CHAPTER 2

Preliminaries: Numbers and Functions

What exactly is a number?

If you think about it, to give a precise answer to this question is surprisingly difficult. As is often the case, the word 'number' reflects a concept of which we have some intuitive understanding, but no concrete definition. In this introduction, we will attempt to pin down what we mean by a number by describing exactly what we should expect from a number system. In fact, though we will not prove it, the only collection that satisfies are of our demands is $\mathbb{R}$, the collection of **real numbers**. Thus we conclude that a number is an element of the set $\mathbb{R}$. Just as with numbers, most of us have probably heard the term 'real numbers,' but may not be exactly sure what they are. Studying real numbers will be one of the important purposes of this course.

As mentioned above, we all know the things that we should expect from a number system. Think back to when you first met the idea of a number. Probably the very first purpose of numbers in your life is that they allowed you to count things: 50 states, 32 professional football teams, 7 continents, 5 golden rings, etc. Needing to count things leads us to the invention (or discovery depending on your point of view) of the natural numbers (the numbers $1, 2, 3, 4, 5, \cdots$). Mathematicians typically denote the collection of natural numbers by the symbol '$\mathbb{N}$.' Though this collection can be constructed quite rigorously from the standard axioms of mathematics, we will assume that we are all familiar with the natural numbers and their basic properties (such as the concept of mathematical induction; see the appendices). The natural numbers fulfill quite successfully our goal of being able to count.

The next thing that we expect of our number system is that it should be able to answer questions like the following: "If the Big Twelve has 10 football teams and the Big Ten has 12 (shockingly it's true), how many teams do the conferences have between them?" In other words we will need to add. We will also multiply. The natural numbers are already well-suited for these tasks. Really this should not come as a surprise. After all, adding natural numbers is really just a different way of looking at counting (i.e., adding three and five is the same as taking three dogs and five cats and counting the total number of animals). As we all know, multiplication of natural numbers is really just repeated addition.

Having addition naturally leads us to subtraction. This is the first place the natural numbers will fail us. Subtracting 7 from 2 is an operation that cannot be performed within $\mathbb{N}$. The need for subtraction, therefore, is one of the reasons that $\mathbb{N}$ will not work as our entire number system. Thus we need to expand the set of natural numbers to the integers. As we all probably know, the integers are comprised of the natural numbers, the number zero, and the negatives of the natural numbers (at this point, you might protest and say that zero should be included as a natural number as it allows us to count collections which contain no objects; in fact many mathematicians do include zero in $\mathbb{N}$, but the distinction is of little importance). The collection of integers is denoted by $\mathbb{Z}$. Again we will assume we know all the basic properties of $\mathbb{Z}$.

The integers are a very good number system for most purposes, but they still have an obvious defect: we cannot divide. Surely any reasonable number system allows division: if you and I have a sandwich and we each want an equal share, a number should describe the portion we each get. Needing division, we throw in fractions: symbols which are comprised of two integers, one in the numerator and one in the denominator (of course the denominator is not allowed to be zero). A fraction will represent the number which results when the numerator is divided by the denominator.

Combining all the numbers we have so far gives $\mathbb{Q}$, the collection of rational numbers. Again, we will assume that we are familiar with all its basic properties. Before we go on to justify our assertion that $\mathbb{Q}$ is not a sufficient number system, we have another property to point out. Notice that most of our properties so far involve **operations** among our numbers: namely addition, subtraction, multiplication, and division. We call these types of properties **algebraic** (in mathematics, the word **algebra** describes the study of operations). The property we are going to discuss next is not algebraic.

Suppose then that I pick a rational number and you pick another. We can easily decide which is bigger: Namely $\frac{a}{b}$ is bigger than $\frac{c}{d}$ (where a, b, c, and d are integers) if ad is bigger than bc (assuming b and d both positive; we can easily assure both denominators are positive by moving any negative into the numerator). Since ad and bc are integers, we know how to compare them (because we know how to compare natural numbers and how to take negatives into account). Since we can always compare any two rational numbers in this way, we say that $\mathbb{Q}$ is totally **ordered**.

In retrospect, we should have demanded this property of our number system from the beginning. Numbers should come with some notion of size. Fortunately, we got it for free. Moreover, it is interesting to notice that our expectation that a number system should include the natural numbers and that it should have certain algebraic properties is enough to lead us to include all of $\mathbb{Q}$. We did not need to insist that our system be ordered to find $\mathbb{Q}$. The order properties turn out to be more important in telling us which potential numbers we should *not* include (such as the imaginary number i).

