1 + 5x_2 = 3$$
$$2x_1 + 4x_2 = 4$$

This $A\vec{x} = \vec{b}$

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -5/2 \\ -1 & 3/2 \end{bmatrix} \quad \text{(from above)}$$

$$A^{-1}\vec{b} = \begin{bmatrix} 2 & -5/2 \\ -1 & 3/2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \text{So } \vec{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \begin{matrix} x_1 = -4 \\ x_2 = 3 \end{matrix}$$

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1) How can we tell whether an $n \times n$ matrix is <u>invertible</u>?
2) And what its inverse is?

Answer to 1): It's invertible if and only if $A\vec{x} = \vec{b}$ has unique sol^n for all $\vec{b}$
— which is the same as:

Fact: A is invertible $\Longleftrightarrow$ the row reduction of A has n pivots.

Another way to say that:

Fact: A is invertible $\Longleftrightarrow$ the RREF of A is

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

i.e. the RREF of A is I

Algorithm for finding A^{-1}:

Start with the matrix $(n \times 2n)$

$$n\left\{\left[\,\overbrace{A}^{n}\;\middle|\;\overbrace{I}^{n}\,\right]\right.$$

and calculate its RREF. If A is invertible, you get

$$\left[\, I \;\middle|\; A^{-1} \,\right]$$

Ex Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ if it exists.

Use the algorithm above:

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \overset{\text{many steps}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & 1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & 1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$

Check: $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}\begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & 1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Inverses of 1×1 matrices: $A=[a]$ $\qquad A^{-1}=\left[\frac{1}{a}\right]$

Ex Find inverse of $A=\begin{bmatrix}4&3&1\\2&1&0\\6&4&1\end{bmatrix}$ if it exists.

$$[A\,|\,I]=\left[\begin{array}{ccc|ccc}4&3&1&1&0&0\\2&1&0&0&1&0\\6&4&1&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&*&*&*&*\\0&1&*&*&*&*\\0&0&0&*&*&*\end{array}\right]$$

So here A is not invertible.

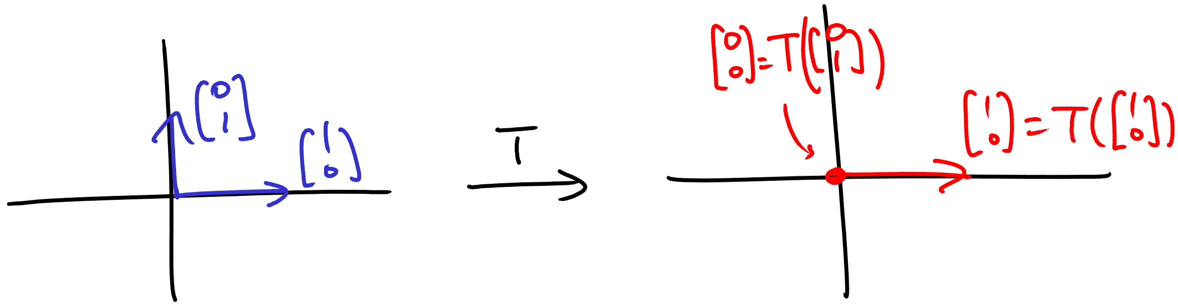
Suppose we look at $\quad A\vec{x}=\vec{0}$
and A is invertible.
Then

$$\vec{x}=A^{-1}\vec{0}$$
$$=\vec{0}$$

So, Fact: A is invertible $\iff$ the equation $A\vec{x}=\vec{0}$ has only the trivial solution.

Facts: A is invertible $\iff$ $A\vec{x}=\vec{b}$ has unique sol^n for any $\vec{b}$
$\iff$ $T(\vec{x})=A\vec{x}$ is 1-1
$\iff$ $T(\vec{x})=A\vec{x}$ has image the whole $\mathbb{R}^n$
$\iff$ A has n pivots
$\iff$ columns of A span $\mathbb{R}^n$
$\iff$ columns of A are linearly independent

Ex The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by <u>projection</u> on the x-axis



Standard matrix of T is $A = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$

A is <u>not</u> invertible —

and correspondingly, T is not 1-1

T doesn't have range = all of $\mathbb{R}^2$

$\vdots$