$\mathbb{Q}$ comes very close to satisfying everything we want in a number system. Unfortunately it is still lacking. Suppose we draw a circle whose diameter is 1. The area of a circle with radius one (usually called the **unit circle**) should certainly be a number. If, however, we restrict ourselves to the rational numbers, this area will not be a number (the number is of course usually denoted π and it is not a rational number). The same could be said of the length of one of the sides of a square whose area is 2 (this number is usually denoted $\sqrt{2}$).

These two examples merely comprise our attempt to give a (geometric) demonstration that $\mathbb{Q}$ is lacking as a number system. The real (more general) property that we seek, called 'completeness', is actually quite subtle and has to do with the presence of something like 'gaps' in $\mathbb{Q}$ (the absence of the number $\sqrt{2}$ or of the number π is an example of such a gap). These gaps have to do with something called a 'monotone sequences' which we will study in detail in this course. One consequence of filling in these gaps is that we are able to perform calculus (showing this might be viewed as the main mathematical purpose of this course). This, in turn, allows us to express all the lengths, areas, volumes, etc. of geometric objects like the examples above as real numbers.

In that we have been a little bit vague in the preceding discussion, we formulate our demands precisely in the appendices (with the exception of the completeness axiom as it is a major object of study in this course). Once again, one of the fundamental results in mathematics is that the collection of real numbers is the *only* system of numbers which satisfies all of our demands. We thus conclude that the real numbers comprise the only possible choice of a number system (at least in the sense we have given; there are a surprising number of close competitors if we relax some of our demands).

To give an exact definition of the real number is surprisingly complicated. In fact, the first rigorous construction of the real numbers was given by Georg Cantor as late as 1873 (by comparison, the rational numbers where constructed in ancient times). For our purposes, we will first take it on faith that the real numbers exist as a number system and that they satisfy the demands we have described. Later in the book (towards the ends of Chapter 3), we will describe a way to define the real numbers rigorously using decimal expansions (there are actually several well-known ways and the way we choose, though perhaps the most famous, was not the first).

Finally, it is important to realize that the properties given in the appendix (which we will call **axioms**), together with the completeness axiom, are the *only* properties that we assume about $\mathbb{R}$. Strictly speaking, any other statement we want to make must be proven from either from our axioms or from properties we have already assumed about $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ (or, of course, some combination of the two).

In general, however, this can get to be a little bit tedious. Hence, we will allow you to assume all of the 'basic' or 'obvious' properties of the real numbers. Unfortunately, deciding which properties are obvious is a subjective process. Therefore, if there is any doubt about whether a statement is obvious, you should prove it rigorously from the axioms (or at least describe how to prove it rigorously). Actually, the ability to decide when statements are obvious or 'trivial' is an important skill in mathematics. Possessing this ability can often be a reflection of great mathematical maturity and insight. In general, make sure you are prepared to back up all your assertions to your fellow students and to your instructor.

In the appendix to this chapter, we will also derive some properties of $\mathbb{R}$ that follow from our axioms. We may work on some of these in class, but thereafter you may consider them "known." The appendix also contains a discussion of basic set theory, induction, and some supplementary material on cardinality.

1. Functions

Although most people may not realize it, the concept of a function is far more basic to mathematics than is the concept of a number. Roughly speaking, a function f from a set A to a set B is a rule that assigns to each element of A an element of B. In this case we write $f : A \to B$. What do we mean by 'rule'? Let's try to be more precise.

Definition. A **function** f from A to B, denoted by $f : A \to B$, is a subset f of the Cartesian product $A \times B = \{(a, b) : a \in A, b \in B\}$ satisfying

(1) for each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$
(2) for all $a \in A$ and for all $b, b' \in B$ if $(a, b) \in f$ and $(a, b') \in f$ then $b = b'$.

The set A is called the **domain** of f and B is called its **codomain**.

At this point, it is probably a good time for some general advice regarding definitions. When taking a rigorous proof-based courses, many students merely skim over (or ignore) the definitions and go straight to the problems (specifically the ones they have been assigned). It is our recommendation that you avoid this behavior: the more deep thinking that you do about the concepts of a course, the easier time you will have succeeding. This is not to say that thinking deeply is easy. It is typically a very difficult time consuming process, but the results can be very rewarding.

Thus rather than simply read the definitions you come across, you should attempt to think deeply about them, particularly the ones that are confusing or long. At times, when we feel that a definition is particularly confusing, we will try to help you through this process, but you should be doing it all the time.

We will give some general advice now in how to think about definitions. We will also give some more advice later in this chapter (after we have more examples of definition that we may use to illustrate our points). Throughout the course, we will emphasize the fact that most (or perhaps all) definitions have two sides to them: the precise definition in mathematical language and the intuitive notion that the definition is an attempt to express.

For example, in the present case, we, speaking intuitively, stated that a function is a rule. We then proceeded to give the precise mathematical definition. When you have both sides of the definition before you, you should ask yourself how the precise definition captures the intuitive notion. Does it capture it fully? Is anything missing? Is the precise definition more broad than the intuitive notion? Is it more narrow? Ask yourselves these questions about the definition above. We will often give a precise definition without giving an intuitive description. In those cases, you should describe for yourself the intuitive idea that is attempting to be captured.

In general, both sides of the definition are important. When giving a proof of a statement which involves a term that we have defined, you will likely need to use the precise definition in your proof. Nevertheless, it is oftentimes the intuitive notion that *leads* your to the proof. Probably more often than not, mathematicians think about a result intuitively and then write down a rigorous proof to back up their intuition.

Technically, then, a function from A to B is just a special subset of $A \times B$. Mathematicians, however, rarely think of functions in this way. Rather we think of their more intuitive notion: a rule. For this reason, we typically use a different notation when discussing functions. Explicitly, instead of saying that the point (a, b) is an element of our function, we write $f(a) = b$. Translating the definition of a function into this notation gives the following definition for function:

> for each $a \in A$ there exists a unique $b \in B$ such that $f(a) = b$

You should see for yourself why this statement is the same as the above definition.

Reflecting the intuitive notion that they capture, a function is sometimes called a **mapping** or a **transformation**. Correspondingly, if $f(a) = b$, one might say "f maps a to b" or "f sends a to b." We should point out, however, that a function is more than just a rule. It actually has three ingredients: the domain, the codomain, and a rule which sends elements of A to elements of B. For example the function $f : [0, 1] \to \mathbb{R}$ given by $f(x) = x^2$ and the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$ are not the same.

As our next bit of general advice regarding definitions, we point out that some definitions, like the previous one, define terms, such as 'function,' that we have all heard before. If you come across a definition that you have already learned, you should compare the definition given with idea in your own head on the other. Are the two notions the same? Is the definition given more general than the one in your head? Is it more specific?

Along these lines, many beginning students believe that a function is the same things a formula. In other words, many students think that in order to specify a function, they need to find a formula using variables. This is not the case. It is perfectly reasonable to define a function by saying something like:

> Define a function from the set of real numbers to the set $\{0, 1\}$ by assigning the value 1 to all rational numbers and the value 0 to all irrational numbers.

Since every number has been given a value and no number has been given more then one, our rule gives a function.

This function $f : \mathbb{R} \to \{0,1\}$ would probably be more commonly described by saying that for $x \in \mathbb{R}$,

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases},$$

but either description would work. Notice that it would be essentially impossible to find what most people would call a 'formula' to describe this function.

You should come up with some examples of this kind on your own. In other words, give some examples of functions that don't have a formula in this sense.

We now define some more terms related to general functions.

Suppose that $f : A \to B$ is a function and suppose that S is a subset of A. We can define a new function $\widetilde{f} : S \to B$ by using same rule as for f but by restricting ourselves to points in S. That is, $\widetilde{f}$ is defined by $\widetilde{f}(x) = f(x)$ for $x \in S$. $\widetilde{f}$ is called the **restriction** of f to S and is usually denoted $f|_S$.

Definition. If $f : A \to B$ is a function, the **range** of f, denoted by $f(A)$, is

$$f(A) = \{f(a) : a \in A\}.$$

Some functions have certain important properties that we shall name.

Definition. Let $f : A \to B$.

(1) f is **surjective** (or **onto**) if $f(A) = B$. That is, f is onto if, for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$.
(2) f is **injective** (**one-to-one** or **1–1**) if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$. That is, f is 1–1 if, for all $a_1, a_2 \in A$ such that $a_1 \neq a_2$, we have $f(a_1) \neq f(a_2)$.
(3) f is a **bijection** (or a **1–1 correspondence**) if it is 1–1 and onto. This is equivalent to: for all $b \in B$ there exists a unique $a \in A$ with $f(a) = b$. (Note: it is usually simpler to show that a function is a bijection by showing it is 1–1 and onto separately.)

We point out that these notions depend on the domain and codomain of the function just as much as the depend on the rule.

2.1. Give an example of a function, $f : \mathbb{R} \to \mathbb{R}$, and a subset $S \subset \mathbb{R}$, such that f is not injective, but the restriction $f|_S : S \to \mathbb{R}$ is injective.

If a function is a bijection then you can 'reverse it' to obtain a function going the other way. The following theorem makes this precise.

2.2. Let $f : A \to B$ be a bijection. Then there exists a bijection $g : B \to A$ satisfying

(1) for all $a \in A$, $g(f(a)) = a$.
(2) for all $b \in B$, $f(g(b)) = b$.

Furthermore (and this is still part of the problem), this function g is unique; if g_1 and g_2 are bijections satisfying (1) and (2) then $g_1 = g_2$. (Would it be enough to assume g_1 and g_2 both satisfy (1)?)

The bijection g is called the **inverse function** of f and is usually denoted by f^{-1}. Do not confuse this with "$1/f$" (which would mean what?)

Definition. Let $f : A \to B$. Let $D \subseteq A$, and $C \subseteq B$.

(1) The **image** (or **direct image**) of D under f, denoted $f(D)$, is
$$f(D) = \{f(x) : x \in D\}.$$
(2) The **pre-image**, of C under f, denoted $f^{-1}(C)$, is
$$f^{-1}(C) = \{a \in A : f(a) \in C\}.$$

The definition above might lead to some temporary confusion in that we are using the symbol f^{-1} in two different ways. Explicitly, if the function f^{-1} exists, which is not always the case, then there are two different ways of reading $f^{-1}(C)$: it can be read as the direct image of the set C under the function f^{-1} or it can be read as the inverse image of C under the function f. Check for yourself that these two interpretations give the same set and so there is no ambiguity. We point out that $f^{-1}(C)$ always exists even if the function f^{-1} does not.

2.3. Let P be the collection of nonnegative real numbers. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that meets each of the following criteria. You can (and will have to) use different functions for different examples.

(1) $f^{-1}(P) = \varnothing$,
(2) $f(P) = \{-10, 10\}$, and

(3) there is some set $D \subset \mathbb{R}$ so that $f(f^{-1}(D)) \neq D$ (and you should specify the set D as well).

We next given an important way to combine two functions.

Definition. Let $f : A \to B$ and $g : B \to C$. The **composition** $g \circ f : A \to C$ is defined by $(g \circ f)(a) = g(f(a))$.

2.4. Let $f : A \to B$ and $g : B \to C$ be two bijections. Then $g \circ f : A \to C$ is also a bijection.

Caution: If f and g are functions, it is *not* true in general that $f \circ g = g \circ f$. In fact, these two compositions may have completely different domains and codomains!

2.5. Give an example of two function $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g \neq g \circ f$.

If $f : A \to B$ is a bijection then $f^{-1} \circ f : A \to A$ is the identity map on A and $f \circ f^{-1} : B \to B$ is the identity map on B (the identity map on a set S is the map $id : S \to S$ defined by $id(x) = x$ for all $x \in S$.) f^{-1} is the only function with these properties.

2. The Absolute Value

Now that we have given the general framework for functions, we move on to consider real numbers. In this section, we define an extremely important function on the real numbers.

Definition. Given a real number $a \in \mathbb{R}$, we define the absolute value of a, denoted $|a|$ to be a if a is nonnegative and $-a$ if a is negative.

Our definition might appear a bit strange at first. Surely we all know that the absolute value of a number should never be negative. Yet we said that the absolute value of a, at least for some values of a, is $-a$, which appears to be negative. Try some examples to figure out what is going on and explain this apparent contradiction to yourself. This sort of thinking will help you in the proof of the next result.

2.6. For $a \in \mathbb{R}$:

(1) $|a| \geq 0$,
(2) $|a| = 0$ if and only if $a = 0$,

(3) $|a| \geq a$, and
(4) $|-a| = |a|$.

Thus the absolute value gives us a function whose domain is $\mathbb{R}$ and whose codomain is the collection of non-negative real numbers.

The following technical observations will be of assistance in some arguments involving the absolute value.

2.7. For all $a \in \mathbb{R}$, $a^2 = |a|^2$.

2.8. For all $a, b \in \mathbb{R}$ with $a, b \geq 0$ we have $a^2 \leq b^2$ if and only if $a \leq b$. Likewise, $a < b$ if and only if $a^2 < b^2$.

The next statement gives two fundamental properties of the absolute value.

2.9. Let $a, b \in \mathbb{R}$, then

(1) $|ab| = |a||b|$ and
(2) $|a + b| \leq |a| + |b|$

Hint: One can prove these by laboriously checking all the cases (e.g., $a > 0$, $b \leq 0$) but in each case an elegant proof is obtained by using our previous observations to eliminate the absolute value and then proceeding using the properties of arithmetic.

The second inequality above is perhaps the most important inequality in all of analysis. It is called the **triangle inequality**.

The remaining results in this section are important consequences of the triangle inequality.

2.10. Let $a, b, c \in \mathbb{R}$. Then we have

(1) $|a - b| \geq \big||a| - |b|\big|$ and
(2) $|a - c| \leq |a - b| + |b - c|$.

The major importance of the absolute value is that it will give us some notion of 'distance' or 'length.' Indeed, you have probably measured the length of something using a yardstick. Needless to say, you typically line up one end of the object with zero and read the length by looking to see where the other end hits the ruler. In a tight place, however, you might line up one end at $7''$ and the other at $13''$. What is the length of the object in this case? Of course it is $13'' - 7'' = 6''$.

A bit more abstractly, if I told you one end was at x and the other was at y, what would be the length? Well, $y - x$ if $y > x$ and $x - y$ if $x > y$. In other words, it would be $|x - y|$. So, we can regard $|x - y|$ as the **distance** between the numbers x and y. Note this works for all real numbers, even if one or both is negative. It also explains why we call the inequality in Problem 2.9(2) the triangle inequality (why? See Problem 2.10(2)).

The statement of Problem 2.10(1) is often called the **reverse triangle inequality**.

2.11. Let $x, \epsilon \in \mathbb{R}$ with $\epsilon > 0$. Then:

(1) $|x| \leq \epsilon$ if and only if $-\epsilon \leq x \leq \epsilon$, where the double inequality $-\epsilon \leq x \leq \epsilon$ means $-\epsilon \leq x$ and $x \leq \epsilon$.
(2) If $a \in \mathbb{R}$, $|x - a| \leq \epsilon$ if and only if $a - \epsilon \leq x \leq a + \epsilon$.

We point out that, by a similar proof, the same properties hold with $\leq$ replaced by $<$.

3. Intervals

Intervals are a very important type of subset of $\mathbb{R}$. Loosely speaking they are sets which consist of all the numbers between two fixed numbers, called the endpoints. We also (informally) allow the endpoints to be $\pm\infty$. Depending on whether the endpoints are finite and whether we include them in our sets, we arrive at 9 different types of intervals in $\mathbb{R}$.

Definition. An interval is a set which falls into one of the following 9 categories (assume $a, b \in \mathbb{R}$ with $a < b$). We apply the word 'bounded' if both the endpoints, a and b, are finite. Otherwise we use the word 'unbounded'.

(1) Bounded open intervals are sets of the form
$$(a, b) := \{x \in \mathbb{R} : a < x < b\}.$$
(2) Bounded closed interval are sets of the form
$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$
(3) There are two type of half-open bounded intervals. One type is sets of the form
$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}.$$

(4) The other is sets of the form

$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}.$$

(5) There are also two types of unbounded open intervals not equal to $\mathbb{R}$. One type is sets of the form

$$(a, +\infty) := \{x \in \mathbb{R} : a < x\}.$$

(6) The other is sets of the form

$$(-\infty, b) := \{x \in \mathbb{R} : x < b\}.$$

(7) There are two types of unbounded closed intervals not equal to $\mathbb{R}$. One type is sets of the form

$$[a, +\infty) := \{x \in \mathbb{R} : a \leq x\}.$$

(8) The other is sets of the form

$$(-\infty, b] := \{x \in \mathbb{R} : x \leq b\}.$$

(9) The whole real line $\mathbb{R} = (-\infty, \infty)$ is an interval. We count $\mathbb{R}$ as being open, closed, and unbounded.

Some mathematicians include the empty set, $\varnothing$, and single points, $\{a\}$ for some $a \in \mathbb{R}$, as intervals. To distinguish these special sets, people often call them 'degenerate intervals' whereas sets of the above would be 'non-degenerate intervals.' We will reserve the word 'interval' for the non-degenerate case. That is, in our language, an interval is not allowed to be $\varnothing$ or $\{a\}$.

We now give some more general advice regarding definitions. One of the first things that you should do when you come across a definition is to come up with some examples of things that fit into the definition (at least in your own head, but it might help to write down your examples). In the present case, we will get you started: an example of an interval is all the numbers between 2 and 3, not including 2 or 3. More specifically, this is an example of a bounded open interval.

You should also attempt to make your examples as interesting or weird as possible so that you can test the outer reaches of what a definitions entails. For example, if we wanted to give an example of a number, we could certainly say the number 1 or the number 2. However, more exotic examples include numbers like $\frac{1}{3}$, $-\frac{7}{3}$, π or $\pi^2 + 7$ (at least once we prove they exist). The benefit of giving weird examples is that we can catch ourselves thinking too narrowly about a concept. If we say the word "number," many people think of $1, 2, 3, 4, 5 \ldots$, but in fact there are many more types of numbers, and forgetting this fact can sometimes lead to trouble.

Another way of looking at this bit of advice is that you should come up with examples that are not basically the same. In other words, if you want to create three examples of numbers, it would suffice to say 1, 2, and 3, but it might be more informative of the nature of numbers to say 0, $-\frac{21}{8}$ and $-e^2 + \frac{1}{2} + \pi$. In fact, it is important to generate both 'easy' or normal examples and 'weird' elaborate ones.

Perhaps equally important is to think of examples that do NOT fall into the definition. An easy way to do this is to name something that doesn't have anything to do with the definition. For example, the University of Texas football team is not an interval and neither is your roommate. However, it might be a better idea to come up with some examples that are close to the definition, but don't fall into it. For example, can you think of some subsets of the real numbers that are not intervals? Try to come up with some cheap examples and some clever ones. In general, try to get as close as you can to the definition without satisfying it. If a definition has two parts, try to come up with an example which fits one part but not the other.

Another question you should ask yourself: "why does this definition exist?" Why is the concept so important that generations of mathematicians have agree it should have a name? This question is not always to easy to answer, especially if you haven't seen the definition used a few times. Nevertheless, you should keep the question in mind as you go through the exercises and results surrounding the definition. In general the answer to this question can be closely related to the intuitive notion behind the definition.

You will notice that we actually gave this type of explanation for the absolute value: we said that it is important as way to measure length or size. We won't spell out exactly why intervals were given a name (you should try to come up with some reasons on your own), but we will say a word about why the notation exists. If we want to specify a set which includes exactly "all the numbers between 3 and π," we notice that the English phrase necessary is a bit long. It's also ambiguous: do we want to conclude 3 and π or leave them out? Thus to be clear, we really have to say something like "all the numbers between 3 and π, including 3 but not including π."

This terminology is definitely getting very cumbersome and so mathematicians have found it convenient to replace all these words with the symbols $[3, \pi)$. This is definitely much shorter, but it comes at a price: the meaning of the symbols might not be obvious to somebody who was already familiar with them and, more importantly, the notation might disguise some subtlety in the definition. This balance is one that any mathematician has to strike for himself or herself.

2.12. Let $a \in \mathbb{R}$ and $\epsilon > 0$. Write the set

$$\{x \in \mathbb{R} : \ |x - a| \leq \epsilon\}$$

as an interval. Write

$$\{x \in \mathbb{R} : \ |x - a| < \epsilon\}$$

as an interval.

Definition. The closure of an interval I, denoted $\overline{I}$, is the union of I and its finite endpoints.

Thus, for $a < b$,

$$\begin{gathered}\overline{(a,b)} = \overline{[a,b]} = \overline{[a,b)} = \overline{(a,b]} = [a,b] \\ \overline{(a,+\infty)} = \overline{[a,+\infty)} = [a,+\infty) \\ \overline{(-\infty,b)} = \overline{(-\infty,b]} = (-\infty,b] \\ \overline{\mathbb{R}} = \mathbb{R}\end{gathered}$$

Definition. The interior of an interval I, denoted I°, is I minus its endpoints.

Thus

$$\begin{gathered}(a,b)^\circ = [a,b]^\circ = [a,b)^\circ = (a,b]^\circ = (a,b) \\ (a,+\infty)^\circ = [a,+\infty)^\circ = (a,+\infty) \\ (-\infty,b)^\circ = (-\infty,b]^\circ = (-\infty,b) \\ \mathbb{R}^\circ = \mathbb{R}\end{gathered}$$

Now that we have given all the basic terminology, we will begin our study of some of the deeper properties of numbers.